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Operators of Toeplitz and Hankel type

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In the present note the authors investigate abstract analoga of classical Toeplitz and Hankel operators and extend to these more general classes some of the results known from the classical theory. The investigation is based on the use of isometric dilations of contractions and on the properties of their Wold decompositions. In particular the unitary part of the isometric dilation plays a decisive role. To explain the genesis and motivation of our investigation let us recall some of the classical facts which are essential for our considerations.

We denote by $T(\varphi)$ the Toeplitz operator on H^2 defined for $f\in H^2$ by the formula $T(\varphi)f=P_+\varphi f$ where P_+ stands for the projection operator of L^2 onto H^2 and φ is an L^∞ function, the symbol of $T(\varphi)$. The projection onto the orthogonal complement $H^2_-=L^2\ominus H^2$ will be denoted by P_-. Since $P_+\bar{z}P_-=0$ we have $P_+\bar{z}P_+\varphi(z)zf(z)=$ $=P_+\varphi(z)f(z)$ for every $f\in H^2$. If S stands for the shift operator (multiplication by z) on H^2 this relation may be restated in the form

$$S^*T(\varphi)S = T(\varphi)$$

and it turns out that the relation $S^*AS=A$ is characteristic for Toeplitz operators on H^2.

There is another important class of operators which may be characterized by a similar relation. Hyponormal operators are defined by the inequality $TT^*\leqq T^*T$ and may accordingly be characterized by the existence of a contraction C such that $T^*=CT$. Hence

$$CTC^* = T^*C^* = (CT)^* = T$$

so that T satisfies a relation of the same type.

In a paper on hyponormal operators [4] C. FOIAŞ and B. SZ.-NAGY used dilation theory to show that for each hyponormal operator T acting on a Hilbert space $\mathfrak{H}$ there exists a normal operator N on a suitable Hilbert space $\mathfrak{G}$, a unitary operator

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U on $\mathfrak{G}$ and a contraction X: $\mathfrak{H}\to\mathfrak{G}$ such that $T=X^*NX$ while $N^*=UN$ and $\|X^*Ug\|\leqq\|X^*g\|$ for all $g\in\mathfrak{G}$.

The relation $T=X^*NX$ is clearly an analogon of the formula $T=P_+\varphi|H^2$: X replaces the injection operator $H^2\to L^2$, N replaces the symbol φ and X^* plays the role of a projection of $\mathfrak{G}$ onto $\mathfrak{H}$. Starting from this observation these two authors developed a theory of Toeplitz type operators [3] where the L^∞ function φ is replaced by an abstract symbol.

It is interesting to note that the relation $N^*=UN$ implies $N=UNU^*$. Indeed, $N=(UN)^*=N^*U^*=(UN)U^*$; it is one of the purposes of the present note to explain the importance of this relation.

In the classical case there is a parallel theory for Hankel operators. Starting from a $\varphi\in L^\infty$ we define $H(\varphi)$: $H^2\to H^2_-$ by the formula $H(\varphi)f=P_-\varphi f$. Since $P_-zP_+=0$ we have

$$H(\varphi)zf = P_-z\varphi(z)f(z) = P_-zP_-\varphi(z)f(z) = P_-zH(\varphi)f$$

so that

$$H(\varphi)S = ZH(\varphi)$$

if Z denotes the operator $g\to P_-zg$ on H^2_-. Again, this relation turns out to be characteristic for Hankel operators from H^2 into H^2_-.

In the present paper we intend to show that the class of Hankel operator also has an abstract analogon and propose to out line a theory of symbols for operators of Toeplitz and Hankel types.

To obtain the symbol for an operator A on H^2 satisfying the relation $S^*AS=A$ we first use this relation to extend A to the whole of L^2; it turns out that this extension commutes with the shift so that it coincides with the operator of multiplication by an L^∞ function φ. The operator A appears then as a compression to H^2 of this multiplication operator $M(\varphi)$.

In the sequel we shall view the symbol of A as this multiplication operator rather than the function generating it — this is possible in view of the isometric isomorphism between L^∞ taken as an algebra and the corresponding algebra of multiplication operators.

To obtain a symbol for an operator X: $H^2\to H^2_-$ satisfying $XS=ZX$ we use first the theorem on intertwining dilations to obtain an operator from H^2 into L^2 intertwining S and $M(z)$; extending its domain of definition to the whole of L^2 we obtain an operator which commutes with $M(z)$ and which yields the original operator as a compression, this time from H^2 into H^2_-.

Observe that $Z=P_-M(z)|H^2_-$ and that $M(\bar{z})=M(z)^*$ is the minimal isometric dilation of S^*. A similar situation obtains in the general case.

In a manner of speaking the construction of symbols for generalized Toeplitz and Hankel operators proceeds — in its early stages — along similar lines as in the classical theory; at a certain point, however, difficulties present themselves which have no counterpart in the classical case. In particular, a relation of the type $XS=ZX$ alone is not sufficient to characterize a class with satisfactory properties. We intend to show that, in the general case of abstract Toeplitz and Hankel operators, it is also possible to construct a symbol which is characterized by a certain commutativity relation and as a compression of which the given operator may be reconstructed.

The investigations of B. Sz.-Nagy and C. Foiaş indicate the important role played by the space $\mathfrak{R}$, the unitary part of the Wold decomposition of the isometric dilation of the contraction T by means of which the abstract Toeplitz operator is defined. The results of the present note seem to confirm the hypothesis that this space forms the natural domain of definition for operators which should play the role of an abstract symbol both for Toeplitz and Hankel operators. The main difficulty seems to lie in the fact that the Wold decomposition is trivial in the classical case, the isometric dilation of S^* being unitary, so that little help can be expected from immediate analogies.

It turns out that the methods presented below work even in the more general case when a Toeplitz operator X is defined by the relation $X=T_1XT_2^*$ where T_1 and T_2 are two arbitrary contractions acting on the spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$ which may be different from each other in general. In this manner we hope to eliminate results whose validity is essentially based on the equality $T_1=T_2$; at the same time, this generality does not seem to be excessive. We still obtain analoga of the Kronecker theorem as well as of the identity

$$T(\varphi\psi)-T(\varphi)T(\psi) = H(\varphi^*)^* H(\psi).$$

In a paper on operator equations [1] R. G. Douglas considered operators satisfying $X=T_1XT_2^*$. His investigations proceed along different lines; nevertheless, his ideas provided inspiration for some of our methods.

The paper is divided into five sections. In the first section we list some technical facts from dilation theory which will be needed in the main text.

Section two contains a short exposé of the theory of Toeplitz operators. In spite of the fact that the emphasis of this note is on Hankel operators it is, in our opinion, useful to include this short section. Our approach differs in details from that of Sz.-Nagy and Foiaş, the differences being motivated by the necessity to prepare the ground for the theory of Hankel operators. Since we intend to represent Toeplitz and Hankel operators as compressions of their symbols (like in the classical case) we use the term symbol in a slightly different way — nevertheless there is a one-to-one correspondence between symbols in our sense and those used by Sz.-Nagy and Foiaş. This makes it possible to present a unified theory for both types of operators.

Section 3 contains the definition and basic properties of Hankel opreators including a generalization of the Nehari theorem. The last two chapters are devoted to an analogy of analytic symbols and to a generalization of the Kronecker theorem.

1. Preliminaries

We start by recalling some of the properties of the minimal isometric dilation of a contraction. Let $\mathfrak{H}$ be a Hilbert space, $T \in \mathscr{B}(\mathfrak{H})$ be a contraction. We denote by U the minimal isometric dilation of T on the space $\mathfrak{R}^+$, i.e. an isometry U defined on a space $\mathfrak{R}^+ \supset \mathfrak{H}$ satisfying

$$T^n = P(\mathfrak{H})U^n | \mathfrak{H} \quad \text{for} \quad n = 0, 1, \ldots$$

and

$$\mathfrak{R}^+ = \bigvee_{k \geq 0} U^k \mathfrak{H}.$$

We shall denote by $P(\mathfrak{L})$ the orthogonal projection of $\mathfrak{R}^+$ onto a subspace $\mathfrak{L} \subset \mathfrak{R}^+$.

Any two minimal isometric dilations of a given contraction are unitarily equivalent.

We shall frequently use the following facts:

(1) $TP(\mathfrak{H}) = P(\mathfrak{H})U$;

(2) $U^*\mathfrak{H} \subset \mathfrak{H}$ and $U^*|\mathfrak{H} = T^*$;

(3) the subspace $\mathfrak{H}^\perp = \mathfrak{R}^+ \ominus \mathfrak{H}$ can be decomposed as follows

$$\mathfrak{H}^\perp = \mathfrak{L} \oplus U\mathfrak{L} \oplus U^2\mathfrak{L} \oplus \ldots,$$

where $\mathfrak{L} = ((U-T)\mathfrak{H})^-$;

(4) $U\mathfrak{H}^\perp \subset \mathfrak{H}^\perp$ and $U|\mathfrak{H}^\perp$ is a unilateral shift of multiplicity $\dim \mathfrak{L}$;

(5) the sequence $\{P(\mathfrak{H}^\perp)U^{*n}\}$ tends to zero in the strong operator topology;

(6) T^* is an isometry if and only if the minimal isometric dilation of T is a unitary operator;

(7) let W be a unitary operator on a Hilbert space $\mathfrak{G}$, let $\mathfrak{M} \subset \mathfrak{G}$ be a subspace invariant with respect to W; then the restriction of W^* to the W^* invariant subspace of $\mathfrak{G}$ generated by $\mathfrak{M}$ is the minimal isometric dilation of the operator $(W|\mathfrak{M})^*$.

The reader is referred to [2] for proofs of (1)—(4).

For lack of space the proofs of the remaining results in this section have to be left to the reader.

If S is an arbitrary isometry on a Hilbert space $\mathfrak{R}$ then the Wold decomposition applies. In other words, the space $\mathfrak{R}$ can be decomposed into a direct sum of two subspaces reducing with respect to S,

$$\mathfrak{R} = \Big(\bigcap_{n \geq 0} S^n\mathfrak{R}\Big) \oplus \big((\mathfrak{R} \ominus S\mathfrak{R}) \oplus (S\mathfrak{R} \ominus S^2\mathfrak{R}) \oplus \ldots\big)$$

so that the restriction of S to the first subspace is a unitary operator and the restriction to the second one is a unilateral shift.

Now, let $\mathfrak{R}$ be the reducing subspace in the Wold decomposition of the minimal isometric dilation U on $\mathfrak{K}^+$ on which U is unitary, i.e. $\mathfrak{R}=\bigcap_{n\geq 0} U^n\mathfrak{K}^+$. Then we have (see [2]):

(8) $UP(\mathfrak{R})=P(\mathfrak{R})U,\ U^*P(\mathfrak{R})=P(\mathfrak{R})U^*$;

(9) the sequence of projections $\{U^nU^{*n}\}_0^\infty$ is decreasing,
$P(\mathfrak{R})\leqq U^nU^{*n}$ for $n=0,1,\dots$
and
$P(\mathfrak{R})k=\lim_{n\to\infty} U^nU^{*n}k$ for all $k\in\mathfrak{K}^+$;

(10) $P(\mathfrak{R})h=\lim_{n\to\infty} U^nT^{*n}h$ for all $h\in\mathfrak{H}$.

There are two subspaces of the space $\mathfrak{R}$ which play an important role in our investigations, namely, $(P(\mathfrak{R})\mathfrak{H})^-$ and $\mathfrak{H}\cap\mathfrak{R}$. Denote by R the restriction of U onto the subspace $\mathfrak{R}$.

1.1. Lemma (see also [3]). *The operator U^* maps $P(\mathfrak{R})\mathfrak{H}$ into itself and $U^*|(P(\mathfrak{R})\mathfrak{H})^-$ is an isometry. The sequence of linear manifolds $\{R^nP(\mathfrak{R})\mathfrak{H}\}_{-\infty}^\infty$ is increasing and*

$$\mathfrak{R}=\Big(\bigcup_{n\geq 0} R^nP(\mathfrak{R})\mathfrak{H}\Big)^-.$$

If T is a contraction on a Hilbert space $\mathfrak{H}$ then $\mathfrak{H}$ can be uniquely decomposed into an orthogonal sum of two subspaces reducing T, $\mathfrak{H}=\mathfrak{H}_u\oplus\mathfrak{H}_s$ such that $T|\mathfrak{H}_u$ is unitary and $T|\mathfrak{H}_s$ is completely non-unitary. We have

$$\mathfrak{H}_u=\{h\in\mathfrak{H}:\ \|T^nh\|=\|T^{*n}h\|=\|h\|\ \text{for all}\ n\geqq 0\}.$$

See [2].

1.2. Lemma. *We have*

$$\mathfrak{H}\cap\mathfrak{R}=\mathfrak{H}\cap P(\mathfrak{R})\mathfrak{H}=\{h\in\mathfrak{H}:\ \|T^{*n}h\|=\|h\|\ \textit{for all}\ n\geqq 0\}=$$
$$=\{h\in\mathfrak{H}:\ T^nT^{*n}h=h\ \textit{for all}\ n\geqq 0\}.$$

The subspace $\mathfrak{H}\cap\mathfrak{R}$ is invariant with respect to U^* and $U^*|\mathfrak{H}\cap\mathfrak{R}$ is an isometry whose Wold decomposition has the form

$$\mathfrak{H}\cap\mathfrak{R}=\mathfrak{H}_u\oplus(\mathfrak{N}\oplus U^*\mathfrak{N}\oplus U^{*2}\mathfrak{N}\oplus\dots)$$

where $\mathfrak{N}=(\mathfrak{H}\cap\mathfrak{R})\ominus U^*(\mathfrak{H}\cap\mathfrak{R})$.

We close this section with two results of a different character which we shall use later. The first is a technical proposition based on the following observation. We have, for each complex number α,

$$U(1-\alpha T)-(1-\alpha U)T=U-T.$$

If $|\alpha|<1$ this relation implies

$$(1-\alpha U)^{-1}(U-T)(1-\alpha T)^{-1} = (1-\alpha U)^{-1}U|\mathfrak{H}-T(1-\alpha T)^{-1}.$$

1.3. Proposition. *Let T be a contraction on a Hilbert space with the minimal isometric dilation U. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be complex numbers of modulus less than 1. Then*

$$U^n(1-\alpha_1 U)^{-1}\ldots(1-\alpha_n U)^{-1}|\mathfrak{H}-T^n(1-\alpha_1 T)^{-1}\ldots(1-\alpha_n T)^{-1} =$$

$$= \sum_{k=1}^{n} U^{k-1}(1-\alpha_1 U)^{-1}\ldots(1-\alpha_k U)^{-1}(U-T)T^{n-k}(1-\alpha_k T)^{-1}\ldots(1-\alpha_n T)^{-1}.$$

1.4. Proposition. *Let $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{K}_1, \mathfrak{K}_2$ be Hilbert spaces, $X\in\mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_2)$, $A_1\in\mathscr{B}(\mathfrak{H}_1, \mathfrak{K}_1)$, $A_2\in\mathscr{B}(\mathfrak{H}_2, \mathfrak{K}_2)$. If $|(Xh_1, h_2)|\leqq\|A_1h_1\|\cdot\|A_2h_2\|$ for all $h_1\in\mathfrak{H}_1$, $h_2\in\mathfrak{H}_2$ then there exists a contraction operator $C:(\mathrm{Ran}\, A_1)^-\to(\mathrm{Ran}\, A_2)^-$ for which $X=A_2^*CA_1$.*

2. Toeplitz operators and their symbols

Consider two contractions $T_1\in\mathscr{B}(\mathfrak{H}_1)$, $T_2\in\mathscr{B}(\mathfrak{H}_2)$; denote by U_1 and U_2 their minimal isometric dilations acting on the spaces $\mathfrak{K}_1^+$, $\mathfrak{K}_2^+$ respectively. We denote by $\mathfrak{R}_1$ and $\mathfrak{R}_2$ the subspaces of $\mathfrak{K}_1^+$ and $\mathfrak{K}_2^+$ which reduce U_1 and U_2 to their unitary parts R_1 and R_2. We denote by $P(\mathfrak{Z})$ the orthogonal projection of $\mathfrak{K}_j^+$ onto a subspace $\mathfrak{Z}\subset\mathfrak{K}_j^+$.

2.1. Proposition. *Consider the set $\mathscr{S}(T_1, T_2)$ of all operators $Z\in\mathscr{B}(\mathfrak{R}_2, \mathfrak{R}_1)$ satisfying the condition*

$$ZR_2=R_1Z,$$

and the set $\mathscr{S}'(T_1, T_2)$ of all operators $Y\in\mathscr{B}(\mathfrak{K}_2^+, \mathfrak{K}_1^+)$ satisfying

$$Y = U_1YU_2^*.$$

If $Y\in\mathscr{B}(\mathfrak{K}_2^+, \mathfrak{K}_1^+)$ then the following four conditions are equivalent:

1° $Y\in\mathscr{S}'(T_1, T_2)$;

2° $YU_2 = U_1Y$ *and* $Y=YP(\mathfrak{R}_2)$;

3° $YU_2^*=U_1^*Y$ *and* $Y=P(\mathfrak{R}_1)Y$;

4° $Y=\lim U_1^nP(\mathfrak{H}_1)YP(\mathfrak{H}_2)U_2^{*n}$ *in the strong operator topology.*

Furthermore

5° *if* $Z\in\mathscr{S}(T_1, T_2)$ *then* $ZP(\mathfrak{R}_2)\in\mathscr{S}'(T_1, T_2)$;

6° *if* $Y\in\mathscr{S}'(T_1, T_2)$ *then* $Y|\mathfrak{R}_2\in\mathscr{S}(T_1, T_2)$.

Proof. If $Y \in \mathscr{B}(\mathfrak{K}_2^+, \mathfrak{K}_1^+)$ satisfies $Y = U_1 Y U_2^*$ then $YU_2 = U_1 Y$. Also, for each $x \in \mathfrak{K}_2^+$,

$$YP(\mathfrak{R}_2)x = \lim YU_2^n U_2^{*n} x =$$
$$= \lim U_1^n Y U_2^{*n} x = \lim Yx = Yx.$$

On the other hand, $YU_2 = U_1 Y$ and $YP(\mathfrak{R}_2) = Y$ implies

$$U_1 Y U_2^* = Y U_2 U_2^* = YP(\mathfrak{R}_2) U_2 U_2^* = Y$$

since $U_2 U_2^* \geqq P(\mathfrak{R}_2)$. This proves the equivalence of 1° and 2°.

$Y \in \mathscr{S}'(T_1, T_2)$ if and only if $Y^* \in \mathscr{S}'(T_2, T_1)$. The last inclusion is equivalent to $Y^* U_1 = U_2 Y^*$ and $Y^* = Y^* P(\mathfrak{R}_1)$. Taking adjoints we obtain the equivalence of 1° and 3°.

The implication 4°$\Rightarrow$1° is obvious. On the other hand, if $Y = U_1 Y U_2^*$ then, for $n \geqq 0$,

$$Y = U_1^n Y U_2^{*n} =$$
$$= U_1^n P(\mathfrak{H}_1) Y P(\mathfrak{H}_2) U_2^{*n} + U_1^n (1 - P(\mathfrak{H}_1)) Y U_2^{*n} +$$
$$+ U_1^n P(\mathfrak{H}_1) Y (1 - P(\mathfrak{H}_2)) U_2^{*n} =$$
$$= U_1^n P(\mathfrak{H}_1) Y P(\mathfrak{H}_2) U_2^{*n} + U_1^n (1 - P(\mathfrak{H}_1)) U_1^{*n} Y +$$
$$+ U_1^n P(\mathfrak{H}_1) Y (1 - P(\mathfrak{H}_2)) U_2^{*n}.$$

Both $(1 - P(\mathfrak{H}_1)) U_1^{*n}$ and $(1 - P(\mathfrak{H}_2)) U_2^{*n}$ tend to zero in the strong operator topology.

Now suppose $Z \in \mathscr{B}(\mathfrak{R}_2, \mathfrak{R}_1)$ satisfies $ZR_2 = R_1 Z$. Then $Y = ZP(\mathfrak{R}_2)$ satisfies $YU_2 = ZP(\mathfrak{R}_2) U_2 = ZU_2 P(\mathfrak{R}_2) = U_1 ZP(\mathfrak{R}_2) = U_1 Y$ and $YP(\mathfrak{R}_2) = Y$. It follows from 2° that $Y \in \mathscr{S}'(T_1, T_2)$.

If $Y \in \mathscr{S}'(T_1, T_2)$ we have, for each n and each $x \in \mathfrak{K}_2^+$, $Yx = U_1^n Y U_2^{*n} x \in U_1^n \mathfrak{K}_1^+$ so that the range of Y is contained in $\mathfrak{R}_1$. Since $Y = U_1 Y U_2^*$ we have $YU_2 = U_1 Y$ and, in view of the inclusion $Yx \in \mathfrak{R}_1$ for each x, this implies

$$(Y \mid \mathfrak{R}_2) R_2 = R_1 (Y \mid \mathfrak{R}_2)$$

as asserted. The proof is complete.

2.2. Remark. The correspondence between elements of sets $\mathscr{S}$ and $\mathscr{S}'$ described in 5° and 6° is contractive in both directions and so it is an isometric linear mapping.

2.3. Definition. An element of the set $\mathscr{S}'(T_1, T_2)$ will be called a symbol with respect to T_1, T_2.

2.4. Proposition. *Let* $Y = \mathscr{S}'(T_1, T_2)$ *be a symbol. Denote by*

$$A = P(\mathfrak{H}_1) Y \mid \mathfrak{H}_2, \quad B = P(\mathfrak{H}_1^\perp) Y \mid \mathfrak{H}_2.$$

Then

$$A = T_1 A T_2^*, \tag{12}$$

$$(U_1 | \mathfrak{H}_1^\perp)^* B = B T_2^*. \tag{13}$$

Moreover, there exists a positive K such that A satisfies the estimate

$$\|A h_2\| \leqq K \cdot \|P(\mathfrak{R}_2) h_2\| \tag{14}$$

for all $h_2 \in \mathfrak{H}_2$ and similarly, B satisfies

$$(B h_2, h_1^\perp) \leqq K \cdot \|P(\mathfrak{R}_2) h_2\| \cdot \|P(\mathfrak{R}_1) h_1^\perp\| \tag{15}$$

for $h_2 \in \mathfrak{H}_2$, $h_1^\perp \in \mathfrak{H}_1^\perp$.

Proof. Using the relation $T_1 P(\mathfrak{H}_1) = P(\mathfrak{H}_1) U_1$ we have, for $h_2 \in \mathfrak{H}_2$,

$$T_1 P(\mathfrak{H}_1) Y T_2^* h_2 = P(\mathfrak{H}_1) U_1 Y U_2^* h_2 = P(\mathfrak{H}_1) Y h_2$$

which proves (12). Similarly, using the inclusion $U_1^* \mathfrak{H}_1 \subset \mathfrak{H}_1$,

$$\begin{aligned} B T_2^* h_2 &= P(\mathfrak{H}_1^\perp) Y U_2^* h_2 = P(\mathfrak{H}_1^\perp) U_1^* Y h_2 = \\ &= P(\mathfrak{H}_1^\perp) U_1^* P(\mathfrak{H}_1^\perp) Y h_2 + P(\mathfrak{H}_1^\perp) U_1^* P(\mathfrak{H}_1) Y h_2 = \\ &= P(\mathfrak{H}_1^\perp) U_1^* P(\mathfrak{H}_1^\perp) Y h_2 = P(\mathfrak{H}_1^\perp) U_1^* B h_2 = \\ &= (U_1 | \mathfrak{H}_1^\perp)^* B h_2. \end{aligned}$$

The estimates (14) and (15) with $K = \|Y\|$ are immediate consequences of the relation

$$Y = Y P(\mathfrak{R}_2) = P(\mathfrak{R}_1) Y P(\mathfrak{R}_2).$$

It is interesting to observe that the estimate (14) is a consequence of (12). On the other hand, we shall see that condition (13) alone does not imply (15).

2.5. Remark. *If $A = T_1 A T_2^*$ then*

$$\|A h_2\| \leqq \|A\| \, \|P(\mathfrak{R}_2) h_2\|$$

for each $h_2 \in \mathfrak{H}_2$.

Proof. For each $h_2 \in \mathfrak{H}_2$ and each natural number n,

$$A h_2 = T_1^n A T_2^{*n} h_2$$

so that $\|A h_2\| \leqq \|A\| \|T_2^{*n} h_2\| = \|A\| \cdot \|U_2^n T_{2n}^* h_2\|$. Since $P(\mathfrak{R}_2) h_2 = \lim U_2^n T_2^{*n} h_2$ the assertion follows.

2.6. Example. Let $\mathfrak{H}_1, \mathfrak{H}_2$ be Hilbert spaces, $T_1 \in \mathscr{B}(\mathfrak{H}_1)$ be such that T_1^* is a nonunitary isometry. Then the minimal isometric dilation U_1 of T_1 is a unitary operator acting on the space $\mathfrak{K}_1^+$, $\mathfrak{R}_1 = \mathfrak{K}_1^+$ and the operator $V_1^* = (U_1 | \mathfrak{H}_1^\perp)^*$ has a nontrivial kernel.

The operator $T_2=0$ on $\mathfrak{H}_2$ is a contraction whose isometric dilation U_2 is a unilateral shift on the space $\mathfrak{K}_2^+$. The subspace $\mathfrak{R}_2$ reduces to $\{0\}$ and the operator $V_2^*=(U_2|\mathfrak{H}_2^\perp)^*$ again has a nontrivial kernel.

Take an arbitrary nonzero operator $X\in\mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_1^\perp)$ such that $\operatorname{Ran} X\subset\operatorname{Ker} V_1^*$. Then obviously $V_1^*X=0=XT_2^*$ and $\|X^*h_1^\perp\|\leqq\|X^*\|\|h_1^\perp\|=\|X^*\|\|P(\mathfrak{R}_1)h_1^\perp\|$ for all $h_1^\perp\in\mathfrak{H}_1^\perp$. Since $\mathfrak{R}_2=\{0\}$ the operator X does not satisfy $\|Xh_2\|\leqq k\cdot\|P(\mathfrak{R}_2)h_2\|$ for all $h_2\in\mathfrak{H}_2$ with any positive k.

Similarly, let Y be any nonzero operator from $\mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_2^\perp)$ which is zero on $(T_1^*\mathfrak{H}_1)^-$ and with values in $\operatorname{Ker} V_2^*$. Then $V_2^*Y=0=YT_1^*$, $\|Yh_1\|\leqq\|Y\|\|h_1\|=\|Y\|\cdot\|P(\mathfrak{R}_1)h_1\|$ for all $h_1\in\mathfrak{H}_1$, but Y^* does not satisfy $\|Y^*h_2^\perp\|\leqq k\|P(\mathfrak{R}_2)h_2^\perp\|$ on $\mathfrak{H}_2^\perp$ with any positive k.

2.7. Definition. Denote by

$$\mathscr{T}(T_1, T_2) = \{A\in\mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_1)\colon A=T_1AT_2^*\}.$$

Operators from the set $\mathscr{T}(T_1, T_2)$ will be called *Toeplitz operators with respect to* T_1, T_2.

Further, operators $B\in\mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_1^\perp)$ satisfying (13), i.e. $(U_1|\mathfrak{H}_1^\perp)^*B=BT_2^*$ and

$$\|(Bh_2, h_1^\perp)\| \leqq \gamma\|P(\mathfrak{R}_2)h_2\|\,\|P(\mathfrak{R}_1)h_1^\perp\|$$

for all $h_2\in\mathfrak{H}_2$, $h_1^\perp\in\mathfrak{H}_1^\perp$ and a suitable constant γ will be called *Hankel operators with respect to* T_1, T_2. Similarly, the family of all Hankel operators will be denoted by $\mathscr{H}(T_1, T_2)$.

2.8. Lemma. *Suppose* $X\in\mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_1)$ *satisfies the relation*

$$X = T_1XT_2^*.$$

Then there exists exactly one operator $\tilde{X}\colon \mathfrak{K}_2^+\to\mathfrak{H}_1$ *with the following three properties:*

1° $\tilde{X}=T_1\tilde{X}U_2^*$,

2° $X=\tilde{X}|\mathfrak{H}_2$,

3° $\|\tilde{X}\|=\|X\|$.

Conversely, if $\tilde{X}\in\mathscr{B}(\mathfrak{K}_2^+, \mathfrak{H}_1)$ *satisfies* 1° *and if* X *is defined by* 2° *then* $X=T_1XT_2^*$ *and* $\|X\|=\|\tilde{X}\|$.

Proof. Suppose first that $\tilde{X}$ satisfies 1° and X is defined by 2°. Then, for $h_2\in\mathfrak{H}_2$,

$$Xh_2 = \tilde{X}h_2 = T_1\tilde{X}U_2^*h_2 = T_1\tilde{X}T_2^*h_2 = T_1XT_2^*h_2.$$

It follows that $X=T_1XT_2^*$.

Further, given $n\geqq 0$, $h_2\in\mathfrak{H}_2$, we have

$$\tilde{X}U_2^nh_2 = T_1^n\tilde{X}U_2^{*n}U_2^nh_2 = T_1^n\tilde{X}h_2 = T_1^nXh_2.$$

This together with $\mathfrak{K}_2^+ = \operatorname{span}_{n\geq 0} U_2^n h_2$ proves that there is at most one $\tilde{X}$ satisfying 1° and 2° for a given X.

Now, suppose $X \in \mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_1)$ satisfies $X = T_1 X T_2^*$. To prove the existence of $\tilde{X}$ it would be sufficient to define it for finite sums of elements $U_2^k h$ $(k \geq 0, h \in \mathfrak{H}_2)$ which form a dense subset in $\mathfrak{K}_2^+$. Let $m \geq k$, $h \in \mathfrak{H}_2$, then

$$T_1^k X h = T_1^k (T_1^{m-k} X T_2^{*m-k}) h = T_1^m X U_2^{*m-k} h = T_1^m X U_2^{*m-k} (U_2^{*k} U_2^k) h =$$

$$= T_1^m X U_2^{*m} U_2^k h$$

and consequently,

$$\sum_0^m T_1^k X h_k = T_1^M X U_2^{*M} \Big(\sum_0^m U_2^k h_2\Big)$$

for each $M \geq m$ and $h_k \in \mathfrak{H}_2$. In particular,

$$\Big\|\sum_0^m T_1^k X h_k\Big\| \leq \|X\| \, \Big\|\sum_0^m U_2^k h_k\Big\|.$$

It follows that the operator $\tilde{X}$ defined on $\lim_{k\geq 0} U_2^k \mathfrak{H}_2$ by $\tilde{X} \sum_0^m U_2^k h_k = \sum_0^m T_1^k X h_k$ is well defined, $\tilde{X} h = Xh$ for $h \in \mathfrak{H}_2$ and $\|\tilde{X}\| \leq \|X\|$ so that $\|\tilde{X}\| = \|X\|$. Moreover,

$$T_1 \tilde{X} U_2^* \Big(\sum_0^m U_2^k h_k\Big) = T_1 \tilde{X} \sum_1^m U_2^{k-1} h_k + T_1 \tilde{X} U_2^* h_0 =$$

$$= T_1 \sum_1^m T_1^{k-1} X h_k + T_1 X T_2^* h_0 = \sum_1^m T_1^k X h_k + T_1 X T_2^* h_0 =$$

$$= \sum_1^m T_1^k X h_k + X h_0 = \tilde{X} \sum_0^m U_2^k h_k.$$

The proof is complete.

2.9. Remark. The preceding lemma can be reformulated in a dual version. Namely, if $X \in \mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_1)$ satisfies $X = T_1 X T_2^*$ then $X^* \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ satisfies $X^* = T_2 X^* T_1^*$. It follows that there exists exactly one operator $W^* \in \mathscr{B}(\mathfrak{K}_1^+, \mathfrak{H}_2)$ such that

$$W^* = T_2 W^* U_1^*, \quad X^* = W^* | \mathfrak{H}_1, \quad \|X^*\| = \|W^*\|,$$

or equivalently,

$$W \in \mathscr{B}(\mathfrak{H}_2, \mathfrak{K}_1^+), \quad W = U_1 W T_2^*,$$

$$X = P(\mathfrak{H}_1) W, \quad \|X\| = \|W\|.$$

2.10. Theorem. *Suppose* $X \in \mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_1)$ *satisfies*

$$X = T_1 X T_2^*.$$

Then there exists exactly one operator $Y \in \mathscr{B}(\mathfrak{K}_2^+, \mathfrak{K}_1^+)$ *with the following properties*

1° Y *is a symbol with respect to* T_1, T_2,
2° $X = P(\mathfrak{H}_1)Y|\mathfrak{H}_2$,
3° $\|X\| = \|Y\|$.

The operator Y will be called the symbol of X.

Proof. According to 2.8 there exists an $\tilde{X} \in \mathscr{B}(\mathfrak{K}_2^+, \mathfrak{H}_1)$ such that $\tilde{X} = T_1 \tilde{X} U_2^*$, $X = \tilde{X}|\mathfrak{H}_2$ and $\|\tilde{X}\| = \|X\|$. Again, according to 2.9 there exists a $Y \in \mathscr{B}(\mathfrak{K}_2^+, \mathfrak{K}_1^+)$ such that

$$Y = U_1 Y U_2^*, \quad P(\mathfrak{H}_1)Y|\mathfrak{H}_2 = \tilde{X}|\mathfrak{H}_2 = X$$

and

$$\|Y\| = \|\tilde{X}\| = \|X\|.$$

The rest of the proof is straightforward.

Proposition 2.1 and Theorem 2.10 show that there is a one-to-one correspondence between $\mathscr{S}(T_1, T_2)$, $\mathscr{S}'(T_1, T_2)$ and $\mathscr{T}(T_1, T_2)$. Summing up, we have the following

2.11. Theorem. *Let* β: $\mathscr{B}(\mathfrak{K}_2^+, \mathfrak{K}_1^+) \to \mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_1)$ *be defined by*

$$\beta Y = P(\mathfrak{H}_1)Y|\mathfrak{H}_2 \quad \textit{for} \quad Y \in \mathscr{B}(\mathfrak{K}_2^+, \mathfrak{K}_1^+).$$

Then β *maps* $\mathscr{S}'(T_1, T_2)$ *isometrically onto* $\mathscr{T}(T_1, T_2)$.

The inverse mapping α *of the restriction of* β *to* $\mathscr{S}'(T_1, T_2)$ *assigns to a Toeplitz operator* $X \in \mathscr{T}(T_1, T_2)$ *its symbol and*

$$\alpha X = \lim U_1^n X P(\mathfrak{H}_2) U_2^{*n}$$

in the strong operator topology.

Proof. Suppose X belongs to $\mathscr{T}(T_1, T_2)$ and that X is generated by a symbol $Y \in \mathscr{S}'(T_1, T_2)$ so that $X = P(\mathfrak{H}_1)Y|\mathfrak{H}_2$. Since $Y = \lim U_1^n P(\mathfrak{H}_1) Y P(\mathfrak{H}_2) U_2^{*n}$ and $P(\mathfrak{H}_1) Y P(\mathfrak{H}_2) = X P(\mathfrak{H}_2)$ we have $Y = \lim U_1^n X P(\mathfrak{H}_2) U_2^{*n}$.

3. Hankel operators

In this section we intend to develop an analogous theory for generalized Hankel operators. To obtain a symbol for operators of this type we shall apply Lemma 2.8 again, this time to a certain operator of Toeplitz type which we shall construct using the theorem on intertwining dilations; as a consequence of the nonuniqueness of the intertwining dilation a situation analogous to the classical case presents itself: a Hankel operator has more than one symbol in general.

The theory is based on the following lemma, a particular case of which is already contained in [5].

3.1. Lemma. *Let* $\mathfrak{M}_1, \mathfrak{M}_2$ *be two Hilbert spaces,* G_1, G_2 *isometries on* $\mathfrak{M}_1, \mathfrak{M}_2$ *respectively. Denote by* $W_i \in \mathscr{B}(\mathfrak{N}_i)$ *the minimal isometric dilation of* G_i^* *so that the* W_i *are unitary* $(i=1, 2)$.

Suppose $C \in \mathscr{B}(\mathfrak{M}_2, \mathfrak{M}_1)$ *satisfies the relation* $G_1^* C = C G_2$. *Then there exists an operator* $D\colon \mathfrak{N}_2 \to \mathfrak{N}_1$ *such that*

$$D = W_1^* D W_2^*, \quad \|D\| = \|C\|$$

and

$$C = P^{\mathfrak{N}_1}_{\mathfrak{M}_1} D \,|\, \mathfrak{M}_2.$$

Proof. The operator G_2 is its own minimal isometric dilation. By the theorem on intertwining dilations [2] there exists a $\tilde{D}\colon \mathfrak{M}_2 \to \mathfrak{N}_1$ such that $W_1 \tilde{D} = \tilde{D} G_2$, $P^{\mathfrak{N}_1}_{\mathfrak{M}_1} \tilde{D} = C$ and $\|\tilde{D}\| = \|C\|$. Since W_1 is unitary we may write $\tilde{D} = W_1^* \tilde{D} G_2 = W_1^* \tilde{D} (G_2^*)^*$.

Now apply Lemma 2.8 in the situation $T_1 = W_1^*$, $T_2 = G_2^*$, $\mathfrak{H}_1 = \mathfrak{N}_1$, $\mathfrak{H}_2 = \mathfrak{M}_2$. It follows that there exists a $D\colon \mathfrak{N}_2 \to \mathfrak{N}_1$ such that $D = W_1^* D W_2^*$, $\tilde{D} = D | \mathfrak{M}_2$ and $\|\tilde{D}\| = \|D\|$. Hence $C = P^{\mathfrak{N}_1}_{\mathfrak{M}_1} \tilde{D} = P^{\mathfrak{N}_1}_{\mathfrak{M}_1} D | \mathfrak{M}_2$ and $\|D\| = \|C\|$.

A linear transformation A from $\mathfrak{H}_2$ into $\mathfrak{H}_1^\perp$ is said to be $\mathfrak{R}$-bounded if there exists a constant α such that

$$|(Ah, k)| \leqq \alpha \|P(\mathfrak{R}_2) h\| \, \|P(\mathfrak{R}_1) k\|$$

for all $h \in \mathfrak{H}_2$ and all $k \in \mathfrak{H}_1^\perp$. The minimum of all α for which the above inequality holds will be called the $\mathfrak{R}$-norm of A and will be denoted by $\|A\|_{\mathfrak{R}}$. Clearly every $\mathfrak{R}$-bounded operator A is norm bounded and its norm does not exceed the $\mathfrak{R}$-norm.

3.2. Theorem. *Suppose* $X \in \mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_1^\perp)$ *satisfies*

$$V_1^* X = X T_2^*,$$

where V_1 *is the restriction of* U_1 *to* $\mathfrak{H}_1^\perp$ *and the domination condition*

$$|(X h_2, h_1^\perp)| \leqq \|X\|_{\mathfrak{R}} \|P(\mathfrak{R}_2) h_2\| \, \|P(\mathfrak{R}_1) h_1^\perp\|$$

holds for all $h_2 \in \mathfrak{H}_2$ *and* $h_1^\perp \in \mathfrak{H}_1^\perp$.

Then there exists an operator $Y \in \mathscr{B}(\mathfrak{K}_2^+, \mathfrak{K}_1^+)$ *with the following properties*

$$Y = U_1 Y U_2^*, \quad \|Y\| = \|X\|_{\mathfrak{R}}$$

and

$$X = P(\mathfrak{H}_1^\perp) Y | \mathfrak{H}_2.$$

Proof. Introduce the abbreviations $A_1 = P(\mathfrak{R}_1) | \mathfrak{H}_1^\perp$, $A_2 = P(\mathfrak{R}_2) | \mathfrak{H}_2$, $\mathfrak{M}_1 = (P(\mathfrak{R}_1)\mathfrak{H}_1^\perp)^-$, $\mathfrak{M}_2 = (P(\mathfrak{R}_2)\mathfrak{H}_2)^-$. According to Proposition 1.4 there exists an operator $C \in \mathscr{B}(\mathfrak{M}_2, \mathfrak{M}_1)$ such that $\|C\| = \|X\|_{\mathfrak{R}}$ and $X = A_1^* C A_2$. Thus $V_1^* A_1^* C A_2 =$

$=A_1^*CA_2T_2^*$. Consider first the product $V_1^*A_1^*$. We have $A_1V_1=P(\mathfrak{R}_1)U_1|\mathfrak{H}_1^\perp=$ $=(U_1|\mathfrak{M}_1)P(\mathfrak{R}_1)|\mathfrak{H}_1^\perp=(U_1|\mathfrak{M}|)A_1$ so that $V_1^*A_1^*=A_1^*(U_1|\mathfrak{M}_1)^*$. Furthermore, for $h_2\in\mathfrak{H}_2$,

$$A_2T_2^*h_2 = P(\mathfrak{R}_2)U_2^*h_2 = U_2^*P(\mathfrak{R}_2)h_2 = U_2^*A_2h_2.$$

Thus

$$A_1^*(U_1|\mathfrak{M}_1)^*CA_2h_2 = V_1^*A_1^*CA_2h_2 = A_1^*CA_2T_2^*h_2 = A_1^*CU_2^*A_2h_2.$$

Since A_1^* is injective on $\mathfrak{M}_1=(\operatorname{Ran} A_1)^-$ we have

$$(U_1|\mathfrak{M}_1)^*C = C(U_2^*|\mathfrak{M}_2).$$

The minimal isometric dilation W_1 of the coisometry $(U_1|\mathfrak{M}_1)^*$ is unitary. Since $\mathfrak{M}_2\subset\mathfrak{R}_2$ the operator $(U_2^*|\mathfrak{M}_2)$ is an isometry.

Now apply Lemma 3.1 with $G_1=U_1|\mathfrak{M}_1$, $G_2=U_2^*|\mathfrak{M}_2$. Since $G_2^*=P(\mathfrak{M}_2)U_2|\mathfrak{M}_2$, its minimal isometric dilation is U_2 on the smallest U_2 invariant subspace of $\mathfrak{K}_2^+$ containing $\mathfrak{M}_2$: this is $\mathfrak{R}_2$. Thus $W_2=U_2|\mathfrak{R}_2$, $\mathfrak{N}_2=\mathfrak{R}_2$.

Since $G_1^*=P(\mathfrak{M}_1)U_1^*|\mathfrak{M}_1$ we have $W_1=U_1^*$ on the smallest U_1^* invariant subspace of $\mathfrak{K}_1^+$ containing $P(\mathfrak{R}_1)\mathfrak{H}_1^\perp$: thus $\mathfrak{N}_1\subset\mathfrak{R}_1$ but the inclusion may be a strict one. By Lemma 3.1 there exists a D: $\mathfrak{N}_2\to\mathfrak{N}_1$ such that $D=U_1DU_2^*|\mathfrak{N}_2$, $\|D\|=\|C\|$ and

$$C = P^{\mathfrak{N}_1}_{\mathfrak{M}_1}D|\mathfrak{M}_2.$$

Finally, set $Y=DP(\mathfrak{R}_2)$. Then

$$Y = DP(\mathfrak{R}_2) = U_1DU_2^*P(\mathfrak{R}_2) = U_1YU_2^*,$$

$$\|Y\| = \|D\| = \|C\| = \|X\|_{\mathfrak{R}}$$

and

$$X = A_1^*CA_2 = A_1^*CP(\mathfrak{R}_2)|\mathfrak{H}_2 = A_1^*P^{\mathfrak{N}_1}_{\mathfrak{M}_1}DP(\mathfrak{R}_2)|\mathfrak{H}_2 = A_1^*P^{\mathfrak{N}_1}_{\mathfrak{M}_1}Y|\mathfrak{H}_2.$$

To complete the proof it suffices to show that $A_1^*P^{\mathfrak{N}_1}_{\mathfrak{M}_1}=P(\mathfrak{H}_1^\perp)|\mathfrak{N}_1$. Indeed, for $r_1\in\mathfrak{N}_1, h_1^\perp\in\mathfrak{H}_1^\perp$,

$$(A_1^*P^{\mathfrak{N}_1}_{\mathfrak{M}_1}r_1, h_1^\perp) = (P^{\mathfrak{N}_1}_{\mathfrak{M}_1}r_1, P(\mathfrak{R}_1)h_1^\perp) = (r_1, P(\mathfrak{R}_1)h_1^\perp) =$$
$$= (r_1, h_1^\perp) = (P(\mathfrak{H}_1^\perp)r_1, h_1^\perp).$$

Notation: Suppose $Y\in\mathscr{S}'(T_1, T_2)$. We shall denote by $T(Y)$ and $H(Y)$ the corresponding Toeplitz and Hankel operators, i.e.

$$T(Y) = P(\mathfrak{H}_1)Y|\mathfrak{H}_2$$

and

$$H(Y) = P(\mathfrak{H}_1^\perp)Y|\mathfrak{H}_2.$$

The well-known identity for products of Toeplitz operators extends to the abstract case without any change.

3.3. Proposition. *Let T_1, T_2, T_3 be contractions acting on spaces $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3$ respectively. If $Y \in \mathcal{S}'(T_1, T_2)$, $Z \in \mathcal{S}'(T_3, T_1)$ then $ZY \in \mathcal{S}'(T_3, T_2)$ and*

$$T(ZY) - T(Z)T(Y) = H(Z^*)^* H(Y).$$

Proof. Consider $Y \in \mathcal{B}(K_2^+, K_1^+)$, $Z \in \mathcal{B}(K_1^+, K_3^+)$ satisfying $Y = U_1 Y U_2^*$, $Z = U_3 Z U_1^*$. Then

$$ZY = U_3 Z U_1^* U_1 Y U_2^* = U_3 Z Y U_2^*$$

and

$$P(\mathfrak{H}_3) ZY | \mathfrak{H}_2 = P(\mathfrak{H}_3) Z P(\mathfrak{H}_1) Y | \mathfrak{H}_2 + P(\mathfrak{H}_3) Z P(\mathfrak{H}_1^\perp) Y | \mathfrak{H}_2 =$$
$$= T(Z)T(Y) + H(Z^*)^* H(Y).$$

3.4. Definition. The operators $Y \in \mathcal{S}'(T_1, T_2)$ for which the corresponding Hankel operator $H(Y)$ is zero will be called *analytic symbols*. Thus Y is analytic if and only if Y maps $\mathfrak{H}_2$ into $\mathfrak{H}_1$. The set of all analytic symbols with respect to T_1, T_2 will be denoted by $\mathcal{A}(T_1, T_2)$.

Obviously

$$\mathcal{H}(T_1, T_2) = \mathcal{T}(T_1, T_2) / \mathcal{A}(T_1, T_2)$$

in the sense of isomorphism of linear spaces.

The classical theorem of Z. Nehari may be formulated as follows. We denote by $\{e_j\}$ the natural basis of L^2 and consider a linear operator A defined on the algebraic linear span of the $\{e_j\}$ with nonnegative indices taking its values in H^2_-. Furthermore, we assume the existence of a sequence of complex numbers $a_0, a_1, \ldots$ such that

$$(Ae_k, e_j) = a_{k+j}$$

for $k \geqq 0$ and $j < 0$. Then the Nehari theorem asserts that the operator A is the Hankel operator corresponding to some $\varphi \in L^\infty$ if and only if A is bounded.

We intend to show that the Nehari theorem has an analogon in the general situation described in the preceding sections. In the abstract theory, however, the boundedness condition has to be replaced by a stronger one — this boundedness condition reduces to ordinary boundedness in the classical case but is different from it in general. It is only in the present generality that the role played by the spaces $\mathfrak{N}$ as well as their meaning for the theory manifests itself; since $\mathfrak{K}^+ = \mathfrak{N}$ in the scalar case, it is not so easy to see the essential features of the classical results which make the theory work.

Using the notion of $\mathfrak{N}$-boundedness it is possible to formulate the following extension of the Nehari theorem.

3.5. Theorem. *Suppose $\mathfrak{M} \subset \mathfrak{H}_2$ is such that the linear span $\mathfrak{H}_0$ of all elements of the form $T_2^{*k} m$, $k \geqq 0$, $m \in \mathfrak{M}$ is dense in $\mathfrak{H}_2$. Let X: $\mathfrak{H}_0 \to \mathfrak{H}_1^\perp$ be a linear trans-*

formation which satisfies

$$V_1^* Xh = XT_2^* h$$

for all $h \in \mathfrak{H}_0$.

Then the following assertions are equivalent:

1° X *is* $\mathfrak{R}$*-bounded;*

2° X *is a Hankel operator.*

Moreover, if X *satisfies* 1° *or* 2° *and* $X=H(Y)$ *with a* $Y \in \mathcal{S}'(T_1, T_2)$ *then*

$$\|H(Y)\|_{\mathfrak{R}} = \operatorname{dist}(Y, \mathcal{A}(T_1, T_2))$$

and the infimum is attained.

Proof. If 1° is satisfied then X can be regarded as an operator acting on the whole space $\mathfrak{H}_2$. Thus X is a Hankel operator and according to Theorem 3.2 there exists a symbol Y such that $X=H(Y)$ and $\|X\|_{\mathfrak{R}}=\|Y\|$. To complete the proof it is sufficient to observe that $H(Y+A)=H(Y)$ for all $A \in \mathcal{A}(T_1, T_2)$.

4. Symbols

One of the interesting questions to be asked in the context of the abstract theory is a more detailed description of the set of all symbols. We can only give partial results in this direction: we do give, however, a complete characterization of those pairs T_1, T_2, for which nonzero Toeplitz operators exist. This question is equivalent to that of the existence of non zero symbols and will be given in terms of the spaces $\mathfrak{R}_1$ and $\mathfrak{R}_2$, the unitary parts in the Wold decomposition of the minimal isometric dilations of T_1 and T_2. The answer is particularly interesting in the case $T_1=T_2=T$. The nonzero Toeplitz operators exist if and only if $\mathfrak{R} \neq \{0\}$. The situation is considerably more complicated in the case of analytic symbols. More delicate considerations are necessary this time; we show that it is possible to reformulate conditions for the existence of nontrivial analytic symbols in a form which may not be much easier to verify but which provides, in principle, a complete description of the set of all analytic symbols.

Consider now the particular case where $T_1=T_2$; it is interesting to characterize those contractions T for which the corresponding set of Toeplitz operators consists of the zero operator only. In other words, to characterize those contractions $T \in \mathcal{B}(\mathfrak{H})$ for which $X \in \mathcal{B}(\mathfrak{H})$ and $X=TXT^*$ implies $X=0$.

4.1. Proposition. *Let T be a contraction on a Hilbert space $\mathfrak{H}$. Then these are equivalent:*

1° *the only operator* X *satisfying* $X=TXT^*$ *is the zero operator;*

2° $\lim T^{*n}h=0$ *for each* $h \in \mathfrak{H}$;

3° $P(\mathfrak{R})\mathfrak{H}=0$;
4° $P(\mathfrak{H})\mathfrak{R}=0$;
5° $P(\mathfrak{H})P(\mathfrak{R})P(\mathfrak{H})=0$;
6° $\mathfrak{R}=0$.

Proof. Assume 1°. According to 2.1 the projection $P(\mathfrak{R})$ is a symbol so that $X=P(\mathfrak{H})P(\mathfrak{R})|\mathfrak{H}$ is a Toeplitz operator. Since $X=0$ we have also $P(\mathfrak{H})P(\mathfrak{R})P(\mathfrak{H})= =0$. Since $P(\mathfrak{H})P(\mathfrak{R})P(\mathfrak{H})=P(\mathfrak{H})P(\mathfrak{R})(P(\mathfrak{H})P(\mathfrak{R}))^*$ the condition 5° implies 4°. If 4° is satisfied we have $P(\mathfrak{R})P(\mathfrak{H})=0$ as well. Now assume 3°. According to Lemma 1.1 we have $\mathfrak{R}=(\bigcup_{n\geq 0} U^nP(\mathfrak{R})\mathfrak{H})^-$ so that $\mathfrak{R}=0$. The implication $6°\Rightarrow 2°$ follows from (10) and the implication $2°\Rightarrow 1°$ is obvious.

Let us remark that condition 5° appears implicitly in the paper of R. G. Douglas [1]. The ideas used in the proof of Theorem 3 in [1] may be used to describe existence conditions even in the case of operators Toeplitz with respect to possibly different T_1 and T_2. To this end it will be convenient to recall a definition.

Consider two unitary operators $U_1\in\mathscr{B}(\mathfrak{H}_1)$ and $U_2\in\mathscr{B}(\mathfrak{H}_2)$ with spectral measures E_1 and E_2 respectively. Following R. G. Douglas we shall say that the operators U_1 and U_2 are relatively singular if, for each $h_1\in\mathfrak{H}_1$ and $h_2\in\mathfrak{H}_2$, the measures $(E_1(\cdot)h_1, h_1)$ and $(E_2(\cdot)h_2, h_2)$ are mutually singular.

According to R. G. Douglas [1] the set of operators intertwining U_1 and U_2 is trivial if and only if U_1 and U_2 are relatively singular.

Using this notion it is possible to formulate conditions for the existence of Toeplitz operators.

4.2. Proposition. *The following assertions are equivalent:*
1° *the only operator $X\in\mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_1)$ satisfying $X=T_1XT_2^*$ is the zero operator;*
2° *either one of the subspaces $\mathfrak{R}_1, \mathfrak{R}_2$ is trivial or the unitary operators R_1 and R_2 are relatively singular.*

Proof. In view of what has been said above it suffices to observe that, according to 2.11 and Remark 2.2 condition 1° is satisfied if and only if the only operator intertwining R_1 and R_2 is the zero operator.

In the classical theory analytic Toeplitz operators may be characterized by the relation $XS=SX$. The corresponding relation $T_1^*X=XT_2^*$ does not guarantee, in general, that X is (T_1, T_2) Toeplitz; we list below some supplementary condition which, together with the above relation, make X Toeplitz in which case the corresponding symbol is analytic.

4.3. Proposition. *Suppose $X\in\mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_1)$ satisfies*

$$XT_2^* = T_1^*X. \tag{16}$$

Then the operator $P(\mathfrak{H}_1)P(\mathfrak{R}_1)X$ *belongs to* $\mathcal{T}(T_1, T_2)$ *and the following four conditions are equivalent:*

1° $X \in \mathcal{T}(T_1, T_2)$,

2° $X = P(\mathfrak{H}_1)P(\mathfrak{R}_1)X$,

3° $X = P(\mathfrak{H}_1 \cap \mathfrak{R}_1)X$,

4° Ran $X \subset \mathfrak{H}_1 \cap \mathfrak{R}_1$.

Moreover, if X satisfies (16) *and one of the conditions* 2°, 3°, 4° *then X is a Toeplitz operator whose symbol is analytic.*

On the other hand, if $Y \in \mathcal{S}'(T_1, T_2)$ *is analytic then the corresponding Toeplitz operator X satisfies* (16) *and the conditions* 2°, 3°, 4°.

Proof. Consider an $X \in \mathcal{B}(\mathfrak{H}_2, \mathfrak{H}_1)$ satisfying $XT_2^* = T_1^*X$. Then

$$T_1P(\mathfrak{H}_1)P(\mathfrak{R}_1)XT_2^* = P(\mathfrak{H}_1)U_1P(\mathfrak{R}_1)T_1^*X = P(\mathfrak{H}_1)U_1P(\mathfrak{R}_1)U_1^*X = = P(\mathfrak{H}_1)P(\mathfrak{R}_1)X$$

so that the operator $P(\mathfrak{H}_1)P(\mathfrak{R}_1)X$ is Toeplitz.

Now, assume (16) and 1°. Then, for $h_2 \in \mathfrak{H}_2$ and each natural number n,

$$Xh_2 = T_1^n XT_2^{*n}h_2 = T_1^n T_1^{*n}Xh_2 = P(\mathfrak{H}_1)U_1^n T_1^{*n}Xh_2 \xrightarrow[n\to\infty]{} P(\mathfrak{H}_1)P(\mathfrak{R}_1)Xh_2.$$

This proves the implication 1° ⇒ 2°.

If 2° is satisfied then

$$X = \big(P(\mathfrak{H}_1)P(\mathfrak{R}_1)\big)^n P(\mathfrak{H}_1)X \xrightarrow[n\to\infty]{} P(\mathfrak{H}_1 \cap \mathfrak{R}_1)X$$

so that 3° is satisfied. The equivalence of 3° and 4° is obvious as well as the implications 4° ⇒ 2° ⇒ 1°.

Again, assume 4° and (16). Let Y be a symbol corresponding to X. Then, according to 2.1

$$Yh_2 = \lim U_1^n XT_2^{*n}h_2 = \lim U_1^n T_1^{*n}Xh_2 = P(\mathfrak{R}_1)Xh_2 = Xh_2 \in \mathfrak{H}_1$$

for all $h_2 \in \mathfrak{H}_2$, so that Y is an analytic symbol.

It remains to show that the Toeplitz operator X corresponding to an analytic symbol Y satisfies (16). Since $X = Y|\mathfrak{H}_2$ we have Ran X = Ran $Y|\mathfrak{H}_2 \subset \mathfrak{H}_1 \cap \mathfrak{R}_1$ and

$$XT_2^*h_2 = XU_2^*h_2 = YU_2^*h_2 = U_1^*Yh_2 = T_1^*Xh_2$$

for $h_2 \in \mathfrak{H}_2$. The proof is complete.

The following example shows that the condition (16) alone does not imply 2°.

4.4. Example. Let us take $T_i = 0$ on a Hilbert space $\mathfrak{H}_i$ $(i = 1, 2)$. Then any $X \in \mathcal{B}(\mathfrak{H}_2, \mathfrak{H}_1)$ satisfies (16). Since both $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are trivial, the only Toeplitz operator with respect to T_1, T_2 is the zero operator.

Now let us turn to existence conditions for analytic symbols. To this end we introduce some notation. The space $\mathfrak{H}_1 \cap \mathfrak{R}_1$ is invariant with respect to U_1^* and the restriction of U_1^* to it is an isometry. Let us denote by $\mathfrak{M}_1$ and $\mathfrak{N}_1$ the unitary part and the wandering subspace respectively in the Wold decomposition of $U_1^* | \mathfrak{H}_1 \cap \mathfrak{R}_1$. Similarly, U_2^* maps the subspace $P(\mathfrak{R}_2)\mathfrak{H}_2^-$ into itself and the restriction of U_2^* to it is an isometry; we denote by $\mathfrak{M}_2$ and $\mathfrak{N}_2$ the analogous subspaces for the Wold decomposition of $U_2^* | P(\mathfrak{R}_2)\mathfrak{H}_2^-$. Using this notation, we intend to prove the following

4.5. Theorem. *Nontrivial analytic symbols with respect to T_1 and T_2 exist if and only if the following three conditions are satisfied:*

1° *$\mathfrak{M}_1$ and $\mathfrak{M}_2$ are both nontrivial and the unitary operators $U_1^* | \mathfrak{M}_1$ and $U_2^* | \mathfrak{M}_2$ are not relatively singular;*

2° *$\mathfrak{N}_1$ and $\mathfrak{N}_2$ are both nontrivial;*

3° *$\mathfrak{M}_1$ and $\mathfrak{N}_2$ are both nontrivial and the spectral measure E of $U_1^* | \mathfrak{M}_1$ is not concentrated on a set of Lebesgue measure zero.*

Proof. In view of the one-to-one correspondence between the set of all symbols and the set of all Toeplitz operators, the set $\mathscr{A}(T_1, T_2)$ will be nontrivial if and only if the corresponding set $\mathscr{T}^a(T_1, T_2)$ of Toeplitz operators is nontrivial. According to 4.3 this set consists of all $X \in \mathscr{B}(\mathfrak{H}_2, \mathfrak{H}_1 \cap \mathfrak{R}_1)$ satisfying $XT_2^* = T_1^* X$. We shall establish a one-to-one linear correspondence between elements of the set $\mathscr{T}^a(T_1, T_2)$ and certain triangular matrices. To simplify the notation we shall write $\mathfrak{L}_1 = (\mathfrak{H}_1 \cap \mathfrak{R}_1) \ominus \mathfrak{M}_1$, $\mathfrak{L}_2 = (P(\mathfrak{R}_2)\mathfrak{N}_2)^- \ominus \mathfrak{M}_2$. To each $X \in \mathscr{T}^a(T_1, T_2)$ we assign a matrix

$$\vartheta X = \begin{bmatrix} Y_{11} & Y_{12} \\ 0 & Y_{22} \end{bmatrix}$$

defined by the following relations

$$Y_{11} = P(\mathfrak{M}_1)Y | \mathfrak{M}_2, \quad Y_{12} = P(\mathfrak{M}_1)Y | \mathfrak{L}_2, \quad Y_{22} = P(\mathfrak{L}_1)Y | \mathfrak{L}_2,$$

where $Y \in \mathscr{S}'(T_1, T_2)$ is the symbol corresponding to X.

Now denote by $\mathscr{M}$ the set of all matrices of the form

$$\begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}$$

with $M_{11} \in \mathscr{B}(\mathfrak{M}_2, \mathfrak{M}_1)$, $M_{12} \in \mathscr{B}(\mathfrak{L}_2, \mathfrak{M}_1)$, $M_{22} \in \mathscr{B}(\mathfrak{L}_2, \mathfrak{L}_1)$ such that the following relations are satisfied

$$(U_1^* | \mathfrak{M}_1) M_{11} = M_{11} (U_2^* | \mathfrak{M}_2), \tag{17}$$

$$(U_1^* | \mathfrak{M}_1) M_{12} = M_{12} (U_2^* | \mathfrak{L}_2), \tag{18}$$

$$(U_1^* | \mathfrak{L}_1) M_{22} = M_{22} (U_2^* | \mathfrak{L}_2). \tag{19}$$

We intend to show that ϑ is an injective mapping of the set $\mathcal{T}^a(T_1, T_2)$ onto $\mathcal{M}$.

Let us consider an $X\in\mathcal{T}^a(T_1, T_2)$ with the corresponding symbol Y. Since $Y=YP(\mathfrak{R}_2)$ we have

$$YP(\mathfrak{R}_2)\mathfrak{H}_2 = Y\mathfrak{H}_2 = X\mathfrak{H}_2 \subset \mathfrak{H}_1\cap\mathfrak{R}_1.$$

Furthermore, the relation $U_1^*Y=YU_2^*$ implies that $\vartheta X\in\mathcal{M}$.

Consider now the operators $Z=P(\mathfrak{L}_1)Y|\mathfrak{M}_2$ and $S_1=U_1^*|\mathfrak{L}_1$. Using $U_1^*Y=$ $=YU_2^*$ again we have also $S_1Z=Z(U_2^*|\mathfrak{M}_2)$ so that $Z^*S_1^{*n}=(U_2^*|\mathfrak{M}_2)^{*n}Z^*$ for every natural number n. Given $m\in\mathfrak{M}_2$ we have

$$\|Z^*m\| = \|(U_2^*|\mathfrak{M}_2)^{*n}Z^*m\| =$$
$$= \|Z^*S_1^{*n}m\| \leqq \|Z^*\|\,\|S_1^{*n}m\| \to 0,$$

so that $Z^*=0$ and $Z=0$ as well. Thus, for each $h_2\in\mathfrak{H}_2$, X can be decomposed as follows

$$Xh_2 = YP(\mathfrak{R}_2)h_2 = Y_{11}P(\mathfrak{M}_2)P(\mathfrak{R}_2)h_2+$$
$$+Y_{12}P(\mathfrak{L}_2)P(\mathfrak{R}_2)h_2+Y_{22}P(\mathfrak{L}_2)P(\mathfrak{R}_2)h_2.$$

Hence $\vartheta X=0$ implies $X=0$ and ϑ is injective.

On the other hand, each $M\in\mathcal{M}$ defines an operator from $P(\mathfrak{R}_2)\mathfrak{H}_2^-$ into $\mathfrak{H}_1\cap\mathfrak{R}_1$. The relations (17), (18), (19) imply that

$$\begin{bmatrix} U_1^*|\mathfrak{M}_1 & 0 \\ 0 & U_1^*|\mathfrak{L}_1 \end{bmatrix}\begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}\begin{bmatrix} U_2^*|\mathfrak{M}_2 & 0 \\ 0 & U_2^*|\mathfrak{L}_2 \end{bmatrix},$$

so that $U_1^*M=MU_2^*|P(\mathfrak{R}_2)\mathfrak{H}_2^-$. If we set $Xh_2=MP(\mathfrak{R}_2)h_2$ for $h_2\in\mathfrak{H}_2$ then $X\in\mathcal{T}^a(T_1, T_2)$ and $\vartheta X=M$.

In view of the isomorphism between $\mathcal{T}^a(T_1, T_2)$ and $\mathcal{M}$ our problem is equivalent to that of describing conditions for $\mathcal{M}$ to be nontrivial. An element $M\in\mathcal{M}$ is nonzero if and only if at least one of its entries is nonzero.

If $M_{11}\neq 0$ then clearly both $\mathfrak{M}_1$ and $\mathfrak{M}_2$ must be nontrivial subspaces; at the same time M_{11} is a nonzero operator intertwining the unitary operators $U_1^*|\mathfrak{M}_1$ and $U_2^*|\mathfrak{M}_2$ and this yields condition 1°. On the other hand, if condition 1° is satisfied, there exists a nonzero operator $Z\in\mathcal{B}(\mathfrak{M}_2, \mathfrak{M}_1)$ for which

$$(U_1^*|\mathfrak{M}_1)Z = Z(U_2^*|\mathfrak{M}_2);$$

then

$$\begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}\in\mathcal{M}.$$

If $M_{22}\neq 0$ both its domain and range must be nontrivial, hence condition 2°. Conversely, if condition 2° holds, take a vector $g\in\mathfrak{R}_2$ and a vector $h\in\mathfrak{R}_1$. It is easy to see that $((U_2^pg, U_2^qg)=0$ for all integers $p\neq q$. The sequence U_1^kh, $k\in\mathbf{Z}$

possesses the same property so that it is possible to define an operator $Y\in\mathcal{B}(\mathfrak{K}_2^+, \mathfrak{K}_1^+)$ by the formula

$$Yx = \sum_{-\infty}^{\infty} (x, U_2^k g) U_1^k h;$$

clearly $Y=U_1YU_2^*$. If $k\geqq 1$ we have

$$(U_2^k g, \mathfrak{H}_2) = (g, U_2^{*k}\mathfrak{H}_2) = (g, P(\mathfrak{R}_2)U_2^{*k}\mathfrak{H}_2) =$$
$$= (g, U_2^{*k}P(\mathfrak{R}_2)\mathfrak{H}_2) = 0,$$

so that $Yx=\sum_{k\leqq 0}(x, U_2^k g)U_1^k h\subset\mathfrak{H}_1$ for $x\in\mathfrak{H}_2$. Thus $Y\in\mathcal{A}(T_1, T_2)$.

Consider the case $M_{12}\neq 0$; it follows that $\mathfrak{L}_2\neq\{0\}$. The operator $S_2=U_2^*|\mathfrak{L}_2$ is a unilateral shift so that the minimal isometric dilation of S_2^* is a unitary operator W with the following properties (see [2], Ch. 2, Theorem 6.4):

(i) the spectral measure $E_W(\cdot)$ of W is equivalent to Lebesgue measure,

(ii) for each nonzero $z\in\mathfrak{L}_2$, the measure $(E_W(\cdot)z, z)$ is equivalent to Lebesgue measure.

Since M_{12} satisfies (18) we have $M_{12}^*=S_2^*M_{12}^*(U_1^*|\mathfrak{M}_1)$ so that $M_{12}^*\in\mathcal{T}(S_2^*, (U_1^*|\mathfrak{M}_1)^*)$. The corresponding symbol G satisfies $G=WG(U_1^*|\mathfrak{M}_1)$ so that $G(U_1^*|\mathfrak{M}_1)=W^*G$ and this implies condition 3°.

On the other hand, if condition 3° holds there exists a nonzero vector $x\in\mathfrak{M}_1$ and a set M of positive Lebesgue measure for which $(E(M)x, x)>0$. Furthermore, if z is an arbitary vector in $\mathfrak{N}_2$ the measure $(E_W(\cdot)z, z)$ is equivalent to Lebesgue measure. It follows that there exists a nonzero operator K defined on $\mathfrak{M}_1$ which intertwines W and $(U_1^*|\mathfrak{M}_1)$, $K(U_1^*|\mathfrak{M}_1)=WK$. Hence $K=WK(U_1^*|\mathfrak{M}_1)^*$ so that $K\in\mathcal{S}'(S_2^*, U_1^*|\mathfrak{M}_1)$ and the corresponding Toeplitz operator $T(K)$ satisfies (18). Accordingly,

$$\begin{bmatrix}0 & T(K)\\ 0 & 0\end{bmatrix}\in\mathcal{M}.$$

The proof is complete.

4.6. Corollary. *If T_1 is completely nonunitary then $\mathcal{A}(T_1, T_2)$ is nontrivial if and only if both $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are nontrivial.*

Proof. It follows from Lemma 1.2 that $\mathfrak{M}_1=\mathfrak{H}_u(T_1)$. If T_1 is completely nonunitary then $\mathfrak{M}_1=\mathfrak{H}_u(T_1)=\{0\}$. It follows from the preceding theorem that $\mathcal{A}(T_1, T_2)$ is nontrivial if and only if 2° is satisfied.

Of course it is possible to reformulate the existence conditions for analytic symbols in a manner analogous to Proposition 4.2 The problem does not become any easier in this reformulation; nevertheless, it provides some more insight into the structure of these symbols.

4.7. Proposition. *Let* $T_1 \in \mathscr{B}(\mathfrak{H}_1)$, $T_2 \in \mathscr{B}(\mathfrak{H}_2)$ *be two contractions. Let us denote by* $\mathfrak{G}_1$ *the smallest* U_1 *reducing subspace containing* $\mathfrak{H}_1 \cap \mathfrak{R}_1$. *Then these are equivalent:*

1° $\mathscr{A}(T_1, T_2) = \{0\}$,

2° *the unitary operators* R_2 *and* $U_1|\mathfrak{G}_1$ *are relatively singular.*

Proof. If Y is an analytic symbol then $Y = U_1 Y U_2^*$ and Y maps $\mathfrak{R}_2 = \bigvee_{n \geq 0} P(\mathfrak{R}_2) U_2^n \mathfrak{H}_2$ into $\bigvee_{n \geq 0} U_1^n(\mathfrak{H}_1 \cap \mathfrak{R}_1)$ which is nothing more than the smallest reducing subspace $\mathfrak{G}_1$ for U_1 containing $\mathfrak{H}_1 \cap \mathfrak{R}_1$. Thus

$$Y|\mathfrak{R}_2 = (U_1|\mathfrak{G}_1)(Y|\mathfrak{R}_2)(U_2|\mathfrak{R}_2)^*.$$

5. Rational symbols and the theorem of Kronecker

It might seem that there is little hope that a reasonable extension to this generality of the algebraic notion of rational function would be possible. We intend to show in this section that such an extension does exist and that it may be used to obtain a generalization of the theorem of Kronecker.

We shall use an abbreviation: if p is a polynomial of degree n, we shall write p_1 for the polynomial defined by the relation $p_1(x) = x^n p(1/x)$.

5.1. Proposition. *Suppose* $Y \in \mathscr{A}(T_1, T_2)$ *and let* q *be a polynomial of degree* n *with roots of modulus less then* 1, $q(x) = (x - \alpha_1) \dots (x - \alpha_n)$.

Then $q(R_1^*)^{-1} Y$ *is a symbol and the corresponding Hankel operator may be expressed as follows*

$$H(q(R_1^*)^{-1} Y) =$$
$$= \sum_{k=1}^{n} U_1^{k-1}(1 - \alpha_1 U_1)^{-1} \dots (1 - \alpha_k U_1)^{-1}(U_1 - T_1) T_1^{n-k}(1 - \alpha_k T_1)^{-1} \dots (1 - \alpha_n T_1)^{-1} Y|\mathfrak{H}_2$$

or in an equivalent form

$$\sum_{k=1}^{n} (U_1^* - \alpha_{k+1}) \dots (U_1^* - \alpha_n) q_1(U_1)^{-1} U_1^{n-1}(U_1 - T_1) \cdot$$
$$\cdot T_1^{n-1} q_1(T_1)^{-1} Y (T_2^* - \alpha_{k-1}) \dots (T_2^* - \alpha_1).$$

Proof. Since $Y = P(\mathfrak{R}_1) Y$ we have

$$U_1 q(R_1^*)^{-1} Y U_2^* = R_1 q(R_1^*)^{-1} Y U_2^* = q(R_1^*)^{-1} R_1 Y U_2^* = q(R_1^*)^{-1} Y$$

so that $q(R_1^*)^{-1} Y$ is a symbol.

Since R_1 is unitary we have

$$q(R_1^*)^{-1} = \prod_1^n (R_1^{-1}-\alpha_j)^{-1} = R_1^n(1-\alpha_1 R_1)^{-1}\dots(1-\alpha_n R_1)^{-1} = R_1^n q_1(R_1)^{-1}$$

and

$$P(\mathfrak{H}_1^\perp)q(R_1^*)^{-1}Y|\mathfrak{H}_2 = P(\mathfrak{H}_1^\perp)R_1^n q_1(R_1)^{-1}Y|\mathfrak{H}_2 = P(\mathfrak{H}_1^\perp)U_1^n q_1(U_1)^{-1}Y|\mathfrak{H}_2 = U_1^n q_1(U_1)^{-1}Y|\mathfrak{H}_2 - P(\mathfrak{H}_1)U_1^n q_1(U_1)^{-1}Y|\mathfrak{H}_2 = U_1^n q_1(U_1)^{-1}Y|\mathfrak{H}_2 - T_1^n q_1(T_1)^{-1}Y|\mathfrak{H}_2.$$

Now, *it* suffices to apply Proposition 1.3.

5.2. Theorem. *Let H be a Hankel operator, $H\in\mathcal{H}(T_1, T_2)$. Then the following assertions are equivalent:*

$1°$ *the range of H is finite dimensional;*

$2°$ *there exists a polynomial q with roots of modulus less than* 1 *and an analytic symbol $Y\in\mathcal{A}(T_1, T_2)$ such that*

2.1° $$H = H\big(q(R_1^*)^{-1}Y\big)$$

and one of the two following equivalent conditions is satisfied

2.2° $d_j=\dim (U_1-T_1)T_1^j(1-\alpha_1 T_1)^{-1}\dots(1-\alpha_{j+1}T_1)^{-1}Y\mathfrak{H}_2<\infty$ *for* $j=0, \dots, n-1$

where $\alpha_1, \dots, \alpha_n$ are the roots of q, $\deg q=n$,

2.3° $\dim (U_1-T_1)T_1^{n-1}(1-\alpha_1 T_1)^{-1}\dots(1-\alpha_n T_1)^{-1}Y\mathfrak{H}_2<\infty.$

If these conditions are satisfied then

$$\dim \operatorname{Ran} H \leqq d_0+d_1+\dots+d_{n-1}.$$

Proof. The range of H is invariant with respect to V_1^*. If it is finite-dimensional there exists a polynomial q such that $q(V_1^*|\operatorname{Ran} H)=0$ so that $q(V_1^*)H=0$.

Since V_1 is a unilateral shift both V_1 and V_1^* have no eigenvalues on the unit circle. Hence we can assume that all the roots of q lie inside the unit disc. If Z is any symbol for H, i.e. $H=P(\mathfrak{H}_1^\perp)Z|\mathfrak{H}_2$, $Z\in\mathcal{S}'(T_1, T_2)$ we have

$$0 = q(V_1^*)H = Hq(T_2^*) = P(\mathfrak{H}_1^\perp)Zq(U_2^*)|\mathfrak{H}_2 = P(\mathfrak{H}_1^\perp)q(U_1^*)Z|\mathfrak{H}_2.$$

Hence $q(U^*)Z\mathfrak{H}_2\subset\mathfrak{H}_1$. Since the range of Z is contained in $\mathfrak{R}_1$ it follows that $Y=q(U_1^*)Z$ is an analytic symbol and $Y=q(R_1^*)Z$ whence $Z=q(R_1^*)^{-1}Y$ which proves 2.1°.

The range of the operator $P(\mathfrak{L}_1)q_1(U_1)H$ is also finite dimensional and it follows from Proposition 5.1 that it is equal to the space $(U_1-T_1)T_1^{n-1}q_1(T_1)^{-1}Y\mathfrak{H}_2$. Thus condition 2.3° is satisfied.

Let us show now, that, for any polynomial q with roots inside the unit disc and any analytic symbol Y, condition 2.3° implies 2.2°. Since Y is an analytic symbol we have $Y\mathfrak{H}_2 \subset \mathfrak{H}_1 \cap \mathfrak{N}_1$. On the other hand T_1^* is an isometry on $\mathfrak{H}_1 \cap \mathfrak{N}_1$ and $T_1 T_1^* h = h$ for all $h \in \mathfrak{H}_1 \cap \mathfrak{N}_1$. Using these facts we can write, for $|\alpha| < 1$,

$$Y\mathfrak{H}_2 = (1-\alpha T_1)^{-1}(1-\alpha T_1)Y\mathfrak{H}_2 =$$
$$= (1-\alpha T_1)^{-1}(T_1 T_1^* - \alpha T_1)Y\mathfrak{H}_2 = (1-\alpha T_1)^{-1}T_1(T_1^* - \alpha)Y\mathfrak{H}_2 =$$
$$= T_1(1-\alpha T_1)^{-1}Y(T_2^* - \alpha)\mathfrak{H}_2 \subseteqq T_1(1-\alpha T_1)^{-1}Y\mathfrak{H}_2.$$

It is easy to deduce from the just established relation that 2.3^{c} implies 2.2°.

Assume that 2.2° is satisfied for a polynomial q with roots inside the unit disc and some analytic symbol Y. Then, according to Proposition 5.1, the Hankel operator $H(q(R_1^*)^{-1}Y)$ is finite dimensional and $\dim \operatorname{Ran} H(q(R_1^*)^{-1}Y) \leqq d_0 + d_1 + \ldots + d_{n-1}$.

The proof is complete.

5.3. Corollary. *Suppose* $\dim \mathfrak{L}_1 < \infty$. *Given a symbol of the form*

$$q(R_1^*)^{-1}Y,$$

where q is a polynomial of degree n (with roots inside the unit disc) and $Y \in \mathcal{A}(T_1, T_2)$, condition 2.2° *is automatically satisfied and*

$$\dim \operatorname{Ran} H(q(R_1^*)^{-1}Y) \leqq n \dim \mathfrak{L}_1.$$

The corollary applies in particular in the case where $\dim \mathfrak{L}_1 = 1$. Furthermore, for classical Hankel operators it is more natural to view the symbol as an equivalence class in L^∞/H^∞ rather than as an individual function; in conformity with this point of view it seems natural to define a rational symbol as a class which contains a rational function, or equivalently, a class which contains a quotient h/q, $h \in H^\infty$, q a polynomial. In view of this it is not unnatural to use the name rational symbol for operators of the form $q(R_1^*)^{-1}Y$, Y analytic.

Theorem 5.2 appears thus as an extension of the well-known theorem of Kronecker. It is natural to ask whether the assumption 2.2° in Theorem 5.2 is essential for the validity of the generalized Kronecker theorem. We limit ourselves to stating that there exist examples which show that ranges of Hankel operators with rational symbols may be both finite and infinite dimensional if $\dim \mathfrak{L}_1$ is infinite.

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