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On the group of analytic automorphisms of the unit ball of J^*-algebras

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1. Introduction

Our purpose is to present an elementary method to integrate a certain Riccati differential equation that plays an important role in the study of the unit ball of J^*-algebras of operators as symmetric spaces. Our approach consists in the use of Potapov's generalized Möbius transformations together with some elementary facts in the theory of holomorphic functions between Banach spaces. These methods have proved to be successful in the study of J^*-algebras and in some other questions, too ([1], [2], [3]).

Let $\mathfrak{H}$ and $\mathfrak{K}$ be complex Hilbert spaces and denote by $\mathcal{U}$ a J^*-algebra of bounded linear operators $X\colon \mathfrak{H}\to\mathfrak{K}$. That is, by definition, $\mathcal{U}$ is a closed complex subspace of $\mathcal{L}(\mathfrak{H}, \mathfrak{K})$ such that $AB^*C+CB^*A\in\mathcal{U}$ whenever $A, B, C\in\mathcal{U}$. Let $B(\mathcal{U})$ be the open unit ball of $\mathcal{U}$ and assume that $A\in\mathcal{U}$ and $X\in B(\mathcal{U})$ are given. Then, we consider the Riccati initial value problem:

$$(*)\qquad \frac{d}{dt}y(t) = A-y(t)A^*y(t),\quad y(0) = X,\quad y(t)\in B(\mathcal{U}),$$

where A^* stands for the adjoint of A. We give an explicit formula for the maximal solution $y_A(t; X)$ of $(*)$ in terms of the initial value X and the parameter A. See also ([3], page 57) and ([4], page 509) where other (but non elementary) approaches to the problem can be found.

We recall the following principal property of J^*-algebras [1]:

Given $M\in B(\mathcal{U})$, the Möbius transformation

$$(1)\qquad T_M(X) = (1-MM^*)^{-1/2}(X+M)(1+M^*X)^{-1}(1-M^*M)^{1/2},\quad X\in B(\mathcal{U})$$

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is a holomorphic automorphism of $B(\mathcal{U})$. Moreover, we have

$$(2) \qquad T_M(0) = M, \quad T_{-M} = T_M^{-1}, \quad T_M(X)^* = T_{M^*}(X^*)$$

and

$$(3) \qquad dT_M(X)Y = (1-MM^*)^{1/2}(1+XM^*)^{-1}Y(1+M^*X)^{-1}(1-M^*M)^{1/2}$$

for $X \in B(\mathcal{U})$ and $Y \in \mathcal{U}$.

Here, positive and negative square roots are defined by the usual power series expansions and, at each ocurrence, 1 denotes the identity operator on the appropiate underlying Hilbert space.

Furthermore, we recall from [5] or [8] the following basic facts concerning $\operatorname{Aut} B(\mathcal{U})$, the group of holomorphic automorphisms of $B(\mathcal{U})$:

Let the vector field $f(X)\dfrac{\partial}{\partial X}$ be complete in $B(\mathcal{U})$ and denote by $y(t, X)$ the solution of

$$(4) \qquad \frac{d}{dt}y(t) = f[y(t)], \quad y(0) = X, \quad y(t) \in B(\mathcal{U})$$

Then, for each fixed $t \in \mathbf{R}$, the mapping $X \to y(t, X)$ is an element of $\operatorname{Aut} B(\mathcal{U})$. Moreover, the mapping $t \mapsto y(t, \cdot)$ is a continuous one-parameter group of automorphisms of $B(\mathcal{U})$ and we have $f(X) = \left.\dfrac{d}{dt}\right|_0 y(t, X)$ for $X \in B(\mathcal{U})$.

2. The main result: one-parameter groups

Let us fix arbitrarily any operator $A \in \mathcal{U}$. By the polar decomposition [6], there is a partial isometry $W \in \mathcal{L}(\mathfrak{H}, \mathfrak{K})$ such that $A = WP$ where $P =: (A^*A)^{1/2}$ and $E =: W^*W$ is a projector onto the closure of the range of P. Let $\operatorname{tgh}(t) = \sum_{n=0}^{\infty} a_{2n+1} t^{2n+1}$, $t \in \mathbf{R}$, be the power series expansion of the function hyperbolic tangent tgh and define

$$\operatorname{tgh}(tP) =: \sum_{n=0}^{\infty} a_{2n+1}(tP)^{2n+1}, \quad t \in \mathbf{R}.$$

Then $\operatorname{tgh}(tP) \in \mathcal{L}(\mathfrak{H})$ and $\|\operatorname{tgh}(tP)\| \leqq \operatorname{tgh}\|tP\| < 1$ for all $t \in \mathbf{R}$.

2.1. Proposition. *For $t \in \mathbf{R}$, the operator $F(t) =: W \operatorname{tgh}(tP)$ satisfies $F(t) \in B(\mathcal{U})$ and the mapping $t \mapsto F(t)$ is continuous.*

Proof. One has

$$F(t)=W\,\mathrm{tgh}\,(tP)=W\sum_{n=0}^{\infty}a_{2n+1}(tP)^{2n+1}=$$

$$=\sum_{n=0}^{\infty}a_{2n+1}t^{2n+1}WP(A^*A)^n=\sum_{n=0}^{\infty}a_{2n+1}t^{2n+1}A(A^*A)^n\in\mathcal{U}.$$

Moreover, $\|F(t)\|\leq\|W\|\cdot\|\mathrm{tgh}\,(tP)\|<1$ so that $F(t)\in B(\mathcal{U})$. Obviously, $t\mapsto F(t)$ is continuous.

2.2. Proposition. *Let the operators M, $N\in B(\mathcal{U})$ be given with*

$$MN^*=NM^*,\quad M^*N=N^*M. \tag{5}$$

Then we have $T_M\circ T_N=T_{T_M(N)}$.

Proof. By Cartan's uniqueness theorem, it suffices to show that the automorphisms $T_M\circ T_N$ and $T_{T_M(N)}$ have the same image and the same derivative at the origin 0.

From (2) we obtain $(T_M\circ T_N)0=T_M(N)=T_{T_M(N)}(0)$. On the other hand, from (3) we get

$$dT_{T_M(N)}(0)X=\big(1-T_M(N)T_M(N)^*\big)^{1/2}X\big(1-T_M(N)^*T_M(N)\big)^{1/2} \tag{6}$$

where, by ([1], p. 22)

$$1-T_M(N)^*T_M(N)= \tag{7}$$

$$=(1-M^*M)^{1/2}(1+N^*M)^{-1}(1-N^*N)(1+M^*N)^{-1}(1-M^*M)^{1/2}.$$

Using (2) together with (7) we obtain

$$1-T_M(N)T_M(N)^*=1-T_{M^*}(N^*)^*T_{M^*}(N^*)= \tag{7'}$$

$$=(1-MM^*)^{1/2}(1+NM^*)^{-1}(1-NN^*)(1+NM^*)^{-1}(1-MM^*)^{1/2}.$$

Now, from the assumption (5) we see that the operators MM^*, NM^* and NN^* commute; thus the operators $(1-MM^*)^{1/2}$, $(1+NM^*)^{-1}$ and $(1-NN^*)$ also commute and (7′) yields

$$1-T_M(N)T_M(N)^*=(1-MM^*)(1+NM^*)^{-2}(1-NN^*),$$

whence

$$[1-T_M(N)T_M(N)^*]^{1/2}=(1-MM^*)^{1/2}(1+NM^*)^{-1}(1-NN^*)^{1/2}.$$

In a similar manner

$$[1-T_M(N)^*T_M(N)]^{1/2}=(1-N^*N)^{1/2}(1+MN^*)^{-1}(1-M^*M)^{1/2}.$$

Substitution in (6) gives

(8) $$dT_{T_M(N)}(0)X =$$

$$= (1-MM^*)^{1/2}(1+NM^*)^{-1}(1-NN^*)^{1/2}X(1-N^*N)^{1/2}(1+MN^*)^{-1}(1-M^*M)^{1/2}.$$

By the chain rule and (3) we have

$$dT_M \circ T_N(0)X = dT_M(N)\circ dT_N(0)X =$$

$$= (1-MM^*)^{1/2}(1+NM^*)^{-1}(1-NN^*)^{1/2}X(1-N^*N)^{1/2}(1+MN^*)^{-1}(1-M^*N)^{1/2}$$

which is the same as (8).

Let us fix any $A\in\mathcal{U}$ and consider the operator

$$F(t) =: W\,\mathrm{tgh}\,(tP)\in B(\mathcal{U}), \quad t\in\mathbf{R}.$$

2.3. Proposition. *The mapping* $\mathbf{R}\to\mathrm{Aut}\,B(\mathcal{U})$ *given by* $t\mapsto T_{F(t)}$ *is a continuous one-parameter group of Möbius transformations.*

Proof. Since the mappings $\mathbf{R}\to B(\mathcal{U})$ and $B(\mathcal{U})\to\mathrm{Aut}\,B(\mathcal{U})$ given respectively by $t\mapsto F(t)$ and $M\mapsto T_M$ are continuous, so is the composite.

Obviously, we have $T_{F(0)}=\mathrm{id}_{B(\mathcal{U})}$. Let us fix $s, t\in\mathbf{R}$ arbitrarily. As $E=W^*W$ is a projector onto the (closure of) the range of P, the operators $M=:F(s)$ and $N=:F(t)$ satisfy

$$MN^* = W\,\mathrm{tgh}\,(sP)\,\mathrm{tgh}\,(tP)W^* = W\,\mathrm{tgh}\,(tP)\,\mathrm{tgh}\,(sP)W^* = NM^*,$$

$$M^*N = \mathrm{tgh}\,(sP)W^*W\,\mathrm{tgh}\,(tP) = \mathrm{tgh}\,(sP)\,\mathrm{tgh}\,(tP) = \mathrm{tgh}\,(tP)\,\mathrm{tgh}\,(sP) =$$

$$= \mathrm{tgh}\,(tP)W^*W\,\mathrm{tgh}\,(sP) = N^*M,$$

and we can apply Proposition 2.2. Therefore

$$T_{F(s)}\circ T_{F(t)} = T_M\circ T_N = T_{T_M(N)}$$

and, in order to obtain the result, it suffices to show that

$$T_M(N) = W\,\mathrm{tgh}\,(s+t)P.$$

By the spectral calculus we have

$$(N+M)(1+M^*N)^{-1} = W(\mathrm{tgh}\,tP+\mathrm{tgh}\,sP)(1+\mathrm{tgh}\,tP\,\mathrm{tgh}\,sP)^{-1} =$$

$$= W\,\mathrm{tgh}\,(s+t)P.$$

Since the operator $\mathrm{tgh}\,(s+t)P$ obviously commutes with $(1-M^*M)^{1/2}=$

$=(1-\mathrm{tgh}^2\, tP)^{1/2}$, we have

$$T_M(N) = (1-MM^*)^{1/2}(N+M)(1+M^*N)^{-1}(1-M^*M)^{1/2} = \tag{9}$$

$$= (1-MM^*)^{1/2}[W\,\mathrm{tgh}\,(s+t)P](1-M^*M)^{1/2} =$$

$$= (1-MM^*)^{-1/2}W(1-M^*M)^{1/2}\,\mathrm{tgh}\,(s+t)P.$$

As W is a partial isometry, we have $WE=WW^*W=W$ and, as E is a projector onto the range of P, $EP=P=PE$. Therefore

$$E\,\mathrm{tgh}\,(sP) = \mathrm{tgh}\,(sP) = (\mathrm{tgh}\,sP)E.$$

Let us set $Q=:\mathrm{tgh}\,(sP)$. Then

$$1-MM^* = 1-WQ^2W^*;\quad 1-M^*M = 1-Q^2$$

and

$$(1-MM^*)^{1/2}W(1-M^*M)^{1/2} = (1-WQ^2W^*)^{-1/2}W(1-Q^2)^{1/2} =$$

$$= \left[\sum_{n=0}^{\infty}(-1)^n\binom{-1/2}{n}(WQ^2W^*)^n\right]W(1-Q^2)^{1/2} =$$

$$= \left[\sum_{n=0}^{\infty}(-1)^n\binom{-1/2}{n}WQ^{2n}E\right](1-Q^2)^{1/2} =$$

$$= \left[\sum_{n=0}^{\infty}(-1)^n\binom{-1/2}{n}WEQ^{2n}\right](1-Q^2)^{1/2} =$$

$$= WE(1-Q^2)^{-1/2}(1-Q^2)^{1/2} = W.$$

Substitution in (9) gives the result.

2.4. Theorem. *Let $\mathcal{U}$ be any J^*-algebra. Let $A\in\mathcal{U}$ be arbitrarily given and write $P=:(A^*A)^{1/2}$, $A=WP$ and $F(t)=:W\,\mathrm{tgh}\,(tP)$ for $t\in\mathbf{R}$. Then, the mapping $\mathbf{R}\to\mathrm{Aut}\,B(\mathcal{U})$ given by $t\mapsto T_{F(t)}$ is a continuous one-parameter group of automorphisms of $B(\mathcal{U})$ whose associated vector field is $f_A(X)\dfrac{\partial}{\partial X}=(A-XA^*X)\dfrac{\partial}{\partial X}$.*

Proof. By Proposition 2.3, the mapping $t\mapsto T_{F(t)}$ is a continuous one-parameter group of Möbius transformations. Therefore, the mapping

$$X\mapsto \frac{d}{dt}\bigg|_0 T_{F(t)}(X),\quad X\in B(\mathcal{U})$$

is a holomorphic vector field that is complete in $B(\mathcal{U})$. Now an easy calculation gives

$$\frac{d}{dt}\bigg|_0 T_{F(t)}(X) = A-XPW^*X = A-X(WP)^*X = A-XA^*X.$$

2.5. Corollary. *If $\mathcal{U}$ is any J^*-algebra and $A\in\mathcal{U}$, then $t\mapsto T_{F(t)}$ is the maximal solution of the initial value problem*

$$\frac{d}{dt}y(t) = A - y(t)A^*y(t), \quad y(0) = X, \quad Y(t)\in B(\mathcal{U})$$

for $X\in B(\mathcal{U})$.

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