Acta Sci. Math., 52 (1988), 81—84

Examples of local uniformity of congruences

IVAN CHAJDA

Following [6], a congruence Θ on an algebra A is *uniform* if every two congruence classes of Θ have the same cardinality. An algebra A is *uniform* if each $\Theta \in \mathrm{Con}\, A$ has this property. A class of algebras is *uniform* if every algebra of this class has this property.

It is well known that groups and Boolean algebras are uniform. Moreover, every variety generated by quasi-primal algebras (i.e. a discriminator variety, see [7]) is uniform, see [6] or Theorem 2.2 in [7]. Some classes of uniform algebras are depicted also in [3]. Although such "nice" varieties are uniform, W. TAYLOR [6] proved that the class of uniform varieties is not definable by a Mal'cev condition. He introduces the following concept: an algebra A is *weakly uniform* if for every cardinal $\mathfrak{m}$ there exists a cardinal $\mathfrak{n}$ such that whenever B_1 and B_2 are congruence classes of some $\Theta \in \mathrm{Con}\, A$, if $\mathrm{card}\, B_1 \leqq \mathfrak{m}$ then $\mathrm{card}\, B_2 \leqq \mathfrak{n}$. It was proven in [6] that the class of varieties of weakly uniform algebras is definable by a Mal'cev condition.

For algebras with a nullary operation, we can give a local version of uniformity:

Definition. An algebra A with a nullary operation c is *c-locally uniform* if for each element $a \in A$ and each $\Theta \in \mathrm{Con}\, A$, $\mathrm{card}\, [a]_\Theta \leqq \mathrm{card}\, [c]_\Theta$. A class K of algebras of the same type with a nullary operation c is *c-locally uniform* if each $A \in \mathscr{K}$ has this property.

It is clear that every uniform algebra with a nullary operation c is c-locally uniform and every c-locally uniform algebra is weakly uniform with $\mathfrak{n} = \mathrm{card}\, [c]_\Theta$.

Recall that an algebra A is *regular* if every two congruences on A coincide whenever they have a congruence class in common. An algebra A with a nullary operation c is *weakly regular (with respect to c)* if every two congruences $\Theta, \Phi \in \mathrm{Con}\, A$ coincide whenever $[c]_\Theta = [c]_\Phi$. A class $\mathscr{K}$ of algebras is *(weakly) regular* if each $A \in \mathscr{K}$ has this property.

Received January 23, 1985.

Proposition (Lemma 2.6 in [5]). *Every uniform variety is regular.*

We can prove a similar result for c-locally uniform algebras.

Theorem 1. Let $\mathcal{K}$ be a class of algebras of the same type with a nullary operation c closed under homomorphic immages. If $\mathcal{K}$ is c-locally uniform, then $\mathcal{K}$ is weakly regular with respect to c.

Proof. Let c be a nullary operation of a c-locally uniform class $\mathcal{K}$. Let $\mathcal{K}$ be closed under homomorphic immages. Suppose $A\in\mathcal{K}$, $\Theta_1, \Theta_2\in\operatorname{Con} A$ and

$$(*) \qquad [c]_{\Theta_1} = [c]_{\Theta_2}.$$

In this case we have clearly $[c]_{\Theta_1\wedge\Theta_2}=[c]_{\Theta_1\vee\Theta_2}=[c]_{\Theta_1}=[c]_{\Theta_2}$; without loss of generality, we can assume $\Theta_1\leqq\Theta_2$. Denote by ω the identity relation on A/Θ_1. By $(*)$, the congruences $\omega=\Theta_1/\Theta_1$ and Θ_2/Θ_1 of $A/\Theta_1\in\mathcal{K}$ have the same congruence class containing the nullary operation $[c]_{\Theta_1}$ of A/Θ_1. Thus

$$\operatorname{card}[c]_{\Theta_2/\Theta_1} = \operatorname{card}[c]_\omega = 1.$$

Since A/Θ_1 is c-locally uniform, we have

$$1 \leqq \operatorname{card}[a]_{\Theta_2/\Theta_1} \leqq \operatorname{card}[c]_{\Theta_2/\Theta_1} = 1$$

for each $a\in A$, thus $\Theta_2/\Theta_1=\omega$, i.e. $\Theta_2=\Theta_1$.

The aim of this paper is to show that there exist important classes of finite algebras which are c-locally uniform but not uniform. By Theorem 1, they must be weakly regular. By [4], such algebras can be found among Heyting algebras, implication algebras and other types of lattice ordered algebras with pseudocomplementation.

An algebra $\langle L; \vee, \wedge, \cdot, 1\rangle$ with three binary and one nullary operations is an *rp-algebra* if $\langle L; \vee, \wedge, 1\rangle$ is a lattice with greatest element 1 and $\cdot$ satisfies the following identities:

$$(**) \qquad x\cdot x = 1, \quad (x\cdot y)\wedge y = y, \quad (x\cdot y)\wedge x = x\wedge y.$$

Theorem 2. *The class of all finite rp-algebras is* 1*-locally uniform but not uniform.*

Proof. Let $\mathcal{K}$ be a class of all finite rp-algebras. Clearly $\mathcal{K}$ is not uniform, because, e.g. the three-element chain $C=\{0, a, 1\}$, $0<a<1$, with a binary operation $\cdot$ defined by

$a\cdot 0=0$, $1\cdot 0=0$ and $x\cdot y=1$ for all other combinations of variables

is an rp-algebra but the partition $\{0\}$, $\{a, 1\}$ forms a congruence on C which is not uniform.

We prove that $\mathscr{K}$ is 1-locally uniform. Let $A\in\mathscr{K}$, $z\in A$, $\Theta\in\operatorname{Con} A$. Since A is a finite lattice, the congruence class $[z]_\Theta$ contains a greatest element a. Put $\varphi(x)=a\cdot x$. We prove that φ is an injection of $[z]_\Theta$ into $[1]_\Theta$. If $x\in[z]_\Theta$, then $\langle x, a\rangle\in\Theta$. Since φ is an algebraic function over A, it follows that $\langle\varphi(x), \varphi(a)\rangle=\langle\varphi(x), a\cdot a\rangle=\langle\varphi(x), 1\rangle\in\Theta$, i.e. $\varphi(x)\in[1]_\Theta$. Thus $\varphi\colon [z]_\Theta\to[1]_\Theta$. Suppose $\varphi(x)=\varphi(y)$ for $x, y\in[z]_\Theta$. Then $a\cdot x=a\cdot y$, whence $a\wedge(a\cdot x)=a\wedge(a\cdot y)$. By $(**)$, this yields $a\wedge x=a\wedge y$. Since $x\leqq a$, $y\leqq a$, we obtain $x=y$. Thus φ is an injection, and therefore $\operatorname{card}[z]_\Theta\leqq\operatorname{card}[1]_\Theta$.

Let L be a lattice and $a, b\in L$. An element $x\in L$ is called a *relative pseudocomplement of a with respect to b if* x is the greatest element satisfying $a\wedge x=a\wedge b$; denote it by $a*b$. A lattice L is *relatively pseudocomplemented* if $a*b$ exists for each $a, b\in L$. Then clearly L has a greatest element 1, and $a*a=1$ for each $a\in L$. Clearly the operation $*$ satisfies the identities $(**)$, i.e. we obtain the following

Corollary 1. *Every finite relatively pseudocomplemented lattice is* 1*-locally uniform.*

Note that a finite lattice is relatively pseudocomplemented if and only if it is distributive. Corollary 1 implies immediately (for the definition, see e.g. [7])

Corollary 2. *Every finite Heyting algebra is* 1*-locally uniform.*

Remark. By [4], a Heyting algebra is regular if and only if it is a Boolean algebra. Every three-element chain $0<a<1$ with a pseudocomplementation is a Heyting algebra which is not uniform.

Following [1], an algebra $\langle A;\cdot\rangle$ with one binary operation is an *implication algebra* if it satisfies

$$(x\cdot y)\cdot x=x,\quad (x\cdot y)\cdot y=(y\cdot x)\cdot x,\quad x\cdot(y\cdot z)=y\cdot(x\cdot z).$$

As it was proven in [1], every implication algebra A has a nullary operation 1 such that $a\cdot a=1$ for each $a\in A$.

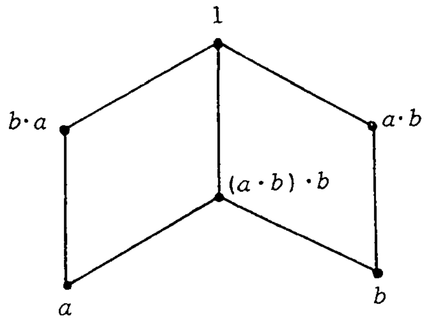
Lemma 1. *Every implication algebra is a* $\vee$*-semilattice with greatest element* 1 *with respect to the operation* $a\vee b=(a\cdot b)\cdot b$.

For the proof, see Theorem 3 and Theorem 4 in [1].

Lemma 2. (Theorem 5 in [1]). *Let A be an implication algebra and $a, b\in A$. If p is any lower bound for a and b (with respect to the semilattice ordering), then the infimum $a\wedge b$ of a and b exists, and* $a\wedge b=[a\cdot(b\cdot p)]\cdot p$.

Theorem 3. *The class of all finite implication algebras is* 1*-locally uniform but not uniform.*

Proof. Let A be free implication algebra with two free generators a, b. By the Corollary of Theorem 2 in [1], A has the following diagram (as a $\vee$-semilattice):



Clearly the equivalence Θ given by the partition

$$\{b \cdot a, 1\}, \quad \{a, (a \cdot b) \cdot b\}, \quad \{a \cdot b\}, \quad \{b\}$$

is a congruence on A which is not uniform.

Let A be a finite implication algebra, $\Theta \in \mathrm{Con}\, A$ and $z \in A$. By Lemma 1, there exists a greatest element a in $[z]_\Theta$. Put $\varphi(x) = a \cdot x$. Clearly $\varphi(a) = a \cdot a = 1$. If $x \in [z]_\Theta$, then $\langle x, a \rangle \in \Theta$ which implies $\langle \varphi(x), \varphi(a) \rangle = \langle \varphi(x), 1 \rangle \in \Theta$, i.e. $\varphi(x) \in [1]_\Theta$. Thus φ is a mapping of $[z]_\Theta$ into $[1]_\Theta$.

We prove that φ is an injection on $[z]_\Theta$. Suppose $x, y \in [z]_\Theta$ and $\varphi(x) = \varphi(y)$. Then $a \cdot x = a \cdot y$. Since $x \leqq a$, $x \leqq a \cdot x$ and $y \leqq a$, $y \leqq a \cdot y$, therefore x is a lower bound of a and $a \cdot x$, y, is a lower bound of a and $a \cdot y$. By Lemma 2, $a \wedge a \cdot x$ and $a \wedge a \cdot y$ exist, and $a \cdot x = a \cdot y$ implies that $a \cdot x \wedge a = a \cdot y \wedge a$. By Lemma 2, $a \cdot x \wedge a = [(a \cdot x)(a \cdot x)] \cdot x = 1 \cdot x = x$, and analogously $a \cdot y \wedge a = y$. Hence $x = y$.

References

[1] J. C. ABBOTT, *Semi-Boolean algebra, Mat. Vestnik*, **4** (19) (1967), 177—198.

[2] S. BURRIS, H. P. SANKAPPANAVAR, *A Course in Universal Algebra*, Springer-Verlag (Berlin—Heidelberg—New York, 1981).

[3] I. CHAJDA, Notes on the Big Dipper identity, *Ann. Univ. Sci. Budapest, R. Eötvös Nom.*, **29** (1986), 97—101.

[4] J. C. VARLET, Regularity in p-algebras and p-semilattices, in: *Banach Center Publ.*, Vol. 9, Polish Sci. Publ. (Warszawa, 1982); pp. 369—382.

[5] R. W. QUACKENBUSH, Algebras with minimal spectrum, *Algebra Univesalis*, **10** (1980), 117—129.

[6] W. TAYLOR, Uniformity of congruences, *Algebra Universalis*, **4** (1974), 342—360.

[7] H. WERNER, *Discriminator algebras*, Studien zur Algebra und ihre Anwendungen, Band 6, Akademie-Verlag (Berlin, 1979).

TŘÍDA LIDOVÝCH MILICÍ 22
750 00 PŘEROV, CZECHOSLOVAKIA