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Commutative GW^*-algebras

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GW^*-algebras (i.e. generalized W^*-algebras) were introduced in [2]. In this paper the structure and the spectral properties of commutative GW^*-algebras will be examined in detail.

I. Preliminaries

Here we give a short summary of our former results concerning GW^*-algebras.

The vector space of the linear forms on the $*$-algebra A will be denoted by A^* and the weak $\sigma(A^*, A)$ topology relates to the canonical duality between A^* and A.

If A is a unital $*$-algebra (whose unit is denoted by $\mathbf{1}$ throughout this paper) and P is a set of positive linear forms on A then the set $\{f\in P \mid f(\mathbf{1})\leqq 1\}$ will be denoted by the symbol $P(1)$. Further, assuming that $P(1)$ is non-void and bounded in the $\sigma(A^*, A)$ topology, $\|\cdot\|_P$ denotes the mapping from A into $\mathbf{R}_+$ defined by

$$\|x\|_P := \sup_{f\in P(1)} \sqrt{f(x^*x)}$$

for all $x\in A$. It is obvious that $\|\cdot\|_P$ is a seminorm on A; the dual seminorm is denoted by $\|\cdot\|'_P$.

If S is a subset of A^* then the linear subspace of A^* spanned by S and the convex hull of S is denoted by $\mathrm{sp}\,(S)$ and $\mathrm{co}\,(S)$, respectively, while the $\sigma(A^*, A)$-closed linear subspace of A^* spanned by S and the $\sigma(A^*, A)$-closed convex hull of S is denoted by $\widetilde{\mathrm{sp}}(S)$ and $\widetilde{\mathrm{co}}(S)$, respectively. If the elements of S are $\|\cdot\|_P$-continuous forms (where P is a set of positive linear forms on A such that $P(1)$ is non void and $\sigma(A^*, A)$-bounded) then the $\|\cdot\|'_P$-closed linear subspace of A^* spanned by S and the $\|\cdot\|'_P$-closed convex hull of S is denoted by $\overline{\mathrm{sp}}\,(S)$ and $\overline{\mathrm{co}}\,(S)$, respectively, provided there is no danger of confusion as for P.

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If f is a linear form on the $*$-algebra A then for every $x\in A$ we define the linear forms $x\cdot f$ and $f\cdot x$ on A as the mappings $y\mapsto f(xy)$ and $y\mapsto f(yx)$, respectively. If $f\in A^*$ and $x, y\in A$ then $x\cdot f\cdot y$ stands for $(x\cdot f)\cdot y$.

Definition. The pair (A, P) is called a *weak GW^*-algebra* if A is a unital $*$-algebra and P is a separating set of positive linear forms on A satisfying:

(I) $P(1)$ is non-void and $\sigma(A^*, A)$-bounded;

(II_w) $\mathbf{R}_+P\subset P$ and $x^*\cdot P\cdot x\subset\widetilde{\overline{\text{co}}}(P)$ for all $x\in A$;

(III) $x\cdot P\subset\overline{\text{sp}}(P)$ for all $x\in A$;

(IV) A is sequentially complete with respect to the uniform structure defined by the $\sigma(A, \text{sp}\,(P))$ topology.

The pair (A, P) is called a *GW^*-algebra* if it is a weak GW^*-algebra and instead of (II_w) satisfies the more restrictive condition:

(II) $\mathbf{R}_+P\subset P$ and $x^*\cdot P\cdot x\subset\overline{\text{co}}\,(P)$ for all $x\in A$.

Finally, the pair (A, P) is referred to as a *complete GW^*-algebra* if it satisfies:

(IV_s) A is quasi complete with respect to the uniform structure defined by the $\sigma(A, \text{sp}\,(P))$ topology.

The most important elementary facts concerning weak GW^*-algebras are the following. If (A, P) is a weak GW^*-algebra then:

— A is a C^*-algebra whose C^*-norm coincides with $\|\cdot\|_P$, that is why we refer to $\|\cdot\|_P$ as the C^*-norm of A (cf. [1] and [2]);

— the $\sigma(A, \text{sp}\,(P))$ and $\sigma(A, \overline{\text{sp}}\,(P))$ topologies coincide in every C^*-norm bounded subset of A (cf. [1] Lemma 1);

— the multiplication of A is C^*-norm boundedly left and right continuous in the $\sigma(A, \text{sp}\,(P))$ topology (cf. [1] Lemma 2);

— the involution of A is proper and continuous in the $\sigma(A, \text{sp}\,(P))$ topology;

— the set of projections (i.e. self-adjoint idempotent elements) of A, equipped with the natural ordering: $g\leqq h\Leftrightarrow g=hg$ and the orthocomplementation: $e^\perp:=\mathbf{1}-e$, is a σ-complete orthomodular lattice admitting a separating set of σ-additive states (cf. [2] Theorem 1);

— the partial isometries are countably summable in A and, consequently, the equivalence of projections is countably additive in A (cf. [2] Proposition 2).

Here we deduce an important auxiliary result for general (not necessarily commutative) weak GW^*-algebras.

Proposition 1. *Let (A, P) be a weak GW^*-algebra. Then the order in A defined as $x\leqq y$ iff $f(y-x)\in\mathbf{R}_+$ $(f\in P)$ coincides with the algebraic order of the C^*-algebra A.*

Proof. Since the elements of P are positive linear forms on A, we have obviously $x\geqq 0$ with respect to the order defined by P, if $x\geqq 0$ in the C^*-algebra A.

Conversely, suppose that $x \geqq 0$ with respect to the order defined by P. Since the set of positive linear forms f on A satisfying $f(x) \in \mathbf{R}_+$ is $\sigma(A^*, A)$-closed, we have $f(x) \geqq 0$ for every $f \in \widetilde{\overline{co}}(P)$. Since $f(x) \in \mathbf{R}_+$ $(f \in P)$, we have $f(x^*) = \overline{f(x)} = f(x)$, hence $x = x^*$ since P separates the points of A. We know that A is a C^*-algebra thus we may write $x = x^+ - x^-$, where x^+ and x^- denotes the positive and negative part of the self-adjoint element x, respectively. Then the positive square root $\sqrt{x^-}$ of x^- exists in A and it is well known that the set $\{\sqrt{x^-}, x^+, x^-\}$ is commutative; moreover, $x^+x^- = x^-x^+ = 0$. Fixed a linear form f in P, we have $(\sqrt{x^-}) \cdot f \cdot (\sqrt{x^-}) \in \widetilde{\overline{co}}(P)$ thus

$$0 \leqq ((\sqrt{x^-}).f.(\sqrt{x^-}))(x) = f(\sqrt{x^-}(x^+ - x^-)\sqrt{x^-}) =$$
$$= f(x^- x^+ - (x^-)^2) = -f((x^-)^2) \leqq 0,$$

i.e. $f((x^-)^* x^-) = 0$ $(f \in P)$. Since P separates the points of A and the involution of A is proper, it follows that $x^- = 0$ thus $x = x^+$ is a positive element in the C^*-algebra A.

II. A type of commutative GW*-algebras

If $\mathscr{B}$ is a σ-algebra of subsets of the set T then $\mathscr{F}_{\mathbf{C}}^b(T, \mathscr{B})$ will denote the set of bounded complex valued $\mathscr{B} - \mathscr{B}(\mathbf{C})$ measurable functions defined on T. The set $\mathscr{F}_{\mathbf{C}}^b(T, \mathscr{B})$ will always be thought of equipped with the pointwise defined algebraic structure and the sup-norm on T (denoted by $|||\cdot|||_T$), thus $\mathscr{F}_{\mathbf{C}}^b(T, \mathscr{B})$ will be regarded as a commutative unital C^*-algebra.

It is known that given a σ-algebra of subsets of the set T and a finitely additive mapping $\Theta\colon \mathscr{B} \to \mathbf{C}$, the following statements are equivalent:

— Θ is bounded, i.e. $\sup_{E \in \mathscr{B}} |\Theta(E)| < +\infty$;

— there is a unique continuous linear form $\hat{\Theta}$ on $\mathscr{F}_{\mathbf{C}}^b(T, \mathscr{B})$ (called the integral on $\mathscr{F}_{\mathbf{C}}^b(T, \mathscr{B})$ defined by Θ) such that $\hat{\Theta}(\chi_E) = \Theta(E)$ for all $E \in \mathscr{B}$.

Moreover, Θ is σ-additive if and only if the integral $\hat{\Theta}$ defined by Θ satisfies the condition:

(L) For every uniformly bounded sequence $(\varphi_n)_{n \in \mathbf{N}}$ of functions in $\mathscr{F}_{\mathbf{C}}^b(T, \mathscr{B})$, if $\varphi_n \to 0$ pointwise on T then $\hat{\Theta}(\varphi_n) \to 0$.

Lemma. *Let $\mathscr{B}$ be a σ-algebra of subsets of the set T and P the set of integrals on $\mathscr{F}_{\mathbf{C}}^b(T, \mathscr{B})$ defined by positive σ-additive set functions on $\mathscr{B}$. Then P is a separating set of positive linear forms on the unital $*$-algebra $A := \mathscr{F}_{\mathbf{C}}^b(T, \mathscr{B})$, P satisfies* (I) *and* sp (P) *is a $\|\cdot\|_P'$-closed set.*

Proof. Since $\{\hat{\delta}_t \mid t \in T\} \subset P$, the set P separates the points of A. On the other hand, $P(1) = \{\hat{\mu} \mid \mu\colon \mathscr{B} \to \mathbf{R}_+$ σ-additive and $\mu(T) = \hat{\mu}(1) \leqq 1\}$, thus for every $\varphi \in A$

and $\hat{\mu} \in P$ we have the inequality $|\hat{\mu}(\varphi)| \leqq \mu(T)|||\varphi|||_T$ showing that $P(1)$ is $\sigma(A^*, A)$-bounded and non void.

Now we prove that $\|\cdot\|_P = |||\cdot|||_T$. Indeed, if $\varphi \in A$ then

$$\|\varphi\|_P := \sup_{\hat{\mu} \in P(1)} \sqrt{\hat{\mu}(\varphi^* \varphi)} = \sup_{\hat{\mu} \in P(1)} \sqrt{\hat{\mu}(|\varphi|^2)} \leqq \sup_{\hat{\mu} \in P(1)} \sqrt{\mu(T)}\, |||\varphi|||_T \leqq |||\varphi|||_T,$$

i.e. $\|\cdot\|_P \leqq |||\cdot|||_T$. Conversely, if $\varphi \in A$ and $c < |||\varphi|||_T$ then there is a point t in T such that $c < |\varphi(t)| = \sqrt{\hat{\delta}_t(\varphi^* \varphi)} \leqq \|\varphi\|_P$, i.e. $|||\cdot|||_T \leqq \|\cdot\|_P$.

Let $\Theta \in \overline{\mathrm{sp}}\,(P)$ and choose a sequence $(\Theta_n)_{n \in \mathbf{N}}$ in $\mathrm{sp}\,(P)$ with the property $\|\Theta_n - \Theta\|'_P \to 0$. We have to show that $\Theta \in \mathrm{sp}\,(P)$. With regard to our former considerations, it sufficies to prove that for every uniformly bounded sequence $(\varphi_n)_{n \in \mathbf{N}}$ in A, if $\varphi_n \to 0$ pointwise on T then $\Theta(\varphi_n) \to 0$. If $n, m \in \mathbf{N}$ then

$$|\Theta(\varphi_m)| \leqq |\Theta(\varphi_m) - \Theta_n(\varphi_m)| + |\Theta_n(\varphi_m)| \leqq \|\Theta - \Theta_n\|'_P\, |||\varphi_m|||_T + |\Theta_n(\varphi_m)|.$$

If $\varepsilon > 0$ is arbitrary then there is a number N_0 in $\mathbf{N}$ such that $\|\Theta - \Theta_{N_0}\|'_P \leqq \varepsilon/2(M+1)$ where $M := \sup_{m \in \mathbf{N}} |||\varphi_m|||_T$. Since $\Theta_{N_0} \in \mathrm{sp}\,(P)$ we have $\Theta_{N_0}(\varphi_m) \to 0$ $(m \to +\infty)$ thus there is a number N in $\mathbf{N}$ with the property that $|\Theta_{N_0}(\varphi_m)| \leqq \varepsilon/2$ for $m \in \mathbf{N}$, $m \geqq N$. Then the above inequality implies that $|\Theta(\varphi_m)| \leqq \varepsilon$ for $m \in \mathbf{N}$, $m \geqq N$, i.e. $\Theta(\varphi_m) \to 0$.

Theorem 1. *Let $\mathscr{B}$ be a σ-algebra of subsets of the set T, $A := \mathscr{F}^b_{\mathbf{C}}(T, \mathscr{B})$ and P the set of integrals on A defined by positive σ-additive set functions on $\mathscr{B}$. Then (A, P) is a commutative GW^*-algebra.*

Proof. With regard to our Lemma we have only to prove that the pair (A, P) satisfies (II), (III) and (IV). If $\varphi \in A$ and $\hat{\mu} \in P$ then $\varphi^* \cdot \hat{\mu} \cdot \varphi = |\varphi|^2 \mu$ where $|\varphi|^2 \mu$ is the positive σ-additive set function on $\mathscr{B}$ defined as: $E \mapsto \hat{\mu}(|\varphi|^2 \chi_E)$, thus $\varphi^* \cdot \hat{\mu} \cdot \varphi \in P$ and, consequently, $\varphi \cdot \hat{\mu} \in P - P + iP - iP \subset \mathrm{sp}\,(P)$, i.e. (A, P) verifies (II) and (III).

In order to prove (IV), let $(\varphi_n)_{n \in \mathbf{N}}$ be a sequence in A such that $(\hat{\mu}(\varphi_n))_{n \in \mathbf{N}}$ is convergent for every $\hat{\mu} \in P$. Since $\hat{\delta}_t \in P$ $(t \in T)$, there is a unique function $\varphi \colon T \to \mathbf{C}$ with the property that $\varphi_n \to \varphi$ pointwise on T. From this we infer that φ is necessarily $\mathscr{B} - \mathscr{B}(\mathbf{C})$ measurable. We intend to show that $\varphi \in A$ and $\varphi_n \to \varphi$ in the $\sigma(A, \mathrm{sp}\,(P))$ topology. In order to prove this we first define for all $n \in \mathbf{N}$ the linear form $\tilde{\varphi}_n \colon \mathrm{sp}\,(P) \to \mathbf{C}$; $\Theta \mapsto \Theta(\varphi_n)$. On account of our Lemma, $\mathrm{sp}\,(P)$ will be considered a Banach space whose norm equals $\|\cdot\|'_P$. Then $\tilde{\varphi}_n$ is a continuous linear form on the Banach space $\mathrm{sp}\,(P)$ for every $n \in \mathbf{N}$ and, by our assumption, the sequence $(\tilde{\varphi}_n)_{n \in \mathbf{N}}$ is pointwise convergent in $\mathrm{sp}\,(P)$. Consequently, the theorem of Banach—Steinhaus implies that $\sup_{n \in \mathbf{N}} \|\tilde{\varphi}_n\| < +\infty$. If $n \in \mathbf{N}$ and $c < |||\varphi_n|||_T$ then there is a point t in T such that $c < |\varphi_n(t)| = |\tilde{\varphi}_n(\hat{\delta}_t)| \leqq \|\hat{\delta}_t\|'_P \|\tilde{\varphi}_n\| = \|\tilde{\varphi}_n\|$, since $\|\hat{\delta}_t\|'_P = 1$ holds obviously, thus $|||\varphi_n|||_T \leqq \|\tilde{\varphi}_n\|$ showing that the sequence $(\varphi_n)_{n\in\mathbf{N}}$ is uniformly bounded in T. From this we obtain that the mapping φ is bounded, i.e. $\varphi \in A$.

Finally, if $\hat{\mu} \in P$ then the theorem of Lebesgue applied to the measure μ and the uniformly bounded, pointwise convergent sequence $(\varphi_n)_{n\in\mathbf{N}}$ result in $\hat{\mu}(\varphi_n) \to \hat{\mu}(\varphi)$, i.e. $\varphi_n \to \varphi$ in the $\sigma(A, \operatorname{sp}(P))$ topology.

This theorem provides a great deal of commutative GW^*-algebras that are not $*$-isomorphic to any W^*-algebra.

III. On the Gelfand representation of commutative GW*-algebras

If T is a compact Hausdorff space then $\mathscr{C}_\mathbf{C}(T)$ and $\mathscr{M}_\mathbf{C}(T)$ will denote the vector space of complex continuous functions defined on T and the vector space of complex Radon measures on T, respectively. Then $\mathscr{C}_+(T)$ and $\mathscr{M}_+(T)$ denote the convex cone of positive elements in $\mathscr{C}_\mathbf{C}(T)$ and $\mathscr{M}_\mathbf{C}(T)$, respectively. The complex vector space $\mathscr{C}_\mathbf{C}(T)$ will always be thought of equipped with the pointwise defined multiplication and conjugation, i.e. $\mathscr{C}_\mathbf{C}(T)$ will be considered a commutative unital $*$-algebra. It is well known that $\mathscr{C}_\mathbf{C}(T)$ is a C^*-algebra whose C^*-norm equals the sup-norm $|||\cdot|||_T$ on T.

Given a commutative unital C^*-algebra A, the celebrated representation theorem of Gelfand and Naimark assures that A and $\mathscr{C}_\mathbf{C}(X(A))$ are isometrically $*$-isomorphic C^*-algebras, where $X(A)$ denotes the compact Hausdorff space whose underlying set is the set of non zero multiplicative linear forms on A and whose topology is the well known Gelfand topology (cf. [3] ch. I, §6, Theorem 1). The Gelfand isomorphism between A and $\mathscr{C}_\mathbf{C}(X(A))$ is denoted usually by $\mathscr{G}_A$; we have $(\mathscr{G}_A(x))(\chi) = \chi(x)$ for all $x \in A$ and $\chi \in X(A)$.

In this section the structure of the compact Hausdorff space $X(A)$ will be examined in the case when (A, P) is a commutative GW^*-algebra.

Proposition 2. *Let T be a compact Hausdorff space, $P \subset \mathscr{M}_+(T)$ and suppose that $(\mathscr{C}_\mathbf{C}(T), P)$ is a weak GW^*-algebra. Then*

(i) $T = (\bigcup_{\mu\in P} \operatorname{Supp} \mu)^-$ *and* $\sup_{\mu\in P} \mu(G) > 0$ *for every non-void open subset G of T.*

(ii) *The interior of a closed G_δ-set in T is closed.*

(iii) *If F is a closed G_δ-set in T and there is a measure μ in P such that $\mu(F) > 0$ then the interior $\mathring{F}$ of F is non-void, i.e. F is not nowhere dense in T.*

Proof. (i) Let G be a non-void open subset of T. Then there is a function $\varphi \in \mathscr{C}_+(T)$ such that $0 \leqq \varphi \leqq 1$, $\operatorname{Supp} \varphi \subset G$ and $\varphi \neq 0$. Since P is a separating set, there exists a measure μ in P with the property $\mu(\varphi) > 0$. Then we have $\mu(G) \geqq$

$\geqq\mu(\varphi)>0$. This proves the second part of (i) and the first part of our assertion is an easy consequence of the second part.

(ii) Let F be a closed G_δ-set in T. Then there is a sequence of functions $(\varphi_n)_{n\in\mathbf{N}}$ in $\mathscr{C}_+(T)$ such that $\varphi_n\geqq\varphi_{n+1}$ $(n\in\mathbf{N})$ and $\varphi_n\to\chi_F$ pointwise on T. If $\mu\in P$ then $(\mu(\varphi_n))_{n\in\mathbf{N}}$ is a decreasing sequence of positive real numbers thus the sequentially completeness of $\mathscr{C}_{\mathbf{C}}(T)$ in the $\sigma(\mathscr{C}_{\mathbf{C}}(T), \operatorname{sp}(P))$ topology now gives the existence of a function φ in $\mathscr{C}_{\mathbf{C}}(T)$ such that $\mu(\varphi_n)\to\mu(\varphi)$ for all $\mu\in P$. Since $\mu\in P$ implies $\mu(\varphi)\geqq 0$ and $\mu(\varphi_n)\geqq\mu(\varphi)$ $(n\in\mathbf{N})$, by Proposition 1 we obtain that $\varphi_n\geqq\varphi\geqq 0$ $(n\in\mathbf{N})$. From this we conclude that $\varphi\leqq\chi_F$. If $\varphi'\in\mathscr{C}_+(T)$ and $\varphi'\leqq\chi_F$ then $\varphi'\leqq\varphi_n$ $(n\in\mathbf{N})$ thus $\mu(\varphi')\leqq\mu(\varphi_n)$ and $\mu(\varphi')\leqq\lim_n\mu(\varphi_n)=\mu(\varphi)$ for every $\mu\in P$, i.e. applying again Proposition 1, we find that $\varphi'\leqq\varphi$. This means that

$$\varphi = \sup\{\varphi'\colon \varphi'\in\mathscr{C}_+(T), \varphi'\leqq\chi_F\}. \tag{1}$$

If $n\in\mathbf{N}$ then $\inf(n\varphi, 1)\leqq\chi_F$ and $\inf(n\varphi, 1)\in\mathscr{C}_+(T)$ thus by (1) we obtain $\inf(n\varphi, 1)\leqq\varphi$. Then we have

$$\chi_{[\varphi>0]} = \sup_{n\in\mathbf{N}}\big(\inf(n\varphi, 1)\big) \leqq \varphi \leqq \chi_F$$

showing that $\varphi=1$ on the set $[\varphi>0]$ thus $\varphi=1$ on the set $\operatorname{Supp}\varphi=[\varphi>0]^-$ as well. Since $\varphi=0$ on $T\setminus\operatorname{Supp}\varphi$ we deduce that $\chi_{\operatorname{Supp}\varphi}=\varphi\in\mathscr{C}_+(T)$, i.e. $\operatorname{Supp}\varphi$ is an open-closed subset of T and $\operatorname{Supp}\varphi\subset F$ thus $\operatorname{Supp}\varphi\subset\mathring{F}$. We claim that $\mathring{F}$ equals $\operatorname{Supp}\varphi$. On the contrary, suppose that $\operatorname{Supp}\varphi\neq\mathring{F}$. Then $\mathring{F}\setminus\operatorname{Supp}\varphi$ is a non-void open subset of T thus there is a mapping $\varphi'\in\mathscr{C}_+(T)$ such that $0\leqq\varphi'\leqq 1$, $\operatorname{Supp}\varphi'\subset\mathring{F}\setminus\operatorname{Supp}\varphi$ and $\varphi'\neq 0$. Then $\varphi+\varphi'\in\mathscr{C}_+(T)$ and $\varphi+\varphi'\leqq\chi_F$ thus by (1) we have $\varphi+\varphi'\leqq\varphi$ in contradiction to $\varphi'\neq 0$. This proves that $\operatorname{Supp}\varphi=\mathring{F}$, i.e. the interior of the closed G_δ-set F is closed in T.

(iii) If F is a closed G_δ-set in T and $\mu\in P$ is a measure such that $\mu(F)>0$ then, applying the notations introduced in the proof of (ii), we obtain

$$\mu(\varphi) = \lim_n\mu(\varphi_n) = \mu(\chi_F) = \mu(F)$$

thus $\varphi\neq 0$, i.e. $\emptyset\neq\operatorname{Supp}\varphi=\mathring{F}$.

Corollary 1. *Let T be a compact Hausdorff space and let $P\subset\mathscr{M}_+(T)$ be a set such that $(\mathscr{C}_{\mathbf{C}}(T), P)$ is a weak GW^*-algebra. Then the open-closed subsets of T form a basis for the topology of T and the closure of every open F_σ-set is open in T. Particularly,* Supp φ *is open-closed for all $\varphi\in\mathscr{C}_{\mathbf{C}}(T)$.*

Proof. Let t be an arbitrary point of T and G an open neighbourhood of t. Then we can choose a function $\varphi\in\mathscr{C}_+(T)$ with the property that $0\leqq\varphi\leqq 1$, $\operatorname{Supp}\varphi\subset G$ and t is in the interior of $[\varphi=1]$. Since $[\varphi=1]$ is G_δ in T, by Proposition

2 we deduce that the interior of $[\varphi=1]$ is open-closed and contained in G. This means that at every point of T there is a basis consisting of open-closed sets, or equivalently, the topology of T has a basis formed by open-closed sets.

The second part of our assertion is a simple reformulation of (ii) in Proposition 2.

Theorem 2. *Let (A, P) be a commutative weak GW^*-algebra. Then A is a C^*-algebra whose underlying $*$-algebra is a Rickart $*$-algebra. Consequently, the set of projectors in A is total in the topology defined by the C^*-norm of A.*

Proof. Compare Corollary 1 with Theorems 1, ch. I, § 6. in [3] and 1.8 in [4].

IV. Spectral theorem for commutative GW*-algebras

If T is a compact Hausdorff space then $\mathscr{B}_0(T)$ denotes the σ-algebra in T generated by the closed G_δ subsets of T; $\mathscr{B}_0(T)$ is usually referred to as the Baire σ-algebra of T. On the other hand, a mapping $\varphi\colon T\to\mathbf{C}$ is called a Baire function if $\varphi^{-1}(E)\in\mathscr{B}_0(T)$ for every Borel set E in $\mathbf{C}$. It can be shown without difficulty that $\mathscr{B}_0(T)$ coincides with the least σ-algebra in T with respect to which every continuous complex valued function defined on T is measurable.

Let T be a compact Hausdorff space; for every countable ordinal number α we define by ω_1-induction the function space $\mathscr{C}_{\mathbf{C}}^{\alpha}(T)$ as follows:

— $\mathscr{C}_{\mathbf{C}}^{0}(T):=\mathscr{C}_{\mathbf{C}}(T)$,

— if $0<\alpha<\omega_1$ then $\varphi\in\mathscr{C}_{\mathbf{C}}^{\alpha}(T)$ if and only if φ is a function $T\to\mathbf{C}$ such that there is a sequence $(\varphi_n)_{n\in\mathbf{N}}$ in $\bigcup_{\beta<\alpha}\mathscr{C}_{\mathbf{C}}^{\beta}(T)$ which is uniformly bounded and pointwise converges to φ in T.

Then we define $\mathscr{C}_{\mathbf{C}}^{\infty}(T):=\bigcup_{\alpha<\omega_1}\mathscr{C}_{\mathbf{C}}^{\alpha}(T)$. It is easy to show that $\mathscr{C}_{\mathbf{C}}^{\infty}(T)=\mathscr{F}_{\mathbf{C}}^{b}(T,\mathscr{B}_0(T))$, i.e. $\mathscr{C}_{\mathbf{C}}^{\infty}(T)$ consists of the bounded complex valued Baire functions defined on T and a subset E of T belongs to $\mathscr{B}_0(T)$ if and only if $\chi_E\in\mathscr{C}_{\mathbf{C}}^{\infty}(T)$. In the sequel the sequence of function spaces $(\mathscr{C}_{\mathbf{C}}^{\alpha}(T))_{\alpha<\omega_1}$ will be referred to as the *standard graduation* of $\mathscr{C}_{\mathbf{C}}^{\infty}(T)$.

According to Theorem 1 and the fact that $\mathscr{C}_{\mathbf{C}}^{\infty}(T)=\mathscr{F}_{\mathbf{C}}^{b}(T,\mathscr{B}_0(T))$, the pair $(\mathscr{C}_{\mathbf{C}}^{\infty}(T), P)$ is a commutative GW^*-algebra, where P is the set of integrals on $\mathscr{C}_{\mathbf{C}}^{\infty}(T)$ defined by positive σ-additive set functions on $\mathscr{B}_0(T)$.

Lemma 2. *If T is a compact Hausdorff space, $P\subset\mathscr{M}_{\mathbf{C}}(T)$ and φ is a universally integrable complex valued function defined on T then the relation $\int_T\varphi\,\mathrm{d}\mu=0$ $(\mu\in P)$ implies that $\int_T\varphi\,\mathrm{d}\mu=0$ for all $\mu\in\overline{\mathrm{sp}}(P)$, where $\overline{\mathrm{sp}}(P)$ is the closure of $\mathrm{sp}(P)$ in $\mathscr{M}_{\mathbf{C}}(T)$ in the measure norm topology.*

Proof. Since the mapping $\mathcal{M}_{\mathbf{C}}(T)\to\mathbf{C}$, $\Theta\mapsto\int_T \varphi\,\mathrm{d}\Theta$ is a measure-norm continuous linear form on $\mathcal{M}_{\mathbf{C}}(T)$, the assertion is obviously true.

Lemma 3. *Let T be a compact Hausdorff space and let $P\subset\mathcal{M}_+(T)$ be a set such that $(\mathscr{C}_{\mathbf{C}}(T), P)$ is a GW*-algebra. If $\varphi\in\mathscr{C}_{\mathbf{C}}^{\infty}(T)$, $\varphi^b\in\mathscr{C}_{\mathbf{C}}(T)$ and $\int_T\varphi\,\mathrm{d}\mu=\mu(\varphi^b)$ for all $\mu\in P$ then we have $|||\varphi^b|||_T\leqq|||\varphi|||_T$.*

Proof. Let t be a fixed point of T and $\mathscr{B}_t$ denote the basis at t of T consisting of open-closed subsets of T (see Proposition 2, Corollary 1). With regard to (i) in Proposition 2, to every $E\in\mathscr{B}_t$ there is a measure μ_E in P such that $\mu_E(E)>0$. Let μ_E be such a measure and put $\lambda_E:=\chi_E\mu_E/\mu_E(E)$ for every $E\in\mathscr{B}_t$. Then $\lambda_E\in\overline{\mathrm{sp}}(P)$ by (III), and it is easy to see that the continuity of φ^b in t implies that $\lim_{E,\mathscr{B}_t}\lambda_E(\varphi^b)=\varphi^b(t)$. Now Lemma 2 yields that $\int_T\varphi\,\mathrm{d}\lambda_E=\lambda_E(\varphi^b)$ for all $E\in\mathscr{B}_t$, since the measure-norm closure of $\mathrm{sp}\,(P)$ in $\mathcal{M}_{\mathbf{C}}(T)$ equals $\overline{\mathrm{sp}}\,(P)$ (viz. $|||\cdot|||_T=\|\cdot\|_P$). From this we infer that

$$|\varphi^b(t|=\lim_{E,\mathscr{B}_t}|\lambda_E(\varphi^b)|=\lim_{E,\mathscr{B}_t}\Big|\int_T\varphi\,\mathrm{d}\lambda_E\Big|\leqq|||\varphi|||_T,$$

i.e. $|||\varphi^b|||_T\leqq|||\varphi|||_T$.

Proposition 3. *Let T be a compact Hausdorff space, $P\subset\mathcal{M}_+(T)$ and suppose that $(\mathscr{C}_{\mathbf{C}}(T), P)$ is a GW*-algebra. Then to every bounded complex valued Baire function φ defined on T there is a unique continuous function φ^b defined on T with the property that $\varphi=\varphi^b$ a.e. for all $\mu\in P$.*

Proof. Since P separates the points of $\mathscr{C}_{\mathbf{C}}(T)$, the uniqueness of φ^b is obvious. The existence of φ^b will be shown by the use of the standard graduation of $\mathscr{C}_{\mathbf{C}}^{\infty}(T)$. Assume that $\varphi\in\mathscr{C}_{\mathbf{C}}^{\infty}(T)$ and by ω_1-induction we show that for every $\alpha<\omega_1$, if $\varphi\in\mathscr{C}_{\mathbf{C}}^{\alpha}(T)$ then there is a function $\varphi^b\in\mathscr{C}_{\mathbf{C}}(T)$ such that $\varphi=\varphi^b$ a.e., for all $\mu\in P$.

The assertion holds for $\alpha=0$, evidently. Suppose that $0<\alpha<\omega_1$ and the assertion is true for every $\beta<\alpha$. Since $\varphi\in\mathscr{C}_{\mathbf{C}}^{\alpha}(T)$, there is a uniformly bounded sequence $(\varphi_n)_{n\in\mathbf{N}}$ in $\bigcup_{\beta<\alpha}\mathscr{C}_{\mathbf{C}}^{\beta}(T)$ such that $\varphi_n\to\varphi$ pointwise on T. With regard to our induction hypothesis, for every $n\in\mathbf{N}$ we can define a function φ_n^b in $\mathscr{C}_{\mathbf{C}}(T)$ such that $\varphi_n=\varphi_n^b$ a.e., for all $\mu\in P$. Now Lemma 3 gives that $|||\varphi_n^b|||_T\leqq|||\varphi_n|||_T$ $(n\in\mathbf{N})$ so the sequence $(\varphi_n^b)_{n\in\mathbf{N}}$ in $\mathscr{C}_{\mathbf{C}}(T)$ is also uniformly bounded.

If $\mu\in P$ then the theorem of Lebesgue applied to μ and the sequence $(\varphi_n)_{n\in\mathbf{N}}$ implies $\int_T\varphi_n\,\mathrm{d}\mu\to\int_T\varphi\,\mathrm{d}\mu$. On the other hand, $\int_T\varphi_n\,\mathrm{d}\mu=\mu(\varphi_n^b)\,(n\in\mathbf{N})$ thus we obtain

$$\lim_n\mu(\varphi_n^b)=\int_T\varphi\,\mathrm{d}\mu\quad(\mu\in P). \tag{2}$$

The sequentially completeness of $\mathscr{C}_{\mathbf{C}}(T)$ in the $\sigma(\mathscr{C}_{\mathbf{C}}(T), \operatorname{sp}(P))$ topology now results in the existence of a function $\varphi^b \in \mathscr{C}_{\mathbf{C}}(T)$ such that

$$\lim_n \mu(\varphi_n^b) = \mu(\varphi^b) \quad (\mu \in P). \tag{3}$$

Comparing (2) and (3) we deduce that $\mu(\varphi^b) = \int_T \varphi \, d\mu$ for every $\mu \in P$. According to Lemma 3, $\Theta(\varphi^b) = \int_T \varphi \, d\Theta$ for all $\Theta \in \overline{\operatorname{sp}}(P)$. If $\mu \in P$ and $\psi \in \mathscr{C}_{\mathbf{C}}(T)$ then by (III) we have $\psi\mu \in \overline{\operatorname{sp}}(P)$ thus $(\varphi\mu)(\psi) = \int_T \varphi \, d(\psi\mu) = (\psi\mu)(\varphi^b) = (\varphi^b\mu)(\psi)$, i.e. $\varphi\mu = \varphi^b\mu$ for all $\mu \in P$. This shows that $\varphi = \varphi^b$ a.e., for every $\mu \in P$.

We call the attention to the fact that Proposition 3 holds only for commutative GW^*-algebras and not for commutative *weak* GW^*-algebras.

Theorem 3. *Let (A, P) be a commutative GW^*-algebra. Then there is a unique $*$-homomorphism $\Theta^P \colon \mathscr{C}_{\mathbf{C}}^{\infty}(X(A)) \to A$ preserving the unit elements satisfying*

$$f(\Theta^P(\varphi)) = \int_{X(A)} \varphi \, d(f \circ \mathscr{G}_A^{-1}) \tag{4}$$

for all $f \in P$ and $\varphi \in \mathscr{C}_{\mathbf{C}}^{\infty}(X(A))$.

Remark. Note that $f \circ \mathscr{G}_A^{-1} \in \mathscr{M}_+(X(A))$ for every positive linear form f on A.

Proof. The uniqueness of Θ^P follows from (4) and the fact that P separates the points of A. In order to prove the existence of Θ^P we first mention that the pair $(\mathscr{C}_{\mathbf{C}}(X(A)), P \circ \mathscr{G}_A^{-1})$ is a commutative GW^*-algebra. Then, by Proposition 3, we can define the mapping

$$\mathscr{C}_{\mathbf{C}}^{\infty}(X(A)) \to \mathscr{C}_{\mathbf{C}}(X(A)), \quad \varphi \mapsto \varphi^b$$

satisfying $\varphi = \varphi^b$ a.e., for every $\mu \in P \circ \mathscr{G}_A^{-1}$ and $\varphi \in \mathscr{C}_{\mathbf{C}}^{\infty}(X(A))$. It is routine to check that this mapping is a $*$-homomorphism between $\mathscr{C}_{\mathbf{C}}^{\infty}(X(A))$ and $\mathscr{C}_{\mathbf{C}}(X(A))$ preserving the unit elements. For every $\varphi \in \mathscr{C}_{\mathbf{C}}^{\infty}(X(A))$ we define $\Theta^P(\varphi) := \mathscr{G}_A^{-1}(\varphi^b)$. Then Θ^P is a unit preserving $*$-homomorphism between $\mathscr{C}_{\mathbf{C}}^{\infty}(X(A))$ and A, evidently. If $f \in P$ and $\varphi \in \mathscr{C}_{\mathbf{C}}^{\infty}(X(A))$ then $f \circ \mathscr{G}_A^{-1} \in P \circ \mathscr{G}_A^{-1}$ thus $\varphi = \varphi^b$ a.e., for $f \circ \mathscr{G}_A^{-1}$, showing that the equality holds for φ and f.

Of course, Theorem 3 can be appreciated as the global (or better to say, collective) spectral theorem for commutative GW^*-algebras. In order to formulate an individual version of the spectral theorem, we note that the spectrum of an element x in a unital algebra A is usually denoted by $\operatorname{Sp}_A(x)$, or, if no confusion arises as for the algebra, the letter A is omitted. It is well known that given a unital C^*-algebra A, to every normal element x of A there is a unique unit preserving $*$-homomorphism $\Theta_x \colon \mathscr{C}_{\mathbf{C}}(\operatorname{Sp}(x)) \to A$ such that $\Theta_x(\mathrm{id}_{\operatorname{Sp}(x)}) = x$ and Θ_x is an isometry whose range equals the C^*-subalgebra of A generated by the set $\{1, x, x^*\}$ (cf. [3] ch. I, § 6, Proposition 5).

Theorem 4. *Let (A, P) be a commutative GW^*-algebra and $x \in A$. Then there exists a unique unit preserving $*$-homomorphism Θ_x^P: $\mathcal{F}_{\mathbf{C}}^b(\mathrm{Sp}(x), \mathcal{B}(\mathrm{Sp}(x))) \to A$ which is an extension of Θ_x and satisfies*

$$f(\Theta_x^P(\varphi)) = \int_{\mathrm{Sp}(x)} \varphi \, d(f \circ \Theta_x) \quad (f \in P), \tag{5}$$

for every bounded complex valued Borel function φ defined on $\mathrm{Sp}(x)$.

Remark. Note that $f \circ \Theta_x \in \mathcal{M}_+(\mathrm{Sp}(x))$ for every positive linear form on A.

Proof. The set P separates the points of A, thus the uniqueness of Θ_x^P follows from (5), evidently.

Since $\mathrm{Sp}(x)$ is a metrisable compact topological space, the σ-algebra $\mathcal{B}(\mathrm{Sp}(x))$ of Borel sets in $\mathrm{Sp}(x)$ coincides with the σ-algebra $\mathcal{B}_0(\mathrm{Sp}(x))$ of Baire sets in $\mathrm{Sp}(x)$. Consequently, we have $\mathcal{F}_{\mathbf{C}}^b(\mathrm{Sp}(x), \mathcal{B}(\mathrm{Sp}(x))) = \mathcal{C}_{\mathbf{C}}^\infty(\mathrm{Sp}(x))$. Since the mapping $\mathcal{G}_A(x)$ is a continuous function from $X(A)$ onto $\mathrm{Sp}(x)$, the operator

$$\mathcal{G}_A(x)^\# : \mathcal{C}_{\mathbf{C}}^\infty(\mathrm{Sp}(x)) \to \mathcal{C}_{\mathbf{C}}^\infty(X(A)), \quad \varphi \mapsto \varphi \circ \mathcal{G}_A(x)$$

is an injective unit preserving $*$-homomorphism between the C^*-algebras $\mathcal{C}_{\mathbf{C}}^\infty(\mathrm{Sp}(x))$ and $\mathcal{C}_{\mathbf{C}}^\infty(X(A))$. Then we put

$$\Theta_x^P := \Theta^P \circ \mathcal{G}_A(x)^\#,$$

where Θ^P denotes the $*$-homomorphism between $\mathcal{C}_{\mathbf{C}}^\infty(X(A))$ and A, introduced in Theorem 3. Thus Θ_x^P is a unit preserving $*$-homomorphism between $\mathcal{C}_{\mathbf{C}}^\infty(\mathrm{Sp}(x))$ and A. It remained to prove the equality (5). Let there be given a linear form $f \in P$ and a function $\varphi \in \mathcal{C}_{\mathbf{C}}^\infty(\mathrm{Sp}(x))$. Then, by the definition of Θ_x^P, we have

$$\begin{aligned} f(\Theta_x^P(\varphi)) &= f(\Theta^P(\mathcal{G}_A(x)^\#(\varphi))) = f(\Theta^P(\varphi \circ \mathcal{G}_A(x))) = \\ &= \int_{X(A)} \varphi \circ \mathcal{G}_A(x) \, \mathrm{d}(f \circ \mathcal{G}_A^{-1}) = \int_{X(A)} \varphi \, \mathrm{d}(\mathcal{G}_A(x)(f \circ \mathcal{G}_A^{-1})), \end{aligned} \tag{6}$$

where $\mathcal{G}_A(x)(f \circ \mathcal{G}_A^{-1})$ denotes the Radon measure on $\mathrm{Sp}(x)$, which is the image of the measure $f \circ \mathcal{G}_A^{-1} \in \mathcal{M}_+(X(A))$ established by the continuous function $\mathcal{G}_A(x)$. It is obvious that the mapping

$$\mathcal{C}_{\mathbf{C}}(\mathrm{Sp}(x)) \mapsto A, \quad \psi \mapsto \mathcal{G}_A^{-1}(\psi \circ \mathcal{G}_A(x))$$

is a unit preserving $*$-homomorphism between $\mathcal{C}_{\mathbf{C}}(\mathrm{Sp}(x))$ and A which assigns x to $\mathrm{id}_{\mathrm{Sp}(x)}$, so the uniqueness of Θ_x results in $\Theta_x(\psi) = \mathcal{G}_A^{-1}(\psi \circ \mathcal{G}_A(x))$ for all $\psi \in \mathcal{C}_{\mathbf{C}}(\mathrm{Sp}(x))$. Thus we obtain $(f \circ \Theta_x)(\psi) = (f \circ \mathcal{G}_A^{-1})(\psi \circ \mathcal{G}_A(x))$ for every

$\psi\in\mathscr{C}_{\mathbf{C}}(\mathrm{Sp}\,(x))$ showing that $f\circ\Theta_x=\mathscr{G}_A(x)(f\circ\mathscr{G}_A^{-1})$. Comparing this equality with (6), we finally deduce that (5) holds for every $f\in P$ and $\varphi\in\mathscr{C}_{\mathbf{C}}^{\infty}(\mathrm{Sp}\,(x))$.

At last we mention that both the $*$-homomorphisms Θ^P and Θ_x^P introduced in Theorem 3 and Theorem 4, respectively, depend essentially on P.

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