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# Abstract spectral theory. II: Minimal characters and minimal spectrums of multiplicative lattices

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## 1. Introduction

A multiplicative lattice is a complete lattice in which there is defined a commutative, associative multiplication which distributes over arbitrary joins (i.e., $a(\vee_\alpha b_\alpha)=\vee_\alpha ab_\alpha$), $ab\leqq a\wedge b$ and the greatest element 1 acts as a multiplicative identity. Throughout this paper, let $L$ denote a multiplicative lattice. In $L$ an element $p$ different from 1 is called prime if $ab\leqq p$ implies $a\leqq p$ or $b\leqq p$. A minimal element in the set of prime elements of $L$ will be called a minimal prime element of $L$. A character of $L$ is a homomorphism of $L$ onto a two element chain $\mathbf{C}_2$. It was shown in [9] that an element $a$ of $L$ is prime if and only if there is a homomorphism $\varphi$ of $L$ onto $\mathbf{C}_2$ with $a=\vee\{x\colon \varphi(x)=0\}$. This means that a prime element of $L$ can now be equivalently associated with a character of $L$, and so a prime element itself will be called a character of $L$. We denote by $\sigma(L)$ and $\pi(L)$ the sets of characters and minimal characters of $L$ respectively.

This work is a continuation of the work initiated by THAKARE and MANJAREKAR [9]. Here we are concerned mainly with minimal characters of $L$ and with the topology on the set $\pi(L)$ which is the restriction of the hull kernel topology introduced on the set $\sigma(L)$ (see [9]).

The studies of minimal prime ideals for commutative rings, commutative semigroups, distributive lattices, lattice ordered groups, $f$-rings and recently 0-distributive semilattices (THAKARE and PAWAR [11], [7]) have been carried out extensively. An attempt to unify these scattered studies was nicely made by KEIMEL [4]. Our study in this paper is close in spirit to the study [4], though however we carry out investigations to include many more novel notions the motivation for which stems from the desire to abstract available notions in commutative rings on the lines of DILWORTH [2].

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The notion $a^*$ of an element $a$ of $L$ is defined as the join of annihilators of powers of $a$, and this concept plays an important role in the investigations of minimal characters in Sections 2 and 5. The concept of minimal characters belonging to an element, appeared in MURATA [5] and ANDERSON [1], is discussed in Section 3. We abstract the notion of an ideal $B$ of a commutative ring $R$ that is related to an ideal $A$ of $R$, and this concept is used in the arguments on primary decompositions of elements of $L$ in Section 4.

In the previous paper [9], we assumed that $L$ always satisfies the following condition which is equivalent to the ascending chain condition:

(K) *Every element of $L$ is compact.*

In this paper, we assume that condition (K) or some weaker ones according to the need.

We remark that for any $p\in\sigma(L)$, the existence of a maximal character $q$ with $p\leqq q$ can be proved under the assumption that $L$ satisfies (K) (see [9]) but the existence of a minimal character $r$ with $r\leqq p$ can be proved without this assumption (because, if $Q$ is a chain of characters then $p=\wedge Q$ is also a character).

## 2. Characters and minimal characters

A subset $S$ of $L$ is called *multiplicatively closed* if $a, b\in S$ implies $ab\in S$, and $S$ is called *submultiplicatively closed* if for $a, b\in S$ there exists $c\in S$ with $c\leqq ab$. Without assuming the condition (K), the Separation Lemma can be stated as follows (cf. [9], Lemma 2.2):

Separation Lemma. *Let $S$ be a submultiplicatively closed subset of $L$, and assume that every element of $S$ is compact. If $S\cap[0, a]=\emptyset$ for some $a\in L$, then there exists a character $p$ of $L$ which is a maximal element of the set $\{x\in L\colon a\leqq x$ and $S\cap[0, x]=\emptyset\}$.*

In fact, this set has a maximal element $p$ by Zorn's lemma since every element of $S$ is compact, and we can prove that $p$ is a character since $S$ is submultiplicatively closed.

An element $a$ of $L$ is called *M-compact* if $a^n$ are compact for infinitely many integer $n$. Every nilpotent element is $M$-compact. An idempotent is $M$-compact if and if it is compact.

Proposition 2.1. *If $a$ is an M-compact element of $L$ and if $a^n\nleqq b$ for every integer $n$, then there exists $p\in\sigma(L)$ such that $b\leqq p$ and $a\nleqq p$. Especially, if $a$ is M-compact and is not nilpotent then there exists $p\in\sigma(L)$ such that $a\nleqq p$.*

Proof. The set $S=\{a^n: a^n \text{ is compact}\}$ is submultiplicatively closed and $S\cap[0,b]=\emptyset$. Hence, by the Separation Lemma there is $p\in\sigma(L)$ such that $b\leqq p$ and $S\cap[0,p]=\emptyset$. Then, $a\not\leqq p$.

Corollary 2.2. *If the greatest element* 1 *of* $L$ *is compact, then for any* $b\in L$ *with* $b<1$ *there exists* $p\in\sigma(L)$ *such that* $b\leqq p$.

Proof. Put $a=1$ in Proposition 2.1.

We need to introduce the following notation which is important in the arguments on minimal characters. For $a\in L$,

$$a^* = \bigvee\{x\in L:\ a^n x = 0 \text{ for some integer } n\}.$$

Evidently, $0^*=1$, $1^*=0$, and $a\leqq b$ implies $b^*\leqq a^*$.

Lemma 2.3. (i) *If* $a^*$ *is compact, then* $a^n a^*=0$ *for some* $n$, *and* $a\wedge a^*$ *is nilpotent.*

(ii) *In the case that* 1 *is compact,* $a\in L$ *is nilpotent if and only if* $a^*=1$.

Proof. (i) The set $S=\{x\in L:\ a^n x=0 \text{ for some } n\}$ is an ideal, since $a^{m+n}(x\vee y)\leqq \leqq a^m x\vee a^n y$. Hence, if $a^*$ is compact then $a^*\in S$. Thus, $a^n a^*=0$ for some $n$, and $(a\wedge a^*)^{n+1}=0$.

(ii) If $a^*=1$ then $a$ is nilpotent by (i). The converse is evident.

Lemma 2.4. *Let* $a\in L$ *and* $p\in\sigma(L)$. $a\not\leqq p$ *implies* $a^*\leqq p$. *(Hence,* $a\wedge a^*\leqq p$ *always.)*

Proof. Assume $a\not\leqq p$. If $a^n x=0$ then we have $a^n x\leqq p$ and $a^n\not\leqq p$. Hence, $x\leqq p$. Therefore, $a^*\leqq p$.

Using the condition (K), we now get a fundamental result with some interesting corollaries.

Theorem 2.5. *Assume that* $L$ *satisfies* (K). *For* $a\in L$ *and* $p\in\sigma(L)$ *the following statements are equivalent:*

(1) $a^*\leqq p$;

(2) *there is some* $q\in\pi(L)$ *with* $q\leqq p$ *and* $a\not\leqq q$.

Proof. (1)⇒(2): Let $S=\{a^n x:\ x\not\leqq p,\ n=1,2,\ldots\}$. Then $S$ is multiplicatively closed. We have $0\notin S$, because if $a^n x=0$ then $x\leqq a^*\leqq p$ by (1). By the Separation Lemma there exists $r\in\sigma(L)$ such that $S\cap[0,r]=\emptyset$. Take $q\in\pi(L)$ such that $q\leqq r$. We have $r\leqq p$, since otherwise $ar\in S\cap[0,r]$, a contradiction. Also, $a\not\leqq r$, since $a\in S$. Hence, $q\leqq p$ and $a\not\leqq q$.

(2)⇒(1): If $q\leqq p$ and $a\not\leqq q$, then $a^*\leqq q\leqq p$ by Lemma 2.4.

Corollary 2.6. *Assume that $L$ satisfies* (K), *and let* $p\in\sigma(L)$. *If* $p^*\leqq p$ *then* $p$ *is not minimal.*

Proof. If $p^*\leqq p$, there is $q\in\pi(L)$ with $q\leqq p$ and $p\nleqq q$ by Theorem 2.5. Thus, $q<p$, and $p$ is not minimal.

As stated in the previous paper [9], the *hull kernel topology* on $\sigma(L)$ is given as follows. For $a\in L$ we put

$$V(a) = \{p\in\sigma(L)\colon a \leqq p\}.$$

Since $V(0)=\sigma(L)$, $V(1)=\emptyset$, $V(a)\cup V(b)=V(ab)$ $(=V(a\wedge b))$ and $\bigcap_\alpha V(a_\alpha)=$ $=V(\bigvee_\alpha a_\alpha)$, we obtain a topology on $\sigma(L)$ such that $\{V(a)\colon a\in L\}$ is the family of all closed sets. It is easy to verify that the closure $\bar{R}$ of a subset $R$ of $\sigma(L)$ coincides with $V(\wedge R)$.

Corollary 2.7. *Assume that $L$ satisfies* (K), *and let* $a\in L$. $V(a^*)$ *is equal to the closure of the open set* $\sigma(L)-V(a)$.

Proof. By Lemma 2.4, we have $\sigma(L)-V(a)\subset V(a^*)$. Hence, it suffices to show that if $\sigma(L)-V(a)\subset V(x)$ then $V(a^*)\subset V(x)$. Let $p\in V(a^*)$. By Theorem 2.5 there is $q\in\pi(L)$ with $q\leqq p$ and $a\nleqq q$. Then, $q\in\sigma(L)-V(a)\subset V(x)$, and hence $x\leqq q\leqq p$. Hence $p\in V(x)$, and we obtain $V(a^*)\subset V(x)$.

The concept of regular characters was introduced by [3], [8] and [9], while its dual concept, coregular characters, appeared in [8] for bounded distributive lattices.
A character $r\in\sigma(L)$ is called *coregular* if for $p, q\in\sigma(L)$, $r\leqq p$ and $q\leqq p$ together imply $r\leqq q$. The companion of Theorem 2.7 of [9] would now be proved.

Theorem 2.8. *Assume that $L$ satisfies* (K). *For* $r\in\sigma(L)$ *the following five statements are equivalent:*

(1) *$r$ is coregular;*
(2) *the set $V(r)$ is open;*
(3) $V(r)\cap V(r^*)=\emptyset$;
(4) $r\vee r^*=1$;
(5) *there is* $x\in L$ *such that* $x\vee r=1$ *and* $r^n x=0$ *for some integer $n$.*

(We remark that (5)⇒(4)⇒(3)⇒(2)⇒(1) can be proved without the assumption (K).)

Proof. (5)⇒(4) is evident. (4)⇒(3): If $V(r)\cap V(r^*)$ had an element $p$ then $r\vee r^*\leqq p<1$, contradicting (4). (3)⇒(2): We have $V(r)\cup V(r^*)=\sigma(L)$ by Lemma 2.4. Hence, by (3) we have $V(r)=\sigma(L)-V(r^*)$, and then $V(r)$ is an open set. (2)⇒ (1): Let $r\leqq p$ and $q\leqq p$, and put $G=\sigma(L)-V(r)$. Since $G$ is closed by (2), we have

$p\notin G=\bar{G}=V(\wedge G)$, and hence $\wedge G\nleqq p$. As $q\leqq p$, we have $\wedge G\nleqq q$, whence $q\notin G$. Hence, $r\leqq q$.

Next, we assume that $L$ satifies (K). (4) implies (5), since $r^n r^*=0$ for some $n$ by Lemma 2.3 (i). (1)⇒(4): If $r\vee r^*<1$, then there is $p\in\sigma(L)$ with $r\vee r^*\leqq p$ by Corollary 2.2. By Theorem 2.5 there is $q\in\pi(L)$ with $q\leqq p$ and $r\nleqq q$, contradicting (1).

Recall the concept of multiplicative normal (i.e. M-normal) lattice introduced in [9]. A multiplicative lattice $L$ is called *M-normal* if each character of $L$ contains a unique minimal character of $L$. We shall have several characterizations of M-normal multiplicative lattices in the following two theorems.

Theorem 2.9. *The following two statements are equivalent:*

(1) *$L$ is M-normal;*

(2) *every minimal character of $L$ is coregular.*

*If* 1 *is compact,* (1) *is also equivalent to the following statement:*

(3) *$q_1\vee q_2=1$ for any distinct minimal characters $q_1, q_2$ of $L$.*

Proof. (1)⇒(2): Let $r\in\pi(L)$, and we take $p, q\in\sigma(L)$ with $r\leqq p$ and $q\leqq p$. There is $q'\in\pi(L)$ with $q'\leqq q$. Then, $r, q'\leqq p$, and hence $r=q'\leqq q$ by (1). Hence, $r$ is coregular. (2)⇒(1): Let $p\in\sigma(L)$, $r_i\in\pi(L)$ $(i=1, 2)$ and $r_i\leqq p$. Since $r_1$ is coregular by (2), we have $r_1\leqq r_2$. Similarly we have $r_2\leqq r_1$, and hence $r_1=r_2$.

(1)⇒(3): Assume that 1 is compact. If $q_1\vee q_2<1$, then there is $p\in\sigma(L)$ with $q_1\vee q_2\leqq p$ by Corollary 2.2, and hence $q_1=q_2$ by (1). (3)⇒(1) is evident.

Recall that a topological space is called *extremally disconnected* if the closure of each open set is open.

Lemma 2.10. *A topological space $X$ is extremally disconnected if and only if for open subsets $G_1, G_2$ of $X$, $G_1\cap G_2=\emptyset$ implies $\bar{G}_1\cap\bar{G}_2=\emptyset$.*

Proof. Assume that $X$ is extremally disconnected. If $G_1\cap G_2=\emptyset$, then $\bar{G}_1\subset X-G_2$, since $X-G_2$ is closed. Hence, $G_2\subset X-\bar{G}_1$, Since $\bar{G}_1$ is open, we have $\bar{G}_2\subset X-\bar{G}_1$, and then $\bar{G}_1\cap\bar{G}_2=\emptyset$.

Next we shall prove the converse. For an open set $G$, we put $U=X-\bar{G}$. Then, $U$ is open and $U\cap G=\emptyset$, and hence $\bar{U}\cap\bar{G}=\emptyset$. Hence, $\bar{U}\subset X-\bar{G}=U$, which implies that $U$ is closed. Hence, $\bar{G}$ is open.

Theorem 2.11. *Assume that $L$ satisfies* (K). *The following five statements are equivalent:*

(1) *$L$ is M-normal;*

(2) *if $G_1$ and $G_2$ are open sets of $\sigma(L)$ with $G_1\cap G_2=\emptyset$ then $\bar{G}_1\cap\bar{G}_2=\emptyset$;*

(3) *$\sigma(L)$ is extremally disconnected;*

(4) *$V(a^*)$ is open for every $a\in L$;*

(5) *if $V(a)\cup V(b)=\sigma(L)$ then $a^*\vee b^*=1$.*

Proof. The equivalences (2)⇔(3) and (3)⇔(4) immediately follow from Lemma 2.10 and Corollary 2.7, respectively.

(1)⇒(2): Let $G_1$ and $G_2$ be open sets with $G_1\cap G_2=\emptyset$. We can put $G_i=\sigma(L)-V(a_i)$ for some $a_i\in L$ $(i=1,2)$. By Corollary 2.7, we have $\bar{G}_i=V(a_i^*)$. If $\bar{G}_1\cap\bar{G}_2$ had an element $p$, then $a_i^*\leqq p$ and by Theorem 2.5 there would exist $q_1, q_2\in\pi(L)$ with $q_i\leqq p$ and $a_i\not\leqq q_i$. By (1), we have $q_1=q_2$, which implies $q_1\in G_1\cap G_2$, a contradiction.

(2)⇒(5): Let $V(a)\cup V(b)=\sigma(L)$. Putting $G_1=\sigma(L)-V(a)$ and $G_2=\sigma(L)-V(b)$, we have $G_1\cap G_2=\emptyset$. By (2) we have $V(a^*)\cap V(b^*)=\bar{G}_1\cap\bar{G}_2=\emptyset$. Hence, $a^*\vee b^*=1$ by Corollary 2.2.

(5)⇒(1): Let $q_1, q_2\in\pi(L)$ with $q_1\neq q_2$, and we shall show $V(q_1^*)\cup V(q_2^*)=\sigma(L)$. For any $p\in\sigma(L)$, there is $q\in\pi(L)$ with $q\leqq p$. If $q\neq q_1$, then since $q_1\not\leqq q$ we have $q_1^*\leqq q\leqq p$ by Lemma 2.4. If $q=q_1$, then $q\neq q_2$ and hence we have $q_2^*\leqq p$. Thus, we get $V(q_1^*)\cup V(q_2^*)=\sigma(L)$, and then $q_1^{**}\vee q_2^{**}=1$ by (5). Since $q_i^*\not\leqq q_i$ by Corollary 2.6, we get $q_i^{**}\leqq q_i$ by Lemma 2.4. Hence, $q_1\vee q_2=1$, and there is no character which contains both $q_1$ and $q_2$.

## 3. Minimal characters belonging to an element

We consider a relation between characters and multiplicatively closed subsets. For $a\in L$, we put

$$C(a) = \{x\in L\colon x \not\leqq a\}.$$

(This notion was introduced in Nemitz [6].) The set of all multiplicatively closed subsets of $L$ is denoted by $\mathcal{M}(L)$.

Lemma 3.1. *$C(p)\in\mathcal{M}(L)$ if and only if $p$ is a character of $L$. The mapping $p\mapsto C(p)$ of $\sigma(L)$ into $\mathcal{M}(L)$ is one-to-one, and $p\leqq q\Leftrightarrow C(p)\supset C(q)$.*

Proof. Evident.

Lemma 3.2. *Let $a\in L$, and take $M\in\mathcal{M}(L)$ with $M\cap[0,a]=\emptyset$.*

(i) *$\mathcal{U}=\{N\in\mathcal{M}(L)\colon N\supset M$ and $N\cap[0,a]=\emptyset\}$ has a maximal element.*

(ii) *$N^*\in\mathcal{U}$ is maximal in $\mathcal{U}$ if and only if for any $x\in L$ with $x\notin N^*$ there exists $y\in N^*$ such that $x^ny\leqq a$ for some integer $n$.*

Proof. (i) For any chain $\mathcal{V}\subset\mathcal{U}$, the union $\bigcup\{N\colon N\in\mathcal{V}\}$ belongs to $\mathcal{U}$. Hence, $\mathcal{U}$ has a maximal element by Zorn's lemma.

(ii) Let $N^*$ be maximal and let $x \notin N^*$. The set $N_1 = \{x^n, y, x^n y: y \in N^*, n = 1, 2, \ldots\}$ is multiplicatively closed since $N^* \in \mathcal{M}(L)$, and $N_1 \supset N^*$. Moreover, $N_1 \neq N^*$, for $x \in N_1$ and $x \notin N^*$. Hence, by the maximality of $N^*$ we have $N_1 \cap \cap [0, a] = \emptyset$. Then, there exists $y \in N^*$ such that $x^n y \leqq a$ for some $n$.

Next, take $N \in \mathcal{M}(L)$ with $N \supsetneqq N^*$, and take $x \in N - N^*$. If $N^*$ satisfies the given condition, there exists $y \in N^*$ such that $x^n y \leqq a$ for some $n$. Then, $x^n y \in N \cap \cap [0, a]$. Hence, $N^*$ is maximal in $\mathcal{U}$.

Recall the concept of minimal characters belonging to an element, which was initiated by Murata [5]. For $a \in L$ with $a < 1$, a minimal element of $V(a) = \{p \in \sigma(L): a \leqq p\}$ is called a *minimal character belonging to a*. The set of all minimal characters belonging to $a$ is denoted by $V_{\min}(a)$. For any chain $Q$ in $V(a)$, we have $\wedge Q \in V(a)$. Hence, for any $p \in V(a)$ there is $q \in V_{\min}(a)$ with $q \leqq p$ by Zorn's lemma. We remark that $V_{\min}(0) = \pi(L)$.

Theorem 3.3. *Let $a \in L$ with $a < 1$ and let $p \in \sigma(L)$. If L satisfies* (K) *then the following statements are equivalent:*

(1) $p \in V_{\min}(a)$;

(2) *$C(p)$ is maximal in the set $\{N \in \mathcal{M}(L): N \cap [0, a] = \emptyset\}$;*

(3) *$a \leqq p$ and there exists $x \in L$ such that $x \nleqq p$ and $p^n x \leqq a$ for some integer $n$.*

*Moreover, without assuming* (K), *the statements* (2) *and* (3) *are equivalent, and* (2) *implies* (1).

Proof. (2)⇔(3): Putting $M = \{1\}$ in Lemma 3.2, (2) is equivalent to the following statement: "$a \leqq p$ and for any $x \leqq p$ there is $y \nleqq p$ such that $x^n y \leqq a$ for some $n$". Evidently, this is equivalent to (3).

(2)⇒(1): If $a \leqq q \leqq p$ with $q \in \sigma(L)$, then $C(q) \in \mathcal{M}(L)$, $C(q) \cap [0, a] = \emptyset$ and $C(q) \supset C(p)$. Hence, $C(q) = C(p)$ by (2), and then $q = p$.

We assume (K) and prove (1)⇒(2). Put $\mathcal{U} = \{N \in \mathcal{M}(L): N \cap [0, a] = \emptyset\}$. $C(p) \in \mathcal{U}$ by $a \leqq p$. If $C(p) \subset N \in \mathcal{U}$, then $N \cap [0, a] = \emptyset$, and by the Separation Lemma there is $q \in \sigma(L)$ with $a \leqq q$ and $N \cap [0, q] = \emptyset$. Then, $C(q) \supset N \supset C(p)$, and hence $p \geqq q$. Hence, $p = q$ by (1), and then $C(p) = N$. Thus, $C(p)$ is maximal in $\mathcal{U}$.

Theorem 3.4. *Let $a \in L$ with $a < 1$. If every finite product of elements of $V_{\min}(a)$ is compact (especially, if L satisfies* (K)*), then $V_{\min}(a)$ is a finite set.*

Proof. Assume that $V_{\min}(a)$ is an infinite set. The set $M$ of all finite products of elements of $V_{\min}(a)$ is multiplicatively closed. If $b \in M$, then $b = p_1 \ldots p_n$ with $p_i \in V_{\min}(a)$, and by the assumption there is $q \in V_{\min}(a)$ which is different from all $p_i$. We have $b \nleqq q$ since $p_i \nleqq q$ for all $i$, and then $b \nleqq a$. Thus, we have $M \cap [0, a] = \emptyset$. By the Separation Lemma there is $r \in \sigma(L)$ with $a \leqq r$ and $M \cap [0, r] = \emptyset$. But, we can take $r_0 \in V_{\min}(a)$ with $r_0 \leqq r$, and then $r_0 \in M \cap [0, r]$, a contradiction.

The concept of radicals is a classical notion of commutative ring theory and its abstract formulation has been attempted long back and is scattered in several papers in various forms (see for example MURATA [5] and ANDERSON [1]). Let us recall this concept in abstract form. The *radical* of an element $a \in L$, denoted by $\sqrt{a}$, is defined by

$$\sqrt{a} = \bigvee \{x \in L:\ x^n \leqq a \text{ for some integer } n\}.$$

Evidently, $a \leqq \sqrt{a}$ for any $a \in L$, and $p = \sqrt{p}$ if $p \in \sigma(L)$. Hence, we have $V(\sqrt{a}) = V(a)$.

Lemma 3.5. (i) *If $\sqrt{a}$ is compact then $\sqrt{a}^n \leqq a$ for some integer $n$.*

(ii) *If $\sqrt{a}$ and $\sqrt{b}$ are compact then $\sqrt{ab} = \sqrt{a \wedge b} = \sqrt{a} \wedge \sqrt{b}$.*

(iii) *If 1 is compact, then $a < 1$ implies $\sqrt{a} < 1$.*

Proof. (i) The set $S = \{x \in L:\ x^n \leqq a$ for some $n\}$ is an ideal, for $(x \vee y)^{m+n} \leqq \leqq x^m \vee y^n$. Hence, if $\sqrt{a}$ is compact then $\sqrt{a} \in S$.

(ii) Evidently, $\sqrt{ab} \leqq \sqrt{a \wedge b} \leqq \sqrt{a} \wedge \sqrt{b}$. By (i), $\sqrt{a}^m \leqq a$, $\sqrt{b}^n \leqq b$ for some $m, n$. Then, $(\sqrt{a} \wedge \sqrt{b})^{m+n} = (\sqrt{a} \wedge \sqrt{b})^m (\sqrt{a} \wedge \sqrt{b})^n \leqq \sqrt{a}^m \sqrt{b}^n \leqq ab$. Hence, $\sqrt{a} \wedge \sqrt{b} \leqq \sqrt{ab}$.

(iii) By Corollary 2.2, there is $p \in \sigma(L)$ with $a \leqq p$. Then, $\sqrt{a} \leqq \sqrt{p} = p < 1$.

Theorem 3.6. *Assume that $L$ is generated by $M$-compact elements, that is, every element of $L$ is a join of $M$-compact elements. For $a \in L$ with $a < 1$,*

$$\sqrt{a} = \wedge \{p:\ p \in V_{\min}(a)\} = \wedge \{p:\ p \in V(a)\}.$$

Proof. Evidently, $\wedge V_{\min}(a) = \wedge V(a)$, and $\sqrt{a} \leqq \wedge V(\sqrt{a}) = \wedge V(a)$. If $\sqrt{a} < < \wedge V(a)$, there would exist an $M$-compact element $x$ such that $x \leqq \wedge V(a)$ and $x \not\leqq \sqrt{a}$. Then, $x^n \not\leqq a$ for every $n$, and by Proposition 2.1 there is $p \in \sigma(L)$ with $a \leqq p$ and $x \not\leqq p$. This contradicts $x \leqq \wedge V(a)$.

Corollary 3.7. *Assume that $L$ is generated by $M$-compact elements, and let $a \in L$ with $a < 1$. $V_{\min}(a)$ contains only one element if and only if $\sqrt{a}$ is a character.*

Proof. The "only if" part follows from the theorem, and the converse is evident.

We remark that the $r$-lattice introduced in [1] satisfies the assumption of this theorem, because any compact element of an $r$-lattice is $M$-compact by Theorem 2.1 of [1].

## 4. Related elements and associated characters of primary elements

We now take up a notion of one more related concept which is found in ring theory. The notion so far has not been pulled down to lattice theory nor has been abstracted in the sense of DILWORTH [2].

Let $a\in L$ with $a<1$. An element $b\in L$ is said to be *related to a* if there exists $x\in L$ such that $x\nleqq a$ and $bx\leqq a$. If $b$ is related to $a$ and $b'\leqq b$ then evidently $b'$ is related to $a$. Hence, the set of all elements of $L$ which are related to $a$ is multiplicatively closed. Next, let $p\in\sigma(L)$. Evidently, $b$ is related to $p$ if and only if $b\leqq p$. Hence, the set of all elements of $L$ which are unrelated to $p$ coincides with $C(p)$ and hence it is multiplicatively closed.

Lemma 4.1. *Let $a\in L$ with $a<1$, and let $b\in L$.*

(i) *If $a=a_1\wedge\ldots\wedge a_n$ $(a_i<1)$ and if $b$ is related to $a$, then $b$ is related to $a_i$ for some $i$.*

(ii) *If there exists $x\in L$ such that $x\nleqq a$ and $b^n x\leqq a$ for some integer $n$, then $b$ is related to $a$.*

(iii) *Assume that $\sqrt{a}$ is compact. If $b$ is related to $\sqrt{a}$ then $b$ is related to $a$. Especially, $\sqrt{a}$ is related to $a$.*

Proof. (i) is evident.

(ii) If $x\nleqq a$ and $b^n x\leqq a$, then taking the smallest integer $i$ such that $b^i x\leqq a$, we have $b^{i-1}x\nleqq a$ and $b(b^{i-1}x)\leqq a$ $(b^0=1)$. Hence, $b$ is related to $a$.

(iii) By Lemma 3.5 (i), $\sqrt{a}^n\leqq a$ for some $n$. If $x\nleqq\sqrt{a}$ and $bx\leqq\sqrt{a}$, then $x^n\nleqq a$ and $b^n x^n\leqq\sqrt{a}^n\leqq a$. Hence, $b$ is related to $a$ by (ii).

Theorem 4.2. *Assume that $L$ satisfies* (K), *and let $a\in L$ with $a<1$. Every minimal character $p$ belonging to $a$ is related to $a$.*

Proof. By Theorem 3.3, there is $x\in L$ such that $x\nleqq p$ and $p^n x\leqq a$ for some $n$. Then, we have $x\nleqq a$, for $a\leqq p$. Hence, $p$ is related to $a$ by Lemma 4.1. (ii).

Following DILWORTH [2], an element $q\in L$ with $q<1$ is called *primary* if $xy\leqq q$ implies $x\leqq q$ or $y^n\leqq q$ for some integer $n$.

Lemma 4.3. *If $q\in L$ is primary and if $\sqrt{q}$ is compact, then $\sqrt{q}\in\sigma(L)$ and $V_{\min}(q)=\{\sqrt{q}\}$. Moreover, $b\in L$ is related to $q$ if and only if $b\leqq\sqrt{q}$.*

Proof. This can be proved by using the fact: $\sqrt{q}^n\leqq q$ for some $n$, and the details are omitted.

Hereafter in this section, we assume that

$(*)$ *For every primary element $q$ of $L$ the element $\sqrt{q}$ is compact.*

By this assumption, we have $\sqrt{q}^n \leqq q$ for some integer $n$, and $\sqrt{q}$ is the least element of $V(q)$. We call $\sqrt{q}$ the *character associated with q*.

As stated in [2], we have the following lemma (the proof is omitted).

Lemma 4.4. *If $q_1, q_2$ are primary elements associated with the same character p, then $q_1 \wedge q_2$ is also a primary element with the same associated character p.*

Following [2], an element $a \in L$ is said to have an *irredundant* (or normal) *primary decomposition,* if $a = q_1 \wedge \ldots \wedge q_m$ for some primary elements $q_1, \ldots, q_m$ and if this expression cannot be reduced further. Then, by Lemma 4.4, $q_1, \ldots, q_m$ are associated with distinct characters.

Remark 4.5. If $a \in L$ has an irredundant primary decomposition $a = q_1 \wedge \ldots \ldots \wedge q_m$ $(m \geqq 2)$, then $a$ is not primary. This fact can be proved by the same way as [5], Lemma 7, since $\sqrt{a} = \sqrt{q_1} \wedge \ldots \wedge \sqrt{q_m}$ by Lemma 3.5 (ii).

Lemma 4.6. *Let $a \in L$ have an irredundant primary decomposition $a = q_1 \wedge \ldots \ldots \wedge q_m$ and put $p_i = \sqrt{q_i}$ $(p_i \in \sigma(L))$.*

(i) *For $p \in \sigma(L)$, $a \leqq p$ if and only if $p_i \leqq p$ for some i.*

(ii) *An element $c \in L$ is related to a if and only if $c \leqq p_i$ for some i.*

Proof. (i) Let $a \leqq p$. We have $p_i^{n_i} \leqq q_i$ for some integer $n_i$. Put $b = \prod_{i=1}^{m} p_i^{n_i}$. Since $b \leqq q_i$ for every $i$, we have $b \leqq a \leqq p$. Then, $p_i \leqq p$ for some $i$, since $p$ is a character. The converse is evident.

(ii) If $c$ is related to $a$, then $c \leqq \sqrt{q_i} = p_i$ for some $i$ by Lemma 4.1 (i) and Lemma 4.3. Conversely, let $c \leqq p_i$ for some $i$. Putting $b = \bigwedge_{j \neq i} q_j$, we have $b > a$ since the decompositon is irredundant. Since $p_i^n \leqq q_i$ for some $n$, we have $c^n b \leqq q_i b \leqq \bigwedge_{j=1}^{m} q_j = a$. Hence, $c$ is related to $a$ by Lemma 4.1 (ii).

Theorem 4.7. *Let $a \in L$ have an irredundant primary decomposition $a = q_1 \wedge \ldots \ldots \wedge q_m$ and put $p_i = \sqrt{q_i}$. The set of all minimal elements of $\{p_1, \ldots, p_m\}$ coincides with $V_{\min}(a)$. The set of all maximal elements of $\{p_1, \ldots, p_m\}$ coincides with the set of all maximal elements of the set $\{x \in L$: x is related to a$\}$.*

Proof. These statements immediately follow from Lemma 4.6.

Corollary 4.8. *If $a \in L$ has an irredundant primary decomposition, then every maximal element among all the elements related to a is a character containing a.*

For $a \in L$ and $p \in \sigma(L)$, we put

$$a(p) = \vee \{x \in L \colon xy \leqq a \text{ for some } y \nleqq p\}.$$

We now set ourselves to describe the elements $a(p)$.

Lemma 4.9. *If* $a \leqq p$ *then* $a \leqq a(p) \leqq p$. *If* $a \nleqq p$ *then* $a(p)=1$.

Proof. Let $a \leqq p$. If $xy \leqq a$ and $y \nleqq p$, then we have $x \leqq p$, since $xy \leqq p$. Hence, $a(p) \leqq p$. Moreover, $a \leqq a(p)$, since $a1 \leqq a$ and $1 \nleqq p$. Next, $a \nleqq p$ implies $a(p)=1$, since $1a \leqq a$.

Lemma 4.10. *Let* $a \in L$ *have an irredundant primary decomposition* $a = q_1 \wedge \ldots \wedge q_m$ *and put* $p_i = \sqrt{q_i}$. *For* $p \in \sigma(L)$, *if we put* $I(p) = \{i\colon p_i \leqq p\}$, *then* $a(p) = \wedge \{q_i\colon i \in I(p)\}$. ($a(p)=1$ *if* $I(p)=\emptyset$.)

Proof. Let $i \in I(p)$. If $xy \leqq a$ and $y \nleqq p$, then since $p_i \leqq p$, we have $y \nleqq p_i = \sqrt{q_i}$, and hence $y^n \nleqq q_i$ for every $n$. Since $xy \leqq q_i$, we have $x \leqq q_i$. Thus, $a(p) \leqq q_i$. Put $b = \wedge \{q_i\colon i \in I(p)\}$. As above we get $a(p) \leqq b$. Next, since $p_j^{n_j} \leqq q_j$ for some $n_j$, we put $c = \prod \{p_j^{n_j}\colon j \notin I(p)\}$. Then, $c \nleqq p$, since $p_j \nleqq p$ for every $j \in I(p)$. We have $c \leqq \wedge \{q_j\colon j \notin I(p)\}$, and hence $bc \leqq a$. Therefore, $b \leqq a(p)$. (If $I(p)=\emptyset$ then we may put $b=1$.)

Theorem 4.11. *Let* $a \in L$ *have an irredundant primary decomposition* $a = q_1 \wedge \ldots \wedge q_m$ *and put* $p_i = \sqrt{q_i}$. *For* $p \in \sigma(L)$, $p = p_i$ *for some i if and only if* $a(p) < 1$ *and p is maximal among all the elements related to* $a(p)$.

Proof. Let $p = p_k$ and put $I = \{i\colon p_i \leqq p_k\}$ ($I \neq \emptyset$, since $k \in I$). By Lemma 4.10, $a(p_k)$ has an irredundant primary decomposition $a(p_k) = \wedge \{q_i\colon i \in I\}$. Since $p_k$ is maximal in $\{p_i\colon i \in I\}$, $p_k$ is maximal in $\{x \in L\colon x \text{ is related to } a(p)\}$ by Theorem 4.7.

Conversely, if $a(p) < 1$, then $I(p) = \{i\colon p_i \leqq p\}$ is non-empty and $a(p)$ has an irredundant primary decomposition $a(p) = \wedge \{q_i\colon i \in I(p)\}$. If $p$ is maximal among the elements related to $a(p)$, then $p$ coincides with a maximal element of $\{p_i\colon i \in I(p)\}$.

Corollary 4.12. *Any two irredundant primary decompositions of an element* $a \in L$ *have the same number of components and the same set of associated characters.*

## 5. Minimal spectrum

First we shall introduce a new concept. A character $p \in \sigma(L)$ is called *purely minimal* if $C(p)$ is maximal in the set $\{M \in \mathscr{M}(L)\colon 0 \notin M\}$. It follows from Lemma 3.1 that any purely minimal character is minimal. The set of all purely minimal characters is denoted by $\pi_0(L)$. This is a subset of $\pi(L)$.

Theorem 5.1. (i) *For* $p \in \sigma(L)$ *the following four statements are equivalent:*
(1) *p is purely minimal;*

(2) *there exists* $x\in L$ *such that* $x\nleqq p$ *and* $p^n x=0$ *for some integer* $n$:

(3) $p^*\nleqq p$;

(4) *for any* $x\in L, p$ *contains precisely one of* $x$ *and* $x^*$;

(ii) *if* $L$ *satisfies* (K), *then any minimal character is purely minimal, that is,* $\pi_0(L)=\pi(L)$.

Proof. The equivalence of (1) and (2) follows from Theorem 3.3 by putting $a=0$. The statement (ii) also follows from Theorem 3.3. The equivalence (2)$\Leftrightarrow$(3) and the implication (4)$\Rightarrow$(3) are evident. (3)$\Rightarrow$(4): If $x\nleqq p$, then $x^*\leqq p$ by Lemma 2.4. If $x\leqq p$, then $p^*\leqq x^*$, and hence $x^*\nleqq p$ by (3).

Corollary 5.2. *If* $p\in\sigma(L)$ *is purely minimal then* $p^{**}\leqq p$, *and* $x\leqq p$ *implies* $x^{**}\leqq p$.

Proof. Since $p^*\nleqq p$ by Theorem 5.1, we have $p^{**}\leqq p$ by Lemma 2.4. If $x\leqq p$, then we have $x^*\geqq p^*$, and hence $x^{**}\leqq p^{**}\leqq p$.

The hull kernel topology on $\pi(L)$ is the induced topology of the hull kernel topology on $\sigma(L)$. $\pi(L)$ with this topology will be called the *minimal spectrum* of $L$. For any $a\in L$, the set $h(a)=\{p\in\pi(L)\colon a\leqq p\}$ is called the *hull* of $a$. For any subset $R$ of $\pi(L)$, the element $K(R)=\wedge\{p\colon p\in R\}$ is called the *kernel* of $R$. Then, a subset $R$ of $\pi(L)$ is closed if and only if $R=h(a)$ for some $a\in L$. Evidently, $a\leqq K(h(a))$ for every $a\in L$, and for every $R\subset\pi(L)$, $h(K(R))$ is equal to the closure of $R$.

Now we get an important topological property of purely minimal characters.

Theorem 5.3. *If* $p\in\pi(L)$ *is purely minimal then* $p$ *is an isolated point of* $\pi(L)$.

Proof. Put $G=\pi(L)-h(p^*)$. $G$ is an open set, and $p\in G$ since $p^*\nleqq p$. If $q\in\pi(L)$ and $q\neq p$, then $p\nleqq q$ and hence $p^*\leqq q$ by Lemma 2.4. Hence we have $G=\{p\}$, and $p$ is an isolated point.

Corollary 5.4. *The induced topology on* $\pi_0(L)$ *from* $\pi(L)$ *is discrete. If* $L$ *satisfies* (K) *then the minimal spectrum* $\pi(L)$ *is discrete.*

Proof. These statements follow from Theorem 5.3 and Theorem 5.1 (ii) immediately.

Remark 5.5. If every finite product of elements of $\pi(L)$ is compact (especially, f $L$ satisfies (K)), then $\pi(L)$ is a finite set. This follows from Theorem 3.4 by putting $a=0$.

Finally, we shall obtain several important results about hulls and nilpotent elements, assuming the condition (K).

Lemma 5.6. *Assume that* $L$ *satisfies* (K). *For* $a\in L$ *and* $p\in\pi(L)$, $a\leqq p$ *if aud only if* $a^*\nleqq p$. *Hence,* $h(a)=\pi(L)-h(a^*)=h(a^{**})$.

Proof. This follows from the property (4) in Theorem 5.1.

Theorem 5.7. *Assume that L satisfies* (K), *and let R be a subset of* $\pi(L)$. *If we put* $a=\vee\{p^*: p\in R\}$, *then* $R=h(a^*)=h(K(R))$.

Proof. If $p\in R$, then we have $p^*\leqq a$ and $p^*\nleqq p$, and then $a\nleqq p$. Conversely, if $a\nleqq p\in\pi(L)$, there exists $q\in R$ such that $q^*\nleqq p$. Then, $q\leqq p$ by Lemma 5.6, and hence $p=q\in R$. Therefore, $R=\pi(L)-h(a)=h(a^*)$. Next, we have $a^*\leqq K(h(a^*))=$ $=K(R)$, and hence $h(K(R))\subset h(a^*)=R\subset h(K(R))$.

Lemma 5.8. *Assume that L satisfies* (*K*).

(i) $\sqrt{0}$ *is the greatest nilpotent element and is equal to* $\wedge\{p: p\in\pi(L)\}$.

(ii) $x\in L$ *is nilpotent if and only if* $h(x)=\pi(L)$.

(iii) $x^*$ *is nilpotent if and only if* $h(x)=\emptyset$.

(iv) $x\wedge x^*$ *is nilpotent for every* $x\in L$.

Proof. (i) follows from Lemma 3.5 (i) and Theorem 3.6. Evidently, (ii) follows from (i). (iii) follows from (ii), since $h(x^*)=\pi(L)-h(x)$. (iv) follows from Lemma 2.3 (i).

Theorem 5.9. *Assume that L satisfies* (*K*). *The following eight statements are equivalent:*

(1) *no nonzero element of L is nilpotent;*

(2) $\wedge\{p: p\in\pi(L)\}=0$;

(3) $x^*=\wedge\{p\in\pi(L): x\nleqq p\}$ *for every* $x\in L$;

(4) $x^*=K(h(x^*))$ *for every* $x\in L$;

(5) $x^{**}=K(h(x))$ *for every* $x\in L$;

(6) $x\leqq x^{**}$ *for every* $x\in L$;

(7) $x\wedge x^*=0$ *for every* $x\in L$;

(8) $x^*=1$ *implies* $x=0$.

Proof. The equivalence of (1) and (2) follows from Lemma 5.8 (i). The equivalence of (3) and (4) follows from Lemma 5.6. (2)⇒(4): Putting $y=K(h(x^*))$, we have $x^*\leqq y$. If $x\leqq p\in\pi(L)$, then $y\leqq p$, since $x^*\leqq p$. Hence, $xy\leqq p$ for every $p\in\pi(L)$, and hence $xy=0$ by (2). Thus, $y\leqq x^*$. (4)⇒(5) is evident, since $h(x^{**})=h(x)$. (5)⇒(6) is evident. (6)⇒(8) is evident, since $1^*=0$. (8)⇒(1) follows from Lemma 2.3 (ii). (1)⇒(7) follows from Lemma 5.8 (iv). (7)⇒(8) is evident.

Theorem 5.10. *Assume that L satisfies* (*K*) *and that no nonzero element of L is nilpotent.*

(i) *$L$ is pseudo-complemented and $x^*$ is a pseudo-complement of $x$ for any* $x \in L$.

(ii) *For* $x, y \in L$, $h(x) \subset h(y)$ *if and only if* $x^* \leqq y^*$. *Hence,* $h(x) = h(y)$ *if and only if* $x^* = y^*$.

(iii) $x^{***} = x^*$ *for every* $x \in L$.

(iv) *For* $a \in L$, *the following four statements are equivalent:*

(1) $a = a^{**}$ *(following* [10], *a may be called normal);*

(2) $a = b^*$ *for some* $b \in L$;

(3) $a = K(h(a))$;

(4) *a is the kernel of some subset of* $\pi(L)$.

Proof. (i) If $y \wedge x = 0$, then $xy = 0$ and hence $y \leqq x^*$. Then, by (7) of Theorem 5.9, $x^*$ is the greatest element of the set $\{y \in L: y \wedge x = 0\}$.

(ii) If $h(x) \subset h(y)$, then $h(x^*) = \pi(L) - h(x) \supset \pi(L) - h(y) = h(y^*)$, and hence $x^* = K(h(x^*)) \leqq K(h(y^*)) = y^*$ by (4) of Theorem 5.9. Conversely, if $x^* \leqq y^*$, then $h(x^*) \supset h(y^*)$ and then $h(x) \subset h(y)$.

(iii) By (6) of Theorem 5.9, we have $x^* \leqq (x^*)^{**}$, and moreover $x^{**} \geqq x$ implies $(x^{**})^* \leqq x^*$.

(iv) (1)$\Rightarrow$(2) and (3)$\Rightarrow$(4) are trivial. (1) and (3) are equivalent by (5) of Theorem 5.9. (2)$\Rightarrow$(1): If $a = b^*$ then $a^{**} = b^{***} = b^* = a$ by (iii). (4)$\Rightarrow$(3): If $a = K(R)$ for some $R \subset \pi(L)$, then we have $h(a) = R$ by Theorem 5.7.

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