Acta Sci. Math., 52 (1988), 157—163

The invariance principle for functionals of sums of martingale differences

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1. Introduction. Let $\{(X_{ni}, F_{ni}),\ 1 \leqq i \leqq k_n\}$, $n \geqq 1$, be a double array of square-integrable random variables whose rows are martingale difference sequences (MDS), i.e. for each $n \geqq 1$ the rv's X_{ni}, $1 \leqq i \leqq k_n$, given on some probability space $(\Omega, \mathscr{A}, P)$ with sub-σ-fields $F_{n0} \subset F_{n1} \subset \ldots \subset F_{nk_n}$, are such that X_{ni} is F_{ni}-measurable and $E(X_{ni}|F_{n,i-1})=0$ a.s. for every $1 \leqq i \leqq k_n$. Define

$$S_{nk} = \sum_{i=1}^{k} X_{ni}, \quad \sigma_{ni}^2 = E(X_{ni}^2 | F_{n,i-1}),$$

$s_{nk}^2 = ES_{nk}^2$ and $S_{nk} = s_{nk}^2 = 0$ if $k=0$, $n \geqq 1$. Let us observe that without loss of generality we may and do assume that for every $n \geqq 1$, $EX_{ni}^2 \neq 0$, $1 \leqq i \leqq k_n$, $s_n^2 = s_{nk_n}^2 = 1$, where $k_n \to \infty$ as $n \to \infty$.

Let $D[0, 1]$ be the space of functions defined on $[0, 1]$ that are right-continuous and have left hand limits, endowed with the Skorohod J_1-topology (cf. [1, §14]). By W we will denote the Wiener measure on $D[0, 1]$ with the corresponding Wiener process $\{W(t)\colon 0 \leqq t \leqq 1\}$.

Let F_M be the space of functions defined on $[0, 1] \times (-\infty, \infty)$ satisfying the following condition: there exists an absolute constant M such that if $f \in F_M$, then f and its derivatives satisfy inequalities of the form

(1) $$|Df(s, x)| \leqq M(1+|x|^\alpha),$$

where D denotes either the identity operator or a first derivative and α is some positive constant.

Define a random function $W_n(t)$, $0 \leqq t \leqq 1$, by

(2) $$W_n(t) = S_{n, m_n(t)}, \quad n \geqq 1,$$

where $m_n(t) = \max\{i \leqq k_n\colon s_{ni}^2 \leqq t\}$, $t \in [0, 1]$.

Received November 27, 1984, and in revised form February 21, 1985.

We shall give sufficient conditions for the weak convergence of the process $\{Z_n(t)=\sum_{i=0}^{m_n(t)-1} f_n(s_{ni}^2, S_{ni})X_{n,i+1},\ 0\leqq t\leqq 1\}$, in Skorohod's space $D[0, 1]$, to the process $\left\{\int_0^t f(s, W(s))dW(s):\ 0\leqq t\leqq 1\right\}$ in $D[0, 1]$, which we denote by $\{Z(t),\ 0\leqq t\leqq 1\}$.

The results obtained are generalizations or extensions of those given in [1, Theorem 16.1], [3, p. 179], [2], [4] and [5].

2. Limit theorems. Suppose there exists a double array $\{C_{ni},\ 1\leqq i\leqq k_n,\ n\geqq 1\}$ of nonnegative numbers such that

$$\sigma_{ni}^2 \leqq C_{ni}, \quad \text{a.s.} \quad 1\leqq i\leqq k_n, \quad n\geqq 1. \tag{3}$$

and set

$$W_n^* = \sum_{i=0}^{m_n(t)} C_{ni}, \quad t\in[0, 1], \quad n\geqq 1 \quad (C_{n0}=0).$$

The main result of this paper is given in the following

Theorem 1. *Let* $\{(X_{nk}, F_{nk}),\ 1\leqq k\leqq k_n\}$, $n\geqq 1$, *be a double array of random variables whose rows are martingale difference sequences such that* $s_n^2=1$, $n\geqq 1$. *Assume*

(4) *the finite dimensional distributions of* $\{W_n,\ n\geqq 1\}$ *converge weakly, as* $n\to\infty$, *to those of* $\{W(t),\ 0\leqq t\leqq 1\}$,

(5) *there exists an array of nonnegative numbers satisfying* (3) *such that for every* $t_1, t_2\in[0, 1]$, $t_2-t_1\geqq m(n)$, $n\geqq 1$,

$$W_n^*(t_2)-W_n^*(t_1) \leqq [F(t_2)-F(t_1)]^r,$$

where $m(n)=\min\{EX_{ni}^2:\ 1\leqq i\leqq k_n\}$, *F is a nondecreasing continuous function on* $[0, 1]$ *and* $r>1/2$ *is some positive constant.*

Then $Z_n\to Z$ *as* $n\to\infty$, *in* $D[0, 1]$, *provided that* $f, f_n\in F_M$, $n\geqq 1$, *and for every* $s\in[0, 1]$

$$Df_n(s, x) \to Df(s, x), \quad \text{as} \quad n\to\infty \tag{6}$$

uniformly in x on every finite interval. Here, the stochastic integral in the definition of $Z(t)$ *is taken in the* L^2*-sense.*

From Theorem 1 we get the following

Theorem 2. *Assume* $\{(X_i, F_i),\ i\geqq 1\}$ *is a square-integrable martingale difference sequence such that* $EX_i^2=1$, $i\geqq 1$, *and*

$$\sup_i E(X_i^2\,|\,F_{i-1}) \leqq M, \quad a.s. \tag{7}$$

for some positive constant M. If (4) *holds with* $W_n(t)=\sum_{i=0}^{[nt]} X_i/\sqrt{n}$ *then, in* $D[0, 1]$,

$$(8) \qquad (1/\sqrt{n})\sum_{i=0}^{[nt]-1} f_n(i/n, S_i/\sqrt{n})X_{i+1} \xrightarrow{\mathscr{D}} \int_0^t f(s, W(s))\,dW(s), \quad as \quad n\to\infty,$$

provided (6) *holds as well.*

To prove Theorem 2 we note that, in this case, (5) is satisfied with $X_{nk}=(X_1+\ldots \ldots+X_k)/\sqrt{n}$, $C_{nk}=M/n$, $1\leqq k\leqq k_n=n$, $F(t)=2t$, $r=1$, $m_n(t)=[nt]$. Thus Theorem 2 follows from Theorem 1. It is easy to see that Theorem 16.1 in [1] is a consequence of Theorem 2 (it is enough to put $f_n=f\equiv 1$, $n\geqq 1$).

We note that a necessary and sufficient condition for (4) to hold is given in Theorem 7.7 [1, p. 49]. Furthermore, if $W_n=\{W_n(t)\colon 0\leqq t\leqq 1\}$ converges weakly, in $D[0, 1]$, to a standard Wiener process $W=\{W(t), 0\leqq t\leqq 1\}$, then (4) also holds. On the other hand, the assertion of Theorem 1 implies the weak convergence of W_n, as $n\to\infty$, to W. Thus the assumption (4) is necessary for (6) to hold. For example, it is well known that if $\{(X_{ni}, F_{ni}), 1\leqq i\leqq k_n\}$, $n\geqq 1$, is a double array of square-integrable random variables whose rows are martingale difference sequences satisfying the Lindeberg condition and $\sum_{i=1}^{k_n}\sigma_{ni}^2\xrightarrow{P} 1$, then (4) holds. Moreover, one can easily observe that every sequence $\{X_n, n\geqq 1\}$ of independent random variables, with $EX_n=0$, $EX_n^2=1$, $n\geqq 1$, satisfying the central limit theorem also satisfies the assumptions of Theorem 2. It should also be mentioned here that the assumptions (1) and (6) concerning the functions f_n, $n\geqq 1$, and f are very general. Some examples of such functions can be found in [3, Section 5].

To give a better illustration of the meaning of Theorem 1, let us note that from a very special case of it we immediately obtain the following assertions. If $\{(X_i, F_i), i\geqq 1\}$ is a sequence of random variables with $EX_i=1$, $i\geqq 1$, and satisfy (4) and (7), then in $D[0, 1]$,

$$\Big\{n^{-1}\sum_{1\leqq i<j\leqq[nt]} X_iX_j,\ 0\leqq t\leqq 1\Big\} \xrightarrow{\mathscr{D}} \Big\{\int_0^t W(s)\,dW(s),\ 0\leqq t\leqq 1\Big\}$$

and

$$\Big\{n^{-3/2}\sum_{i=1}^{[nt]}(i-1)X_i,\ 0\leqq t\leqq 1\Big\} \xrightarrow{\mathscr{D}} \Big\{\int_0^t s\,dW(s),\ 0\leqq t\leqq 1\Big\}$$

as $n\to\infty$. The first assertion follows from Theorem 2 with $f_n(t, x)=f(t, x)=x$, and the second one with $f_n(t, x)=f(t, x)=t$. The distributions of the integrals

$$\int_0^t W(s)\,dW(s) \quad \text{and} \quad \int_0^t s\,dW(s)$$

are well known. For example,

$$\int_0^t W(s)\,dW(s) = (W^2(t)-t)/2 .$$

Remark. We note that condition (5) implies

(9) $$W_n^*(1) \leqq K,\ n \geqq 1, \quad \text{for some constant } K.$$

Moreover, by (5),

$$\max_{1 \leqq i \leqq k_n} C_{ni} \leqq \sup \{[F(t_2)-F(t_1)]^r\colon t_2 - t_1 = m(n)\},\ n \geqq 1,$$

and, by (3), $EX_{ni}^2 \leqq C_{ni}$, and $\lim_{n\to\infty} m(n) = 0$, so that

(10) $$\max_{1 \leqq i \leqq k_n} EX_{ni}^2 \to 0, \quad \text{as} \quad n \to \infty,$$

because the function F is uniformly continuous.

3. Auxiliary lemmas. Let for every function $f \in F_M$

$$f^C(s, x) = f(s, x) I([-C, C])(x), \quad s \in [0, 1],$$

where C is a positive constant and $I(A)(\cdot)$ denotes the indicator function of the set A, and set

$$\|(x, y)\|_2 = (x^2+y^2)^{1/2}, \quad (x, y) \in R^2.$$

Lemma 1. *Let* $\{f_n,\ n \geqq 1\}$ *be a sequence of functions such that* $f_n \in F_M$, $n \geqq 1$, *and let* $0 = p_0 < p_1 < \ldots < p_r = t$, $t = t_0 < t_1 < \ldots < t_b = s$, $0 \leqq t < s \leqq 1$, *be partitions of the intervals* $[0, t]$ *and* $[t, s]$, *respectively. Assume that for each* n *the* MDS $\{(X_{ni}, F_{ni}),\ 1 \leqq i \leqq k_n\}$ *satisfies the assumptions of Theorem* 1. *Then, for every* $\varepsilon > 0$ *and each* $C > 0$,

(11) $$\lim_{\gamma\to 0} \lim_{n\to\infty} P_1(\varepsilon, \gamma, n, C) = 0,$$

where

$$\gamma = \max_{1 \leqq i \leqq r} (p_i - p_{i-1}) + \max_{1 \leqq i \leqq b} (t_i - t_{i-1})$$

and

$$P_1(\varepsilon, \gamma, n, C) = P\Big(\Big\|\Big(\sum_{i=0}^{m_n(t)} f_n^C(s_{ni}^2, S_{ni}) X_{n,i+1} - \sum_{j=0}^{r-1} f_n^C(p_j, W_n(p_j))(W_n(p_{j+1}) - W_n(p_j)),$$

$$\sum_{i=m_n(t)+1}^{m_n(s)} f_n^C(s_{ni}^2, S_{ni}) X_{n,i+1} - \sum_{j=0}^{b-1} f_n^C(t_j, W_n(t_j))(W_n(t_{j+1}) - W_n(t_j))\Big)\Big\|_2 > \varepsilon\Big) =$$

$$= P(\|(X(n, \gamma, 0, t), X(n, \gamma, t, s))\|_2 > \varepsilon) = P(\|(X_1, X_2)\|_2 > \varepsilon).$$

Proof. To prove Lemma 1 it is enough to show that

(12) $$\lim_{\gamma\to 0} \lim_{n\to\infty} EX^2(n, \gamma; t, s) = 0,$$

because, in the same way, we can prove that (12) holds with $X(n, \gamma, 0, t)$ and then $P_1(\varepsilon, \gamma, n, C) \leqq \varepsilon^{-2}(EX_1^2 + EX_2^2)$.

Let, for every i $\big(m_n(t_j) < i \leqq m_n(t_{j+1})\big)$, $W_{ij} = W_{ij}^{(n)} = f_n^C(s_{ni}^2, S_{ni}) - f_n^C\big(t_j, W_n(t_j)\big)$. Then we have

$$P_1(\varepsilon, \gamma, n, C) = P\Big(\Big|\sum_{j=0}^{b-1} \sum_{i=m_n(t_j)+1}^{m_n(t_{j+1})} W_{ij} X_{n,i+1}\Big| > \varepsilon\Big).$$

On the other hand, for every $i < i'$ $\big(m_n(t_j) \leqq i < m_n(t_{j+1}),\ m_n(t_{j'}) \leqq i' < m_n(t_{j'+1})\big)$

$$EW_{ij} X_{n,i+1} W_{i'j'} X_{n,i'+1} = EW_{ij} X_{n,i+1} W_{i'j'} E(X_{n,i'+1} \,|\, F_{ni'}) = 0.$$

Thus

$$EX_2^2 = \sum_{j=0}^{b-1} \sum_{i=m_n(t_j)+1}^{m_n(t_{j+1})} EW_{i-1,j}^2 X_{ni}^2 = \sum_{j=0}^{b-1} \sum_{i=m_n(t_j)+1}^{m_n(t_{j+1})} EW_{i-1,j}^2 E(X_{ni}^2 \,|\, F_{n,i-1}) \leqq$$

$$\leqq \sup_{i,j} EW_{ij}^2 \Big(\sum_{i=m_n(t)}^{m_n(s)} C_{ni}\Big) \leqq W_n^*(1) \sup_{i,j} EW_{ij}^2.$$

Hence, by (9),

$$EX_2^2 \leqq K \sup_{i,j} EW_{ij}^2. \tag{13}$$

Let us observe that, by (1), for every $f \in F_M$ and (s, x), $(s_1, x_1) \in [0, 1] \times R$,

$$|f^C(s, x) - f^C(s_1, x_1)| \leqq K_C(|s - s_1| + |x - x_1|), \tag{14}$$

where K_C is an absolute positive constant which depends only on C. Thus, for every $m_n(t_j) < i \leqq m_n(t_{j+1})$, $0 \leqq j \leqq b-1$,

$$\begin{aligned} EW_{ij}^2 &\leqq 2K_C^2\{|s_{ni}^2 - t_j|^2 + E(S_{ni} - S_{nm_n(t_j)})^2\} \leqq \\ &\leqq 2K_C^2\Big\{\big(t_{j+1} - t_j + \max_{1 \leqq i \leqq k_n} EX_{ni}^2\big)^2 + \big(t_{j+1} - t_j + \max_{1 \leqq i \leqq k_n} EX_{ni}^2\big)\Big\}. \end{aligned} \tag{15}$$

Taking into account (10) and (15) we obtain (12).

Lemma 2. *Let f, f_n, $n \geqq 1$, be functions satisfying the assumptions of Theorem* 1. *If the assumptions of Lemma* 1 *are also satisfied, then for every* $C > 0$

$$\lim_{\gamma \to 0} \lim_{n \to \infty} P_2(\varepsilon, \gamma, n, C) = 0, \tag{16}$$

where

$$\begin{aligned} P_2(\varepsilon, \gamma, n, C) = P\Big(\Big\|\Big(&\sum_{j=0}^{r-1} \{f_n^C(p_j, W_n(p_j)) - f^C(p_j, W_n(p_j))\}(W_n(p_{j+1}) - W_n(p_j)), \\ &\sum_{j=0}^{b-1} \{f_n^C(t_j, W_n(t_j)) - f^C(t_j, W_n(t_j))\}(W_n(t_{j+1}) - W_n(t_j))\Big)\Big\|_2 > \varepsilon\Big) = \\ &= P\big(\|(X_1', X_2')\|_2 > \varepsilon\big), \quad say. \end{aligned}$$

Proof. Again, it is enough to show that

$$\lim_{\gamma\to 0}\lim_{n\to\infty} E(X_2')^2 = 0.$$

Let, for every $0\leqq j\leqq b$, $V_{nj}(x)=f_n^C(t_j, x)-f^C(t_j, x)$. We have

$$E(X_1')^2 = \sum_{j=0}^{b-1} E\{(V_{nj}(W_n(t_j)))^2 \sum_{i=m_n(t_j)+1}^{m_n(t_{j+1})} E(X_{ni}^2 | F_{nm_n}(t_j))\}.$$

Let $R_n = \max_{0\leqq j\leqq b-1} \sup_x V_{nj}^2(x)$. Then, by (6), $R_n \to 0$ as $n\to\infty$.
Thus

$$EX_1'^2 \leqq R_n \sum_{j=0}^{b-1} \sum_{i=m_n(t_j)+1}^{m_n(t_{j+1})} EX_{ni}^2 = R_n(EW_n^2(s)-EW_n^2(t)) \leqq R_n \to 0, \quad \text{as} \quad n\to\infty.$$

Lemma 3. *Let the assumptions of Lemma* 1 *and Theorem* 1 *be satisfied. Then for any given* $C>0$,

$$(17)\quad \begin{aligned} &\Big(\sum_{j=0}^{r-1} f^C(p_j, W_n(p_j))(W_n(p_{j+1})-W_n(p_j)),\ \sum_{j=0}^{b-1} f^C(t_j, W_n(t_j))(W_n(t_{j+1})-W_n(t_j))\Big) \xrightarrow{\mathcal{D}} \\ &\xrightarrow{\mathcal{D}} \Big(\sum_{j=0}^{r-1} f^C(p_j, W(p_j))(W(p_{j+1})-W(p_j)),\ \sum_{j=0}^{b-1} f^C(t_j, W(t_j))(W(t_{j+1})-W(t_j))\Big) \\ &\text{as} \quad n\to\infty, \end{aligned}$$

where $\{W(t)\colon 0\leqq t\leqq 1\}$ *is a standard Wiener process in* $D[0, 1]$.

The assertion of Lemma 3 follows from (4) and Theorem 5.1 [1].

Lemma 4. *If* $f\in F_M$, *then for every* $\varepsilon>0$ *and any given* $C>0$

$$(18)\quad \begin{aligned} &P\Big(\Big\|\Big(\sum_{j=0}^{r-1} f^C(p_j, W(p_j))(W(p_{j+1})-W(p_j)) - \int_0^t f^C(x, W(x))\,dW(x), \\ &\sum_{j=0}^{b-1} f^C(t_j, W(t_j))(W(t_{j+1})-W(t_j)) - \int_t^s f^C(x, W(x))\,dW(x)\Big\|_2 > \varepsilon\Big) \to 0 \end{aligned}$$

as

$$\gamma = \max_{0\leqq i\leqq r-1}(p_{i+1}-p_i) + \max_{0\leqq j\leqq t-1}(t_{j+1}-t_j) \to 0,$$

where $0=p_0<p_1<\ldots<p_r=t$, $t=t_0<t_1<\ldots<t_b=s$, $0\leqq t<s\leqq 1$ *are partitions of the intervals* $[0, t]$ *and* $[t, s]$, *respectively.*

The proof of Lemma 4 is essentialy the same that is given in [4].

4. Proof of Theorem 1. Let us observe that

$$(19)\qquad P(\max_{1\leqq i\leqq k_n} |S_{ni}| > C) \leqq C^{-2}.$$

Furthermore

(20) $$P(\sup_{0\leqq t\leqq 1}|W(t)|>C)\to 0 \quad \text{as} \quad C\to\infty.$$

Thus, taking into account (11) and (16)—(20) we get

(21) $$(Z_n(t), Z_n(s)-Z_n(t)) \xrightarrow{\mathscr{D}} (Z(t), Z(s)-Z(t)) \quad \text{as} \quad n\to\infty,$$

for every $0\leqq t<s\leqq 1$. Clearly, using this method we may prove that the finite dimensional distributions of $\{Z_n, n\geqq 1\}$ converge weakly, as $n\to\infty$, to those of $\{Z(t):\ 0\leqq t\leqq 1\}$.

To complete the proof, we have to verify the tightness condition. We use Theorem 15.6 in [1]. From this theorem and (19) we infer that it suffices to show

(22) $$E(Z_n^C(t)-Z_n^C(t_1))^2(Z_n^C(t_2)-Z_n^C(t))^2 \leqq [F(t_2)-F(t_1)]^{2r},$$

for any $t_1\leqq t\leqq t_2$, $n\geqq 1$, $C>0$, where

$$Z_n^C(t) = \sum_{i=0}^{m_n(t)-1} f_n^C(s_{ni}^2, S_{ni})X_{n,i+1}, \quad t\in[0, 1].$$

We first note that, by (3) and (1),

$$E(Z_n^C(t)-Z_n^C(t_1))^2(Z_n^C(t_2)-Z_n^C(t))^2 \leqq$$
$$\leqq K(C)(W_n^*(t)-W_n^*(t_1))(W_n^*(t_2)-W_n^*(t)) \leqq 4^{-1}K(C)(W_n^*(t_2)-W_n^*(t_1))^2,$$

where $K(C)$ is some positive constant which depends only on C. Hence, by assumption (5) condition (22) holds, because in the case $t_2-t_1<m(n)$, $Z_n^C(t)=Z_n^C(t_1)$ or $Z_n^C(t)=Z_n^C(t_2)$.

Acknowledgement. I would like to thank the referee for helpful comments.

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