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Free product of ortholattices

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The purpose of this paper is to prove a structure theorem for the free product of ortholattices. The method of BRUNS [1] for constructing a free ortholattice is combined with GRÄTZER's method for constructing the free product of lattices [2].

An ortholattice is a lattice L with a smallest element 0 and a largest element 1 and with an orthocomplementation $':L\to L$ such that

(i) $a''=a,\ \ a\in L,$

(ii) $a\leqq b$ implies $b'\leqq a',\ \ a,b\in L,$

(iii) $a\vee a'=1,\ \ a\wedge a'=0,\ \ a\in L.$

The free product of ortholattices is defined as follows.

Definition 1. Let $(L_i, 0_i, 1_i, '^i)$, $i\in I$, be a set of ortholattices. An ortholattice $(L, 0, 1, ')$ is a free product of the ortholattices L_i, $i\in I$, if

(i) for any $i\in I$, there is an injective homomorphism u_i: $L_i\to L$ which preserves the lattice operations and orthocomplementation so that each L_i can be considered as a subalgebra of L, and for $i,j\in I$, $i\neq j$, $L_i-\{0_i, 1_i\}$ and $L_j-\{0_j, 1_j\}$ are disjoint;

(ii) L is generated by $\bigcup\{u_i(L_i)\colon i\in I\}$;

(iii) for any ortholattice A and for a family of homomorphisms φ_i: $L_i\to A$, $i\in I$, there exists a homomorphism φ: $L\to A$ such that $\varphi\circ u_i$ agrees with φ_i for all $i\in I$.

Definition 2. Let X be an arbitrary set. The set $P(X)$ of polynomials over X is the smallest set satisfying (i) and (ii), where

(i) $X\subset P(X)$,

(ii) if $p,q\in P(X)$, then $p\vee q$ and $p\wedge q\in P(X)$.

For a lattice A we define $A^b=A\cup\{0^b, 1^b\}$, where $0^b, 1^b\notin A$, and we order A^b by the rules: $0^b<x<1^b$ for any $x\in A$, $x\leqq y$ in A^b if $x,y\in A$ and $x\leqq y$ in A. Thus

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$A^b \neq A$ and we have $a \wedge b = 0^b$ only if $a = 0^b$ or $b = 0^b$, and $a \vee b = 1^b$ only if $a = 1^b$ or $b = 1^b$.

Let $\{L_i : i \in I\}$ be a set of ortholattices. Put $Q = \bigcup \{L_i : i \in I\}$. We suppose that L_i and L_j are disjoint provided $i \neq j$, $i, j \in I$.

Definition 3. Let $P(Q)$ be the set of polynomials over Q. The upper i-cover of $p \in P(Q)$, $p^{(i)}$, is an element of $(L_i)^b$, defined as follows:

(i) for $a \in Q$ (i.e. $a \in L_i$ for exactly one $i \in I$), $a^{(j)} = a$ if $j = i$, $a^{(j)} = 1^b$ if $j \neq i$.

(ii) $(p \wedge q)^{(i)} = p^{(i)} \wedge q^{(i)}$ and $(p \vee q)^{(i)} = p^{(i} \vee q^{(i)}$, where $\wedge$ and $\vee$ is taken in $(L_i)^b$.

The definition of lower i-cover, $p_{(i)}$, is analogous, with 0^b replacing 1^b in (i).

It is clear that $p^{(i)} \neq 0_b$ and $p_{(i)} \neq 1^b$. An upper or lower i-cover is proper if it is not 1^b or 0^b.

Corollary 4. [2] *For any* $p \in P(Q)$ *and* $i \in I$ *we have that* $p_{(i)} \leqq p^{(i)}$, *and if* $p_{(i)}$ *and* $p^{(j)}$ *are proper and* $p_{(i)} \leqq p^{(j)}$, *then* $i = j$.

Definition 5. For $p, q \in P(Q)$, we put $p \subseteqq q$ if one of the following cases (i)—(vi) below occurs:

(i) $p = q$,

(ii) for some $i \in I$, $p^{(i)} \leqq q_{(i)}$,

(iii) $p = p_0 \wedge p_1$ where $p_0 \subseteqq q$ or $p_1 \subseteqq q$,

(iv) $p = p_0 \vee p_1$ where $p_0 \subseteqq q$ and $p_1 \subseteqq q$,

(v) $q = q_0 \wedge q_1$ where $p \subseteqq q_0$ and $p \subseteqq q_1$,

(vi) $q = q_0 \vee q_1$ where $p \subseteqq q_0$ or $p \subseteqq q_1$.

The rank $r(p)$ of a $p \in P(Q)$ is defined as follows: for $p \in Q$, $r(p) = 1$ and $r(p) = r(p_1) + r(p_2)$ if $p = p_1 \wedge p_2$ or $p = p_1 \vee p_2$.

Lemma 6. [2] *Let* $p, q, r \in P(Q)$ *and* $i \in I$. *Then*

(i) $p \subseteqq q$ *implies* $p_{(i)} \leqq q_{(i)}$ *and* $p^{(i)} \leqq q^{(i)}$.

(ii) $p \subseteqq q$ *and* $q \subseteqq r$ *imply* $p \subseteqq r$.

Since by 5 (i), $p \subseteqq p$ for any $p \in P(Q)$, the relation $\subseteqq$ is a quasiordering, and so we can define $p \equiv q$ iff $p \subseteqq q$ and $q \subseteqq p$, $p, q \in P(Q)$. We put

$$R(p) = \{q : q \in P(Q) \text{ and } p \equiv q\}, \quad R(Q) = \{R(p) : p \in P(Q)\},$$

$$R(p) \leqq R(q) \quad \text{if} \quad p \subseteqq q.$$

Lemma 7. [2] *$R(Q)$ is a lattice, and we have*

$$R(p) \wedge R(q) = R(p \wedge q), \quad R(p) \vee R(q) = R(p \vee q).$$

Furthermore, if $a, b, c, d \in L_i$, $i \in I$, *and if* $a \wedge b = c$, $a \vee b = d$ *in* L_i, *then* $R(a) \wedge R(b) = R(c)$ *and* $R(a) \vee R(b) = R(d)$.

As a consequence of Lemma 7 we get that $p \mapsto R(p)$, $p \in L_i$, is an embedding of L_i into $R(Q)$. Therefore, identifying $p \in L_i$ with $R(p)$ we get each L_i as a sublattice of $R(Q)$, and hence $Q \subset R(Q)$. It is also obvoius that the partial ordering induced by $R(Q)$ on Q agrees with the original partial ordering.

Let us add the set $\{0, 1\}$ to $P(Q)$ and let us define $0 \subseteqq p \subseteqq 1$ for any $p \in P(Q)$, $p \vee 0 = p$, $p \wedge 0 = 0$, $p \vee 1 = 1$, $p \wedge 1 = p$. Let us further define the map $'$ on $P(Q) \cup \{0, 1\}$ as follows: if $x \in L_i$ for some $i \in I$, put $x' = x'^i$; $1' = 0$, $0' = 1$, and recursively, $(a \wedge b)' = a' \vee b'$, $(a \vee b)' = a' \wedge b'$.

We note that the elements 0, 1 are different from the auxiliary elements 0^b and 1^b used in the definition of the lower and upper covers. In the following lemma we put $(0^b)' = 1^b$, $(1^b)' = 0^b$.

Lemma 8. *For any* $p \in P(Q)$, $(p')^{(i)} = (p_{(i)})'$ *and* $(p')_{(i)} = p^{(i)\prime}$.

Proof. We shall proceed by induction on $r(p)$. If $r(p) = 1$, then $p \in L_i$ for some $i \in I$, and $p^{(i)} = p_{(i)} = p$, $p^{(j)} = 1^b$, $p_{(j)} = 0^b$ for $j \neq i$. Therefore, $p^{(j)\prime} = (1^b)' = 0^b = p_{(j)}$ for $j \neq i$, and as $p' = p'^i$ is the orthocomplement of p in L_i, we have $(p')^{(j)} = 1^b$, $(p')_{(j)} = 0^b$ for $j \neq i$. From this we obtain that $(p^{(j)})' = 0^b = (p')_{(j)}$, $(p_{(j)})' = 1^b = (p')^{(j)}$ for $j \neq i$. Further, $(p')^{(i)} = p' = (p_{(i)})'$, $(p')_{(i)} = p' = (p^{(i)})'$. Now let $p = q \vee r$, then $p' = q' \wedge r'$, and $(p')^{(i)} = (q')^{(i)} \wedge (r')^{(i)} = (q_{(i)})' \wedge (r_{(i)})'$ by the induction hypothesis, so that $(p')^{(i)} = (q_{(i)} \vee r_{(i)})' = (p_{(i)})'$, and dually for $p = q \wedge r$. The proof of $(p')_{(i)} = (p^{(i)})'$ is similar.

Lemma 9. $a'' = a$ *for any* $a \in P(Q) \cup \{0, 1\}$, *and* $a \subseteqq b$ *implies* $b' \subseteqq a'$ *for any* $a, b \in P(Q) \cup \{0, 1\}$.

Proof. By the definition, $0'' = 1' = 0$, $1'' = 0' = 1$. If $a \in L_i$ for some $i \in I$, then obviously $a'' = a$. Let $a = b \wedge c$. Then $a' = b' \vee c'$, and $a'' = b'' \wedge c''$. By induction we obtain that $a'' = a$. For $a = b \vee c$ the situation is dual.

Now we shall prove the second statement. If $a = 0$ or $b = 1$, it is obvious. We shall suppose that $a, b \notin \{0, 1\}$ and proceed by induction on $r(a) + r(b)$. If $r(a) + r(b) = 2$, then $a \subseteqq b$ holds by 5 (i) or 5 (ii), so that $a, b \in L_i$ for some $i \in I$, and $a' = a'^i$, $b' = b'^i$, which implies that $b' \subseteqq a'$. Now let $r(a) + r(b) = r$, and let the statement hold for all $r(a) + r(b) < r$. If $a \subseteqq b$ holds by 5 (i), then $a' = b'$. If $a \subseteqq b$ holds by 5 (ii), then $a^{(i)} \leqq b_{(i)}$ for some $i \in I$. By Lemma 8, $(a^{(i)})' = (a')_{(i)}$ and $(b_{(i)})' = (b')^{(i)}$. Therefore $a^{(i)} \leqq b_{(i)}$ implies $(b')^{(i)} \leqq (a')_{(i)}$, which in turn implies that $b' \subseteqq a'$ by 5 (ii). If $a \subseteqq b$ by 5 (iii) with $a = a_0 \wedge a_1$, then $a_0 \subseteqq b$ or $a_1 \subseteqq b$, which implies by the induction hypothesis that $b' \subseteqq a_0'$ or $b' \subseteqq a_1'$. As $a' = a_0' \vee a_1'$, we get that $b' \subseteqq a'$ by 5 (vi). If $a \subseteqq b$ by 5 (iv) and $a = a_0 \vee a_1$, where $a_0 \subseteqq b$ and $a_1 \subseteqq b$, then $b'\subseteqq a'_0$ and $b'\subseteqq a'_1$ and this implies that $b'\subseteqq a'_0\wedge a'_1=a'$ by 5 (v). If $a\subseteqq b$ by 5 (v), then $b=b_0\wedge b_1$ and $a\subseteqq b_0$ and $a\subseteqq b_1$. This implies that $b'_0\subseteqq a'$ and $b'_1\subseteqq a'$, which implies that $b'=b'_0\vee b'_1\subseteqq a'$ by 5 (iv). If $a\subseteqq b$ by 5 (vi), where $b=b_0\vee b_1$ with $a\subseteqq b_0$ or $a\subseteqq b_1$, then $b'_0\subseteqq a'$ or $b'_1\subseteqq a'$, and therefore $b'_0\wedge b'_1\subseteqq a'$ by 5 (iii).

Following Bruns [1], we shall define the subset S of reduced elements in $P(Q)\cup\{0, 1\}$.

Definition 10. Define a subset S of $P(Q)\cup\{0,1\}$ recursively as follows: a is in S if

(i) $a\in\{0, 1\}$ or $a\in\bigcup\{L_i-\{0_i, 1_i\}: i\in I\}$,

(ii) $a=b\vee c$ with $b, c\in S$ and $b'\nsubseteq a$, $c'\nsubseteq a$,

(iii) $a=b\wedge c$ with $b, c\in S$ and $a\nsubseteq b'$, $a\nsubseteq c'$.

Lemma 11. *The set S is closed under $'$.*

Proof. If $a\in\{0, 1\}$, then obviously $a'\in\{0, 1\}$. If $a\in L_i-\{0_i, 1_i\}$ for some $i\in I$, then $a'\in L_i-\{0_i, 1_i\}$ so that $a'\in S$. If $a=b\vee c$, $b, c\in S$ and $b'\nsubseteq a$, $c'\nsubseteq a$, then $a'=b'\wedge c'$ and $a'\nsubseteq b$, $a'\nsubseteq c$. By induction, $b', c'\in S$, and $a'\in S$ by 10 (iii). If $a=b\wedge c$ with $b, c\in S$ and $a\nsubseteq b'$, $a\nsubseteq c'$, then by induction, $b', c'\in S$, and $b\nsubseteq a'$, $c\nsubseteq a'$ implies that $a'\in S$ by 10 (ii).

Lemma 12. *If $a\in S-\{0, 1\}$ then $a^{(i)}\neq 0_i$ and $a_{(i)}\neq 1_i$ for all $i\in I$.*

Proof. We shall proceed by induction. If $a\in L_i-\{0_i, 1_i\}$ then $a^{(i)}=a_{(i)}=$ $=a\notin\{0_i, 1_i\}$, and $a_{(j)}=0^b$, $a^{(j)}=1^b$ for $j\neq i$. Now let $a=b\vee c$. Let us suppose that $a^{(i)}=0_i$ for some $i\in I$. Then $a^{(i)}=b^{(i)}\vee c^{(i)}$ implies that $b^{(i)}$ and $c^{(i)}$ are proper, and $b^{(i)}=c^{(i)}=0_i$, which contradicts the induction hypothesis. Now let $a=b\wedge c$, $b, c\in S$, $a\nsubseteq b'$, $a\nsubseteq c'$. If $a^{(i)}=0_i$, then $a^{(i)}=b^{(i)}\wedge c^{(i)}$ implies that $b^{(i)}$ or $c^{(i)}$ are proper, and $a^{(i)}=0_i\subseteqq(b^{(i)})'=(b')_{(i)}$ implies by 5 (ii) that $a\subseteqq b'$, a contradiction. Now let us suppose that $a_{(i)}=1_i$ for $a\in S$, $i\in I$. By Lemma 11, $a'\in S$, and by Lemma 8, $(a_{(i)})'=(a')^{(i)}=0_i$, which contradicts the above part of the proof.

Lemma 13. *For any $a\in S-\{1\}$, $a'\nsubseteq a$. If $a\in P(Q)$ and $b\in S-\{1\}$, then $a\nsubseteq b$ or $a'\nsubseteq b$.*

Proof. If $a\in S-\{1\}$ by 10 (i), then $a=0$ or $a\in L_i-\{0_i, 1_i\}$ for some $i\in I$. In both cases $a'\nsubseteq a$ holds. Now let us suppose that $a\in S-\{1\}$ and $a'\subseteqq a$ holds by 5 (ii). Then $(a')^{(i)}\leqq a_{(i)}$ for some $i\in I$. This implies that $(a')^{(i)}=(a_{(i)})'\leqq a_{(i)}$, but this is impossible by Lemma 12. Now let $a\in S$ by 10 (ii) with $a=b\vee c$. If $b'\wedge c'\subseteqq$ $\subseteqq b\vee c$ holds by 5 (iii), then $b'\subseteqq b\vee c$ or $c'\subseteqq b\vee c$, which contradicts 10 (ii). If $b'\wedge c'\subseteqq b\vee c$ by 5 (vi), then $b'\wedge c'\subseteqq b$ or $b'\wedge c'\subseteqq c$. From this it follows that $b'\subseteqq a$ or $c'\leqq a$, contradicting 10 (ii). If $a=b\wedge c$, then if $b'\vee c'\subseteqq b\wedge c$ by 5 (iv), then

$b' \subseteqq b \wedge c$ and $c' \subseteqq b \wedge c$. But this implies that $b' \subseteqq b$ and $c' \subseteqq c$ by 5 (v), contradicting the induction hypothesis. If $b' \vee c' \subseteqq b \wedge c$ by 5 (v), then $b' \vee c' \subseteqq b$ and $b' \vee c' \subseteqq c$, and this implies by 5 (iv) that $b' \subseteqq b$ and $c' \subseteqq c$, contradicting the induction hypothesis. Thus the first part of Lemma 13 is proved.

Finally, if $a \subseteqq b$ and $a' \subseteqq b$ with $a \in P(Q)$ and $b \in S - \{1\}$, then $a' \subseteqq b$ implies $b' \subseteqq a$, and this together with $a \subseteqq b$ gives $b' \subseteqq b$, which contradicts the first part of the proof.

Obviously, the relation $\subseteqq$ defined on $S \times S$ by Definition 5 together with the rule $0 \subseteqq x \subseteqq 1$ for all $x \in S$, is a quasiordering on S. Let Θ be the relation defined on $S \times S$ by $a \Theta b$ iff $a \subseteqq b$ and $b \subseteqq a$. We prove now that S/Θ is an ortholattice with $0/\Theta$ as the smallest and $1/\Theta$ as the largest element, $a/\Theta \vee b/\Theta = (a \vee b)/\Theta$ if $a \vee b \in S$ and $a/\Theta \vee b/\Theta = 1/\Theta$ if $a \vee b \notin S$, and, finally, that $a/\Theta \mapsto a'/\Theta$ is an orthocomplementation.

Let us define $a/\Theta \leqq b/\Theta$ iff $a \subseteqq b$, $a, b \in S$. Obviously, $\leqq$ is a partial ordering on S/Θ, and $0/\Theta$ and $1/\Theta$ are the smallest and largest element of S/Θ, respectively. If, for $a, b \in S$, the element $a \vee b \in S$, then $a/\Theta \vee b/\Theta = (a \vee b)/\Theta$ by Lemma 7. If, for $a, b \in S$, the element $a \vee b \notin S$, then $a' \subseteqq a \vee b$ or $b' \subseteqq a \vee b$ holds, and for every c in S such that $a, b \subseteqq c$ we get by 5 (iv) that $a, a' \subseteqq c$ or $b, b' \subseteqq c$. This implies by Lemma 13 that $c = 1$. Thus $1/\Theta$ is the supremum of a/Θ and b/Θ. For meets the situation is dual. Therefore, S/Θ is a lattice. For every $a \in S - \{0, 1\}$ the elements $a \vee a'$ and $a \wedge a'$ are not in S, and this implies that a'/Θ is the complement of a/Θ in S.

Theorem 14. *Let $\{L_i \colon i \in I\}$ be a set of ortholattices and let $Q = \bigcup \{L_i \colon i \in I\}$. Denote by $P(Q)$ the set of all polynomials over Q and by S the subset of $P(Q) \cup \{0, 1\}$ given by Definition 10. Finally, let Θ be the congruence relation defined by $a \Theta b$ iff $a \subseteqq b$ and $b \subseteqq a$. Then S/Θ is a free product of L_i, $i \in I$.*

Proof. Put $L = S/\Theta$. We have to prove that

(i) each L_i, $i \in I$, is a subalgebra of L and for $i, j \in I$, $i \neq j$, $L_i - \{0_i, 1_i\}$ and $L_j - \{0_j, 1_j\}$ are disjoint,

(ii) L is generated by $\bigcup \{L_i \colon i \in I\}$,

(iii) for any ortholattice A and for a family of homomorphisms $\varphi_i \colon L_i \to A$, $i \in I$, there exists a homomorphism $\varphi \colon L \to A$ such that φ agrees on L_i with φ_i for all $i \in I$.

(i) We have already proved that L is an ortholattice. Define $\psi_i \colon L_i \to L$ by $\psi_i(x) = x/\Theta \equiv R(x)$ if $x \in L_i - \{0_i, 1_i\}$, and $\psi_i(1_i) = 1/\Theta$, $\psi_i(0_i) = 0/\Theta$. Clearly, we have $\psi_i(x') \equiv \psi_i(x)'$, and $\psi_i(x \vee y) = \psi_i(x) \vee \psi_i(y)$ for $x, y \in L_i$. If $x \in L_i$, $x \neq 0_i$, then $x/\Theta \neq 0/\Theta$, which implies that ψ_i is an embedding.

(ii) is clear.

(iii) We define inductively a map $v \colon P(Q) \to A$ as follows: for $p \in Q$ we set

$\nu(p)=\varphi_i(p)$ if $p\in L_i$, $i\in I$. If $p=p_0\wedge p_1$ or $p=p_0\vee p_1$, $\nu(p_0)$ and $\nu(p_1)$ have already been defined, we set $\nu(p)=\nu(p_0)\wedge\nu(p_1)$ or $\nu(p)=\nu(p_0)\vee\nu(p_1)$, respectively.

We need the following lemma.

Lemma 15. *For $p\in P(Q)$ and $i\in I$, the following hold.*

(i) *If $p_{(i)}$ is proper, then $\nu(p_{(i)})\leqq\nu(p)$.*

(ii) *If $p^{(i)}$ is proper, then $\nu(p)\leqq\nu(p^{(i)})$.*

(iii) *$p\subseteqq q$ implies that $\nu(p)\leqq\nu(q)$.*

(iv) *$\nu(p')=\nu(p)'$ in A.*

Proof. (i)—(iii) The proof is the same as the proof of Lemma 9 in [2].

(iv) If $p\in Q$, then $p\in L_i$ for exactly one $i\in I$, and $\nu(p)=\varphi_i(p)$, so that $\nu(p')=$ $=\varphi_i(p')=\varphi_i(p)'=\nu(p)'$. If $p=p_0\wedge p_1$, then $\nu(p')=\nu(p_0'\vee p_1')=\nu(p_0')\vee\nu(p_1')=$ $=\nu(p_0)'\vee\nu(p_1)'$ by the induction hypothesis, which implies that $\nu(p')=(\nu(p_0)\wedge$ $\wedge\nu(p_1))'=\nu(p)'$. The situation for $p=p_0\vee p_1$ is dual.

Now we can complete the proof of Theorem 14. Take a $p\in S$ and define $\varphi(p/\Theta)=\nu(p)$ if $p\in S-\{0,1\}$, and $\varphi(1/\Theta)=1$, $\varphi(0/\Theta)=0$ in A. φ is well-defined since if $p,q\in S-\{0,1\}$ and $p/\Theta=q/\Theta$, then $p\subseteqq q$ and $q\subseteqq p$, which implies by Lemma 15 that $\nu(p)=\nu(q)$. Further, $\varphi(p/\Theta\wedge q/\Theta)=\varphi((p\wedge q)/\Theta)=\nu(p\wedge q)=$ $=\nu(p)\wedge\nu(q)=\varphi(p/\Theta)\wedge\varphi(q/\Theta)$ if $p\wedge q\in S$, $p,q\in S-\{0,1\}$. Clearly, $\varphi(p/\Theta\wedge$ $\wedge 0/\Theta)=\varphi(0/\Theta)=0=\varphi(p/\Theta)\wedge\varphi(0/\Theta)$, and $\varphi(p/\Theta\wedge 1/\Theta)=\varphi(p/\Theta)=\varphi(p/\Theta)\wedge$ $\wedge\varphi(1/\Theta)$. If $p,q\in S$, and $p\wedge q\notin S$, then $p\wedge q\subseteqq p'$ or $p\wedge q\subseteqq q'$, so that $\nu(p\wedge q)\leqq$ $\leqq\nu(p)'$ or $\nu(q)'$, which implies that $\nu(p\wedge q)=\nu(p)\wedge\nu(q)=0$. Hence, $\varphi(p/\Theta)\wedge$ $\wedge\varphi(q/\Theta)=\nu(p)\wedge\nu(q)=0=\varphi(0/\Theta)=\varphi(p/\Theta\wedge q/\Theta)$. Further, $\varphi(p'/\Theta)=\nu(p')=$ $=\nu(p)'=\varphi(p/\Theta)'$ if $p\in S-\{0,1\}$, and $\varphi(1/\Theta)=\varphi(0/\Theta)'$. We see that $\varphi\colon S/\Theta\to A$ is a homomorphism. Finally, for $p\in L_i$, $p\neq 0_i, 1_i$, we have $\varphi(p/\Theta)=\varphi(\psi_i(p))=$ $=\nu(p)=\varphi_i(p)$, $\varphi(\psi_i(0_i))=\varphi(0/\Theta)=0=\varphi_i(0_i)$, $\varphi(\psi_i(1_i))=\varphi(1/\Theta)=1=\varphi_i(1_i)$, so that $\varphi\circ\psi_i=\varphi_i$. This completes the proof.

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