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A note on optimal interpolation with rational functions

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Introduction. This note provides a further application of results derived in [7], which dealt with polynomial interpolation.

Let $\mathbf{Y}$ be the space of rational functions whose numerators are of degree n or less, with denominator

$$Q(t) = (t-t_{n+1})\dots(t-t_{n+m}).$$

If nodes of interpolation $t_0, \dots, t_n$ are chosen on an interval, $[a, b]$ such that

$$a = t_0 < t_1 < \dots < t_n = b,$$

and such that

$$t_{n+k} \notin [t_0, t_n] \quad \text{for} \quad k \in \{1, \dots, n\},$$

it is possible to construct *fundamental functions* $y_0, \dots, y_n$ such that $y_i(t_j) = \delta_{ij}$ (Kronecker delta) for $i \in \{0, \dots, n\}$ and for $j \in \{0, \dots, n\}$, by means of the formula

$$y_i(t) = \frac{Q(t_i)}{Q(t)} \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(t-t_j)}{(t_i-t_j)}.$$

One defines an interpolating projection $L\colon C[a, b] \to \mathbf{Y}$ by

$$Lf = \sum_{i=0}^{n} f(t_i) y_i \quad \text{for} \quad f \in C[a, b].$$

Clearly, L is bounded, and

$$\|L\| = \Big\| \sum_{i=0}^{n} |y_i| \Big\|.$$

Our purpose here is to minimize $\|L\|$.

Notation. We define, for $i \in \{1, \dots, n\}$, X_i to be the function (in $\mathbf{Y}$) which agrees with $\sum_{j=0}^{n} |y_j|$ on the interval $[t_{i-1}, t_i]$, $\lambda_i = X_i(T_i)$, and T_i as the point in (t_{i-1}, t_i)

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at which λ_i is attained. We note that

$$X_i'(T_i) = 0, \quad \text{for} \quad i \in \{1, \dots, n\}.$$

Results.

Theorem. *If interpolation is done on an interval* $[a, b]$ *with rational functions having denominator*

$$Q(t) = (t - t_{n+1}) \dots (t - t_{n+m})$$

and nodes of interpolation $t_0, \dots, t_n$, *such that*

$$a = t_0 < \dots < t_n = b < t_{n+1} < \dots < t_{n+m},$$

then

(i) *interpolation of minimal norm is characterized by the Bernstein condition* [1] *that* $\lambda_1 = \dots = \lambda_n$, *which is produced by a unique choice of nodes;*

(ii) *the quantities* $\lambda_1, \dots, \lambda_n$ *obey the Erdős condition* [3] *that if one of them is greater than the common value given in* (i), *another is less;*

(iii) *the norm of interpolation is governed by the ratio* $(b-a)/(t_{n+1}-b)$. *Specifically, the norm increases without bound as* $b \mapsto t_{n+1}$ *and decreases as* $b \to a$, *with lower limit equal to the norm of optimal Lagrange interpolation with polynomials of degree n or less.*

Corollary 1. *The above theorem also holds when the space of interpolation consists of all multiples of the function*

$$(t - t_{n+1})^{k_1} \dots (t - t_{n+m})^{k_m}$$

by a polynomial of degree n or less, with $k_j < 0$ *for* $j \in \{1, \dots, n\}$.

Corollary 2. *Some or all of the points* $t_{n+1}, \dots, t_{n+m}$ *can be to the left of* t_0 *as well as to the right of* t_n, *and the above results are still valid.*

Proof of Theorem. One notes that the functions

$$\partial\lambda_i/\partial t_j = -y_j(T_i) X_i'(t_j), \quad i \in \{1, \dots, n\}, \quad j \in \{0, \dots, n\}$$

exist and are continuous in $t_0, \dots, t_n$. The points $T_0, \dots, T_n$, of course, depend in an analytic fashion upon the nodes.

All of our results will follow from properties of various submatrices of

$$A = (\partial\lambda_i/\partial t_j)_{i,j=1}^n,$$

which represents the derivative of the function

$$(t_1, \dots, t_n) \mapsto (\lambda_1, \dots, \lambda_n).$$

We define A_i for $i \in \{1, \dots, n\}$ to be the matrix obtained by deleting the i^{th} column and n^{th} row of A. To prove (i) and (ii) of the theorem, it suffices to show

(1) $\det A_i \neq 0$ for $i \in \{1, \dots, n\}$ for arbitrary $t_0, \dots, t_{n+m}$,

and

(2) $\det A_i$ alternates in sign on $\{1, \dots, n\}$.

To prove (iii), it is enough to prove

(3) $\det A \neq 0$.

To show (1) and (2), we first perform some row and column cancellations. For $j \in \{1, \dots, n\}$, the j^{th} row of A is given by

$$-y_i(T_1)X_1'(t_j) \dots -y_j(T_n)X_n'(t_j).$$

It is possible therefore to multiply the j^{th} row by the "denominator" of y_j, namely by

$$\frac{1}{Q(t_j)} \sum_{\substack{l=0 \\ l \neq j}}^{n+m} (t_j - t_l).$$

When this procedure has been completed, the i^{th} column, for $i \in \{1, \dots, n+m\}$ is of the form

$$\frac{1}{Q(T_i)} \prod_{\substack{j=0 \\ j \neq 1}}^{n} (T_i - t_j) X_i'(t_1)$$

$$\dots$$

$$\frac{1}{Q(T_i)} \prod_{\substack{j=0 \\ j \neq n}}^{n} (T_i - t_j) X_i'(t_n),$$

and the non-zero quantity $\prod_{j=0}^{n} (T_i - t_j)$ may be divided from the i^{th} column. Following this operation by multiplication of the i^{th} column by $Q(T_i)$, the matrix is left in the form

$$B = \begin{bmatrix} \dfrac{X_1'(t_1)}{t_1 - T_1} & \dots & \dfrac{X_n'(t_1)}{t_1 - T_n} \\ \dots & \dots & \dots \\ \dfrac{X_1'(t_n)}{t_n - T_1} & \dots & \dfrac{X_n'(t_n)}{t_n - T_n} \end{bmatrix}.$$

Now, it is possible to multiply the j^{th} row by $(Q(t_j))^2$, and the expression

$$q_i(t) = \frac{X_i'(t)}{t - T_i} (Q(t))^2, \quad i \in \{1, \dots, n+m\}$$

is a polynomial of degree $n+m-2$ or less which is evaluated at the successive points $t_1, \dots, t_n$ down the i^{th} column of the matrix.

Clearly, the roots of the polynomials $(t-T_i)q_i(t)$ will strictly interlace on the interval $[T_1, \infty]$, and it is possible to choose points $T_{n+1}, \dots, T_{n+m}$ with

$$T_n < t_n < T_{n+1} < \dots < T_{n+m}$$

such that the following conditions are satisfied by $q_1, \dots, q_n$.

(i) Each polynomial q_i has exactly one root in each of the subintervals (T_j, T_{j+1}), $j \in \{1, \dots, n+m-1\}$ of the interval $[T_1, T_{n+m}]$, except that q_i has no root in (T_i, T_{i+1}) for $i \in \{1, \dots, n+m-1\}$, nor in (T_{i-1}, T_i) for $i \in \{2, \dots, n+m\}$.

(ii) $q_i(T_j) \neq 0$ for $i, j \in \{1, \dots, n+m\}$.

Proposition. *Let polynomials* $q_1, \dots, q_{n+m}$ *and points* $T_1, \dots, T_{n+m}$ *satisfy* (i) *and* (ii), *and let points* $t_1, \dots, t_{n-1}$ *be situated so that*

$$T_1 < t_1 < T_2 < \dots < T_{n-1} < t_{n-1} < T_n.$$

Then, for $k \in \{1, \dots, n\}$,

$$\det(q_i(t_j))_{\substack{1 \le i \le n,\, 1 \le j \le n-1 \\ i \neq k}} \neq 0.$$

A proof of this Proposition appears in [7].

At this point, (1) and (2) follow. To prove (3), we need only to note that, in the present context, $n-1$ may be replaced by n in the above Proposition, with $k=n+1$ and the proposition still holds, permitting one to analyse what occurs as $t_n \to t_0$ or $t_n \to t_{n+1}$, subject to the condition $\lambda_1 = \dots = \lambda_n$.

This completes the proof of the Theorem. Corollary 1 is now established by a re-examination of the steps of matrix cancellation, leading to a similar system of polynomials $q_1, \dots, q_n$. Details of a similar argument appear in [7]. Corollary 3 can clearly be obtained by a slight modification of the above Proposition.

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