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Abstract Galois theory and endotheory. II

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5. Abstract fields and endofields; isomorphism and homomorphism theorems

Let $S=(E, R)$ be a structure, and consider the class $\bar{R}$ of relations preserved by each $\sigma\in G(E/S)$ and the class $\bar{\bar{R}}$ of relations stabilized by each $\delta\in D(E/S)$. These classes are closed with respect to the fundamental and direct fundamental operations, respectively, and they are the smallest classes having this property. Really, if $\varrho\supseteqq R$ is a class closed with respect to the fundamental operations and X^0 is a set such that $\operatorname{card} X^0\geqq\operatorname{card} E$ and R is under X^0 then ϱ includes $R_f^{(X^0)}$. By Remark 1 of Section 4, $R_f^{(X^0)}=\bar{R}^{(X^0)}$. As each $r\in\bar{R}$ belongs to some $\bar{R}^{(X^0)}$, we have $\bar{R}\subseteqq\varrho$. The case of $\bar{\bar{R}}$ can be handled similarly. Therefore, for every set R of relations, the *closure* of R with respect to all fundamental or to all direct fundamental operations is well-defined.

Now let ϱ be a class of relations which is closed with respect to fundamental or, respectively, to direct fundamental operations. Let G be the group of permutations of E that preserve each $r\in\varrho$, and let D be the monoid of self-mappings of E that stabilize each $r\in\varrho$. The semi-regular decomposition R_r of each $r\in\varrho$ (cf. Section 2) is included in ϱ. Further, and arbitrary $\sigma\in S(E)$ preserves r iff it preserves every relation in R_r, and an arbitrary $\delta\in D(E)$ stabilizes r iff it stabilizes every relation in R_r. So G and D are completely determined by the semi-regular relations belonging to ϱ. Let r be a semi-regular relation in ϱ, let $P\in t(r)$, and let $\tilde{P}$: $\tilde{X}\to E$ be a fixed bijective point. Then $(\varepsilon_{P,\tilde{P}})\cdot r$ is an $\tilde{X}_P$-relation belonging to ϱ, where $\tilde{X}_P=\tilde{P}^{-1}P\cdot E\subseteqq\tilde{X}$. Clearly, the same permutations $\sigma\in S(E)$ preserve and the same $\delta\in D(E)$ stabilize $(\varepsilon_{P,\tilde{P}})\cdot r$ as r; and $(\varepsilon_{P,\tilde{P}})\cdot r$ is a relation under $\tilde{X}$. So G and D are already determined by $\varrho\cap R^{(\tilde{X})}$, which is a set of relations under $\tilde{X}$. In fact, $G=G(E/\varrho\cap R^{(\tilde{X})})$ and $D=D(E/\varrho\cap R^{(\tilde{X})})$. Now put $R=\varrho\cap R^{(\tilde{X})}$. As $\varrho\subseteqq p-\operatorname{inv} G$

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or $\varrho \subseteqq s-\mathrm{inv}\, D$, we have $\varrho \subseteqq \bar{R}$ or $\varrho \subseteqq \bar{\bar{R}}$, whence $\varrho = \bar{R}$ or $\varrho = \bar{\bar{R}}$, respectively. So, finally, every closed (with respect to all fundamental or direct fundamental operations) class of relations *is the closure* (with respect to the same operations) *of some set* of relations; this set may even be supposed to be under $\tilde{X}$ where $\tilde{X}$ is an arbitrary set with power card E.

Classes of relations that are closed with respect to all fundamental or all direct fundamental operations will be called *abstract fields* and *abstract endofields* on E (or, in other words, with base set E), respectively. For a structure $S=(E, R)$, $\bar{R}=R_f$ and $\bar{\bar{R}}=R_{df}$ are the smallest abstract field and abstract endofield including R. They will be called the abstract field and abstract endofield generated by S, and will be denoted by $K(S)$ and $K_e(S)$, respectively. If k and K are abstract fields (resp. endofields) and $k \subseteqq K$ then k is said to be a subfield (resp. subendofield) of K or, in other words, K is called an extension or overfield (resp. overendofield) of k. The notation K/k, instead of $k \subseteqq K$, is also used. We have seen that every abstract field or endofield is generated by an appropriate structure S. Two structures, S and S', generate the same abstract field or endofield iff $S \sim S'$ or $S \underset{d}{\sim} S'$, respectively. More generally, $K(S) \subseteqq K(S')$ is equivalent to $S \leqq S'$, while $K_e(S) \subseteqq K_e(S')$ is equivalent to $S \underset{d}{\leqq} S'$. In particular, if K is an abstract field or endofield and $\mathrm{card}\, X^0 \geqq \mathrm{card}\, E$ then $K = K(K \cap R(E; X^0))$ or $K = K_e(K \cap R(E; X^0))$, respectively, and $k \subseteqq K$ is clearly equivalent to $k \cap R(E; X^0) \subseteqq$ $\subseteqq K \cap R(E; X^0)$. Given a set F of structures, we say, by abusing the language[1)], that the corresponding abstract fields $K(S)$ or endofields $K_e(S)$, $S \in F$, form a *set*. In this sense, all the abstract fields and all the abstract endofields on E form sets, denoted by $AF(E)$ and $AEF(E)$, respectively. In particular, if some set X^0 with the property $\mathrm{card}\, X^0 \geqq \mathrm{card}\, E$ is fixed then any abstract field or endofield K is uniquely determined by its part $K^{(X^0)} = K \cap R(E; X^0)$ under X^0, and the mapping $K \to K \cap$ $\cap R(E; X^0)$ preserves the inclusion. This allows us to say that one set of fields or endofields is included in another, and, also, to speak of mappings, the intersection and the join (alias compositium) of a given set of fields or endofields. That is, for example, $k \subseteqq K$ will mean $k \cap R(E; X^0) \subseteqq K \cap R(E; X^0)$, a mapping $K \cap R(E; X^0) \to$ $\to k \cap R(E; X^0)$ will be considered as a mapping $K \to k$, K will be called the intersection or join of a set F of (endo)fields iff $K^{(X^0)}$ is that of $F^{(X^0)} = \{k^{(X^0)}; k \in F\}$. Note that $\bigvee_{K \in F} K$ is the smallest (endo)field that includes every $K \in F$ and $\bigcap_{K \in F} K$ is the greatest (endo)field included in all $K \in F$. Further, $r \in \bigcap_{K \in F} K$ iff $r \in K$ for all $K \in F$.

1) In case we want to remain within the frame of Bernays—Gödel axiomatic system. There are other ways to found mathematics where no abuse or not this kind of abuse would occur in the present situation.

For an abstract field K, let $G(E/K)$ denote the group of all permutations of E that preserve each $r \in K$. If K is an abstract endofield, let $D(E/K)$ denote the monoid of self-mappings of E that stabilize each $r \in K$. Clearly, if $K=K(S)$ or $K=K_e(S)$ then $G(E/K)=G(E/S)$ or $D(E/K)=D(E/S)$, respectively. So K is the class of all relations preserved by every $\sigma \in G(E/K)$ or stabilized by every $\delta \in D(E/K)$, respectively. Thus $K \to G(E/K)$ is a bijection of $AF(E)$ onto the set of permutation groups on E, while $K \to D(E/K)$ is a bijection of $AEF(E)$ onto the set of monoids of mappings $E \to E$. These mappings, called *canonical Galois mappings*, are decreasing, i.e., $k \subseteq K$ implies $G(E/k) \supseteq G(E/K)$ or $D(E/k) \supseteq D(E/K)$, respectively.

Now, if K is an abstract endofield such that $D(E/K)$ happens to be a group then K is an abstract field and $G(E/K)=D(E/K)$. Really, if all $\sigma \in G(E/K)$ stabilize a relation then they preserve it (cf. Remark 1 in Section 1). Further, $AF(E) \subseteq AEF(E)$. Therefore a number of results for endofields that will be proved later are automatically valid for abstract fields, too. On the other hand, if K_e is an abstract endofield defined by a structure S, i.e., $K_e=K_e(S)$, then $K=K(S)$ is completely determined by K_e, i.e., K does not depend on the particular choice of S. Really, by Remark 1 of Section 1, $G(E/K)=G(E/S)$ is the greatest permutation group included in $D(E/S)=D(E/K_e)$. When K_e happens to be an abstract field then $K=K_e$. So the mapping $K_e \to K$, from $AEF(E)$ onto $AF(E)$, can be called the *canonical projection*.

Let K and K' be abstract endofields with respective base sets E and E'. (So, the base sets of points, relations, structures, etc. are no longer assumed to be fixed in the rest of this paragraph.) We shall speak of a mapping of K into another endofield K' only if it is describable, in terms of Bernays—Gödel axiomatism, as a class of pairs $(r, r') \in K \times K'$. This is the case if, for an arbitrary $r \in K$, the corresponding r' can be described in terms of set theory. A mapping (assumed to be admissible in the previous sense) η: $K \to K'$ will be called surjective if for each $r' \in K'$ there is an $r \in K$ such that $r'=\eta \cdot r$, and it is called injective if $r_1 \neq r_2 \in K$ implies $\eta \cdot r_1 \neq \eta \cdot r_2$. This η will be said to be a *homomorphism with respect to a fundamental operation* ω if, with ξ denoting the value of the argument of ω, ω is defined for $\eta \cdot \xi$ if it is defined for ξ and $\eta \cdot \omega(\xi)=\omega(\eta \cdot \xi)$. (Here ξ may be a set of relations in K, then $\eta \cdot \xi$ denotes $\{\eta \cdot r;\ r \in \xi\}$, or a single relation in K.)

Observation 1. If η is a homomorphism with respect to all projections, all contractions and the infinitary union then η is surely a mapping, i.e., η is describable in terms of the Bernays—Gödel system.

To prove this observation, put $D=D(E/K)$ and let $\tilde{P}$: $\tilde{X} \to E$ be a bijective point. By Remark 4 in Section 4, for each $r \in K$ there is a superposition ω of these three kinds of fundamental operations such that $r=\omega(\tilde{D} \cdot \tilde{P})$. But then ω is also defined for $\eta\cdot(\tilde{D}\cdot\tilde{P})$ and $\eta\cdot r=\eta\cdot\omega(\tilde{D}\cdot\tilde{P})=\omega(\eta\cdot(\tilde{D}\cdot\tilde{P}))$, i.e., η is completely determined by $\eta\cdot(\tilde{D}\cdot\tilde{P})$, the image of $\tilde{D}\cdot\tilde{P}$.

In spite of the above argument we should not think that for every $r'\in K'$ there exists a homomorphism with respect to the fundamental operations occurring in Observation 1 that sends $\tilde{D}\cdot\tilde{P}$ to r'. The reason is that $\omega(r')$ is not necessarily defined when $\omega(\tilde{D}\cdot\tilde{P})$ is, or $\omega(\tilde{D}\cdot\tilde{P})=\omega'(\tilde{D}\cdot\tilde{P})$ need not imply $\omega(r')=\omega'(r')$.

Remark 1. If η: $K\to K'$ is a homomorphism with respect to all contractions then the η-image of every X-relation in K is an X-relation again.

Really, a contraction $(\varphi\colon X\to Y)$ is defined only for X-relations and it is defined for all X-relations when it is a floatage (i.e., φ is a bijection), whence the assertion follows easily.

Remark 2. If η: $K\to K'$ is a surjective homomorphism with respect to all projections, all contractions and the infinitary union, as in Observation 1, then there exists a surjective point P': $\tilde{X}\to E'$ belonging to $\eta\cdot(D\cdot\tilde{P})$.

To prove this remark, observe that the operations $\mathrm{pr}_{\bar{X}}^{X}$ and $(\varphi\colon X\to Y)$ are punctual mappings of X-relations. If P: $X\to E$ is an X-point then $(P|\bar{X})\cdot\bar{X}\subseteq P\cdot X$ and $((\varphi)\cdot P)\cdot Y=P\cdot X$. Therefore, if r is an X-relation, $\mathrm{pr}_{\bar{X}}\cdot r$ and $(\varphi\colon X\to Y)\cdot r$ have surjective points only if r has. Similarly, $\bigcup_{r\in R} r$ has some surjective point iff there is an $r\in R$ having one. So any relation obtained from $\eta\cdot(D\cdot\tilde{P})$ by a superposition ω of projections, contractions and infinitary unions has surjective points only if $\eta\cdot(D\cdot\tilde{P})$ has. Since each $r\in K$ is of the form $\omega(D\cdot\tilde{P})$ for such a superposition ω, $\eta\cdot r=\eta\cdot\omega(D\cdot\tilde{P})=\omega(\eta\cdot(D\cdot\tilde{P}))$ and $\eta\cdot r$ has no surjective point when $\eta\cdot(D\cdot\tilde{P})$ does not have. But there are relations in K' having surjectiv points; indeed, the $D(E'/K')$-orbit of any surjective point P' contains P'. This proves Remark 2.

Now let K be an abstract endofield. A non-empty relation $r\in K$ is called *irreducible* in K if $\emptyset\neq r'\subset r$ holds for no relation $r'\in K$. A relation $r\in K$ is said to be *indecomposable* in K if for any set $R\subset K$ $\bigcup\cdot R=r$ implies $r\in R$. Every irreducible relation is clearly indecomposable.

Lemma 1. *A relation $r\in K$ is indecomposable iff it is the D-orbit of some point P: $X\to E$ where $D=D(E/K)$. If the D-orbit of some surjective point P is irreducible in K then D is a permutation group and K is an abstract field. Further, if K is an abstract endofield such that $D=D(E/K)$ is a group then all D-orbits are irreducible.*

Proof. Let $r\in K$ be indecomposable. As $r=\bigcup_{P\in r} D\cdot P$, there exists a point $P\in r$ such that $r=D\cdot P$. It is obvious that $D\cdot P$ is indecomposable. Now let P: $X\to E$ be a surjective point, and suppose D is not a permutation group. If we had $D\delta=D$ for all $\delta\in D$ then each element of the monoid D would have a left inverse and, as it is well-known from the elements of group theory, D would turn out to be a group (of permutations, of course). Hence there is a $\delta \in D$ such that $D\delta$ is a proper subset of D. Then the D-orbit $D \cdot (\delta \cdot P) = D\delta \cdot P$ of $\delta \cdot P$ is a non-empty relation in K and a proper subset of $D \cdot P$. This means that $D \cdot P$ is not irreducible. Hence if $D \cdot P$ is irreducible then D is a subgroup of $S(E)$ and K is an abstract field. Finally, if $D = D(E/K)$ is a group then any two D-orbits are disjoint or coincide, whence every D-orbit is irreducible in K. The proof is complete.

Let K and K' be abstract endofields on E and E', respectively, let $D = D(E/K)$ and $D' = D(E'/K')$ denote the corresponding stability monoids, and let $\eta \colon K \to K'$ be a mapping of K into K'. With these notations fixed, we prove four lemmas.

Lemma 2. *If η is a homomorphism with respect to the infinitary union then it preserves the inclusion $\subseteqq$ between relations and semi-commutes with the infinitary intersection. If, in addition, η is surjective and preserves the argument set of relations then $\eta \cdot \emptyset = \emptyset$, $\eta \cdot I(X, E) = I(X, E')$, and for each point P' on E' the D'-orbit $D' \cdot P'$ is the η-image of $D \cdot P$ for some point P on E.*

Proof. For $r, r' \in K$, $r \subseteqq r'$ we have $\eta \cdot r \cup \eta \cdot r' = \eta \cdot (r \cup r') = \eta \cdot r'$, i.e., $\eta \cdot r \subseteqq \subseteqq \eta \cdot r'$. If R is a set of relations, $r \in R$ and $R \subset K$, then $\bigcap \cdot R \subseteqq r$ for every $r \in R$ and we obtain $\eta \cdot (\bigcap \cdot R) \subseteqq \bigcap_{r \in R} \eta \cdot r = \bigcap \cdot (\eta \cdot R)$. If η is surjective and preserves the argument sets then $\eta \cdot \emptyset = \emptyset$ and $\eta \cdot I(X, E) = I(X, E')$ easily follow from the fact that η preserves the inclusion; the smallest and largest X-relations on E are obviously mapped on the smallest and largest ones on E'. Assume now that $\eta \cdot r = D' \cdot P'$ where $r \in K$ and P' is a point on E'. Then $\eta \cdot r = \eta \cdot \left(\bigcup_{P \in r} D \cdot P\right) = \bigcup_{P \in r} \eta \cdot (D \cdot P)$ and the indecomposability of $\eta \cdot r = D' \cdot P'$ in K' yield the existence of some $P \in r$ such that $D' \cdot P' = \eta \cdot (D \cdot P)$.

Lemma 3. *Suppose η is a homomorphism with respect to the infinitary union and intersection; further let $\eta \cdot \emptyset = \emptyset$ and $\eta \cdot I(X, E) = I(X, E')$ for any X. Then η is also a homomorphism with respect to the negation $\neg$, which is a partially defined operation on K. Moreover, if η is surjective and K happens to be an abstract field then the η-images of D-orbits are D'-orbits and K' is also an abstract field.*

Proof. If r is an X-relation and $r, \neg \cdot r \in K$ then $\eta \cdot r \cup \eta \cdot (\neg \cdot r) = \eta \cdot (r \cup (\neg \cdot r)) = = \eta \cdot I(X, E) = I(X, E')$ and $\eta \cdot r \cap \eta \cdot (\neg \cdot r) = \eta \cdot (r \cap (\neg \cdot r)) = \eta \cdot \emptyset = \emptyset$, whence $\eta \cdot (\neg \cdot r) = \neg \cdot (\eta \cdot r)$ follows. Now let η be assumed surjective and let K be an abstract field. For each $r' \in K'$ there is an $r \in K$ with $r' = \eta \cdot r$. As $\neg \cdot r$ also belongs to K, $\neg \cdot r' = \neg \cdot (\eta \cdot r) = \eta \cdot (\neg \cdot r) \in K'$, showing that K' is also an abstract field. The surjectivity of η readily yields that η sends indecomposable relations to indecomposable ones. Hence Lemma 1 applies and the proof is complete.

Lemma 4. *Assume that η is a homomorphism with respect to all dilatations and $\eta \cdot I(X, E) = I(X, E')$ for any X. Then the η-image of a multidiagonal $I_C(E) \in K$ is $I_C(E')$, a multidiagonal of the same pattern. If, in addition, $\eta \cdot \emptyset = \emptyset$, η is a homomorphism with respect to the intersection and all points of a relation r in K are injective then so are the points of $\eta \cdot r$.*

Proof. Let C be an equivalence relation on an argument set X, and let ψ denote the canonical surjection $X \to X^* = X/C$. We have $I_C(E) = [\psi] \cdot I(X^*, E)$, whence $\eta \cdot I_C(E) = [\psi] \cdot (\eta \cdot I(X^*, E)) = [\psi] \cdot I(X^*, E') = I_C(E')$. In particular, if $x, y \in X$ then $\text{ext}_X \cdot I(\{x, y\}, E)$ is a simple diagonal and $\eta \cdot (\text{ext}_X \cdot I(\{x, y\}, E)) = \text{ext}_X \cdot I(\{x, y\}, E')$. Observe that, for an X-relation r, all $P \in r$ are injective iff $r \cap \text{ext}_X \cdot I(\{x, y\}, E) = \emptyset$ for any two distinct elements x and y in X; and so this property is preserved by η.

Lemma 5. *If η is a homomorphism with respect to the infinitary union then the following two conditions are equivalent:*

(C) *if $R' \subseteq K'$, $r \in K$ and $r' = \bigcup \cdot R'$ equals $\eta \cdot r$ then there exists a mapping θ: $R' \to K$ such that $r = \bigcup \cdot (\theta \cdot R')$ and, for every $\varrho' \in R'$, $\eta \cdot (\theta \cdot \varrho') \subseteq \varrho'$;*

(D) *the η-image of every D-orbit $D \cdot P$ in K is a D'-orbit (on E') in K'.*

Proof. Assume (C) and let $r = D \cdot P$ be a D-orbit in K. Let $R' \subseteq K'$ be a set of relations such that $\eta \cdot r = \bigcup \cdot R'$. Consider a mapping θ according to (C). Then $r = \bigcup \cdot (\theta \cdot R')$ and the indecomposability of r in K (cf. Lemma 1) yield the existence of a $\varrho' \in R'$ such that $r = \theta \cdot \varrho'$. Therefore $\varrho' \subseteq r' = \eta \cdot r = \eta \cdot (\theta \cdot \varrho') \subseteq \varrho'$, i.e., $r' = \varrho'$. Thus r' is indecomposable in K' and Lemma 1 furnishes (D). Conversely, assume (D) and let $\eta \cdot r$ be equal to $r' = \bigcup \cdot R'$ for some $r \in K$ and $R' \subseteq K'$. As $r = \bigcup_{P \in r} D \cdot P$, we can define a mapping θ: $R' \to K$ by putting $\theta \cdot \varrho' = \bigcup_{\eta \cdot (D \cdot P) \subseteq \varrho'} D \cdot P$ for $\varrho' \in R'$. Then $r = \bigcup \cdot (\theta \cdot R')$ and $\eta \cdot (\theta \cdot \varrho') \subseteq \varrho'$ for every $\varrho' \in R'$. The proof of the lemma is done.

For two abstract endofields K and K', a mapping η: $K \to K'$ will be called an *isomorphism* of K onto K' if it is bijective and is a homomorphism with respect to all fundamental operations. (Note that, by Lemma 3 it is sufficient to require that η be a bijective homomorphism with respect to direct fundamental operations only.) As an isomorphism η is uniquely determined by the η-image of the D-orbit $D \cdot \tilde{P}$ of a bijective point $\tilde{P}$: $\tilde{X} \to E$, there are no logical difficulties in considering these mappings. The image $\eta \cdot (D \cdot \tilde{P})$ is a D'-orbit, as it follows from Lemmas 2 and 3. If K is an abstract field then, by Lemma 3, so is K'. Therefore, if K is an abstract field then D' is a permutation group on E' and $\eta \cdot (D \cdot \tilde{P}) = D' \cdot \tilde{P}'$ for some $\tilde{X}$-point $\tilde{P}'$ of E'. We claim that $\tilde{P}'$ is bijective. Since the points of $D \cdot \tilde{P}$ are injective, the same is true for $\eta \cdot (D \cdot \tilde{P}) = D' \cdot \tilde{P}'$ by Lemma 4. In particular, $\tilde{P}'$ is injective. If $\tilde{P}'$ is not surjective then there are a set $\hat{X} \supset \tilde{X}$ and a point $\hat{P} \in E'^{\hat{X}}$ such that $\hat{P}$ is still injective and $\tilde{P}'=(\hat{P}|\tilde{X})$. By applying the previous argument for $D'\cdot\hat{P}$ and η^{-1} we obtain that $\eta^{-1}\cdot(D'\cdot\hat{P})$ consists of injective points. But then $D\cdot\tilde{P}=$ $=\eta^{-1}\cdot(D'\cdot\tilde{P}')=\eta^{-1}\cdot(\mathrm{pr}^{\hat{X}}_{\tilde{X}}\cdot(D'\cdot\hat{P}))=\mathrm{pr}^{\hat{X}}_{\tilde{X}}\cdot(\eta^{-1}\cdot(D'\cdot\hat{P}))$ would contain no surjective point, which is a contradiction. Therefore $\tilde{P}'$ is surjective, whence it is bijective, indeed.

An obvious example of isomorphism of abstract endofields is the *transportation of structures*. For definition, let K be an abstract endofield on E and let s: $E\to E'$ be a bijection. This bijection induces a mapping (s): $r\to s\cdot r$ of K, and the class $s\cdot K=\{s\cdot r;\ r\in K\}$ is visibly closed under all direct fundamental operations, whence it is an abstract endofield. Further, (s) is a bijection of K onto $s\cdot K$, which, by Proposition 1 (1) of Section 3, commutes with all fundamental operations. Therefore (s) is an isomorphism of K onto $s\cdot K$, called the *transportation of structure* induced by s. If K is an abstract field then, clearly, so is $s\cdot K$.

Lemma 6. *If s: $E\to E'$ is a bijection and K is an abstract endofield on E then $D(E'/s\cdot K)=sD(E/K)s^{-1}$.*

Proof. Let $r\in K$ and let δ be a self-mapping of E. Then $\delta\cdot r\subseteq r$ iff $s\delta 1_E\cdot r\subseteq s\cdot r$ iff $s\delta(s^{-1}s)\cdot r\subseteq s\cdot r$ iff $s\delta s^{-1}\cdot(s\cdot r)\subseteq s\cdot r$, which proves the lemma.

Consequence. *When K happens to be an abstract field then $G(E'/s\cdot K)=$ $=sG(E/K)s^{-1}$.*

Theorem (the isomorphism theorem of abstract Galois theory). *Every isomorphism of an abstract field is a transportation of structure.*

Proof. We have seen that each $\tilde{X}$-point of E is of the form $\delta\cdot\tilde{P}$ for a suitable δ: $E\to E$. Considering $\varepsilon_{\delta\cdot\tilde{p},\tilde{p}}=(\tilde{P}^{-1}|\delta\cdot E)(\delta\cdot\tilde{P})$ we have $(\varepsilon_{\delta\cdot\tilde{p},\tilde{p}})\cdot(\delta\cdot\tilde{P})=(\tilde{P}|\tilde{X}_{\delta\cdot\tilde{p}})$, where $\tilde{X}_{\delta\cdot\tilde{p}}=\tilde{P}^{-1}\cdot(\delta\cdot E)$, and $\delta\cdot\tilde{P}=[\varepsilon_{\delta\cdot\tilde{p},\tilde{p}}]\cdot(\tilde{P}|\tilde{X}_{\delta\cdot\tilde{p}})$. Now, if σ is a permutation of E, so $\sigma\cdot E=E$ and $\tilde{X}_{\sigma\cdot\tilde{p}}=\tilde{X}$, these formulas turn into $\sigma\cdot\tilde{P}=[\varepsilon_{\sigma\cdot\tilde{p},\tilde{p}}]\cdot\tilde{P}$; i.e. $\sigma\circ\tilde{P}=P\circ\varepsilon_{\sigma\cdot\tilde{p},\tilde{p}}$ and $\varepsilon_{\sigma\cdot\tilde{p},\tilde{p}}$ is a permutation of $\tilde{X}$. So for any permutation σ of E there exists one (and only one) permutation $\varepsilon(\sigma)=\varepsilon_{\sigma\cdot\tilde{p},\tilde{p}}$ of $\tilde{X}$ such that $\sigma\circ\tilde{P}=$ $\tilde{P}\circ\varepsilon(\sigma)$, $\varepsilon(\sigma)$ being clearly dependent on the choice of $\tilde{P}$; further, for every permutation ε of $\tilde{X}$ there exists one and only one permutation $\sigma(\varepsilon)$ of E such that $\sigma(\varepsilon)\circ\tilde{P}=$ $=\tilde{P}\circ\varepsilon$. We obviously have $\varepsilon(\sigma)=\tilde{P}^{-1}\circ\sigma\circ\tilde{P}$ and $\sigma(\varepsilon)=\tilde{P}\circ\varepsilon\circ\tilde{P}^{-1}$. Let G stand for $G(E/K)$ and put $r=G\cdot\tilde{P}$. For $\varepsilon\in S(X)$ we have $[\varepsilon]\cdot r=[\varepsilon]\cdot(G\cdot\tilde{P})=G\cdot([\varepsilon]\cdot\tilde{P})=$ $=G\cdot(\sigma(\varepsilon)\cdot\tilde{P})=G\sigma(\varepsilon)\cdot\tilde{P}$ and, as $\tilde{P}$ is surjective, $[\varepsilon]\cdot r\cap r=G\sigma(\varepsilon)\cdot\tilde{P}\cap G\cdot\tilde{P}=$ $=(G\sigma(\varepsilon)\cap G)\cdot\tilde{P}$, which is either $r=G\cdot\tilde{P}$ or $\emptyset$ depending on $\sigma(\varepsilon)\in G$ or $\sigma(\varepsilon)\notin G$. But $\sigma(\varepsilon)\in G$ iff $\varepsilon=\varepsilon(\sigma(\varepsilon))=\tilde{P}^{-1}\circ\sigma(\varepsilon)\circ\tilde{P}$ belongs to $\tilde{P}^{-1}\circ G\circ\tilde{P}$. So $[\varepsilon]\cdot r\cap r=r$ if $\varepsilon\in\tilde{P}^{-1}G\tilde{P}$ and $[\varepsilon]\cdot r\cap r=\emptyset$ if $\varepsilon\notin\tilde{P}^{-1}G\tilde{P}$.

Now if η: $K\to K'$ is an isomorphism of an abstract field K on E onto an abstract endofield K' on E' then K' is also an abstract field and $\eta\cdot(G\cdot\tilde{P})=G'\cdot\tilde{P}'$ where $G'=G(E'/K')$ and $\tilde{P}'$: $\tilde{X}\to E'$ is a bijective point. For a permutation ε of E an analogous reasoning shows that $[\varepsilon]\cdot r'\cap r'$ is r' or $\emptyset$ according to $\varepsilon\in\tilde{P}'^{-1}G\tilde{P}'$ or $\varepsilon\notin\tilde{P}'^{-1}G\tilde{P}'$, where $r'=G'\cdot P'$.

Put $s=\tilde{P}'\tilde{P}^{-1}$: $E\to E'$, which is a bijection of E onto E'. We have $(s)\cdot(G\cdot\tilde{P})=$ $=s\cdot(G\cdot\tilde{P})=sG\cdot\tilde{P}=sGs^{-1}\cdot(s\cdot\tilde{P})=sGs^{-1}\cdot(\tilde{P}'\tilde{P}^{-1}\cdot\tilde{P})=sGs^{-1}\cdot\tilde{P}'\tilde{P}^{-1}\tilde{P}=sGs^{-1}\cdot\tilde{P}'$. Since η is a $K\to K'$ isomorphism, we have $[\varepsilon]\cdot r\cap r=r\Rightarrow[\varepsilon]\cdot(\eta\cdot r)\cap(\eta\cdot r)=\eta\cdot r$ and $[\varepsilon]\cdot r\cap r=\emptyset\Rightarrow[\varepsilon]\cdot(\eta\cdot r)\cap(\eta\cdot r)=\eta\cdot\emptyset=\emptyset$. So $\varepsilon\in\tilde{P}^{-1}G\tilde{P}$ iff $\varepsilon\in\tilde{P}'^{-1}G'\tilde{P}'$. Therefore we have $\tilde{P}^{-1}G\tilde{P}=\tilde{P}'^{-1}G'\tilde{P}'$ and $G'=(\tilde{P}'\tilde{P}^{-1})G(\tilde{P}'\tilde{P}^{-1})^{-1}=sGs^{-1}$. Thus $\eta\cdot(G\cdot\tilde{P})=$ $=G'\cdot\tilde{P}'=sGs^{-1}\cdot\tilde{P}'=sG\cdot(s^{-1}\cdot\tilde{P}')=sG\cdot(\tilde{P}\tilde{P}'^{-1}\cdot\tilde{P}')=sG\cdot\tilde{P}=s\cdot(G\cdot\tilde{P})=(s)\cdot(G\cdot\tilde{P})$. As $\eta\cdot(G\cdot\tilde{P})$ determines the isomorphism η, we have $\eta=(s)$, which completes the proof.

Starting from this theorem, it is easy to develop a formalism for abstract field extensions that I have already done in [1], i.e., an analogous counterpart of the classical Galois theory. Indeed, let K/k be an extension of abstract fields. An isomorphism η: $K\to K$ is called an *isomorphism of* K/k or an *isomorphism with respect to* k if its restriction to k is the identical mapping 1_k. If η is an isomorphism of K/k then it is induced by a bijection σ: $E\to E$ which preserves all $r\in k$, i.e., by a $\sigma\in G(E/k)$. Two isomorphisms of K/k, say (σ) and (τ) induced by $\sigma,\tau\in G(E/k)$, coincide if and only if for every $r\in K$ we have $\sigma\cdot r=(\sigma)\cdot r=(\tau)\cdot r=\tau\cdot r$, i.e. $\sigma^{-1}\tau\cdot r=r$, which is equivalent to $\sigma^{-1}\tau\in G(E/K)$ and also to $\tau\in\sigma G(E/K)$. Therefore if $G(K/k)$ denotes the set of isomorphisms of K/k then $\eta\to\{\sigma\in G(E/k);\ (\sigma)=\eta\}$ is a bijection of $G(K/k)$ onto $G(E/k)/G(E/K)$, the set of left residue classes of $G(E/k)$ modulo $G(E/K)$. The cardinal number $[K:k]=\operatorname{card}G(K/k)$ is called the *Galois degree* of K/k. Note that $[K:k]$ is equal to the index $(G(E/k):G(E/K))$ of $G(E/K)$ in $G(E/k)$. In case L, K and k are abstract fields and $L\supseteq K\supseteq k$ then L/k is called an (abstract) overextension of K/k while K/k is an (abstract) subextension of L/k. Every $\eta\in G(K/k)$ is induced by some $\eta'\in G(L/K)$; really, η is a transposition of structures induced by some $\sigma\in G(E/k)$, which induces an appropriate isomorphism η' of L/k. Clearly, $[L:k]=(G(E/k):G(E/L))=$ $=(G(E/k):G(E/K))(G(E/K):G(E/L))=[L:K][K:k]$. An abstract field extension K/k is called *normal* if $\eta\cdot K=K$ holds for every $\eta\in G(K/k)$, i.e., if every isomorphism of K/k is an automorphism. In case K/k is an abstract field extension then K/k is normal iff $\sigma\cdot K=K$ for all $\sigma\in G(E/k)$ (here we put (σ) instead of η), i.e., iff $\sigma G(E/K)\sigma^{-1}=G(E/\sigma\cdot K)=G(E/K)$. So K/k is normal iff $G(E/K)$ is invariant in $G(E/k)$. Let K/k be a normal extension; the second isomorphism theorem of group theory readily yields that the mapping $L\to G(K/L)$ is a decreasing bijection from the set $\{L;\ K\supseteq L\supseteq k\}$ of all intermediate abstract fields onto the set of all subgroups of $G(K/k)$, and L/k is normal iff $G(K/L)$ is invariant in $G(K/k)$. In case L/k is normal then each $\eta\in G(K/k)$ induces an automorphism $\bar{\eta}=(\eta|L)$ of L/k, and the mapping $\eta\to\bar{\eta}$ is a homomorphism of $G(K/k)$ onto $G(L/k)$ with the kernel $G(K/L)$. So $G(L/k)$ is canonically isomorphic to $G(K/k)/G(K/L)$.

Definition. Let K and K' be abstract endofields with respective base sets E and E'. A mapping η: $K \to K'$ is called a *homomorphism* of K *onto* K' if it is surjective, it is a homomorphism with respect to the infinitary union, all projections, all extensions, all contractions and all dilatations, and, further, it satisfies the following condition

(C) If $r' = \eta \cdot r = \bigcup \cdot R'$ for an arbitrary $r \in K$ and a set $R' \subseteqq K'$ then there exists a mapping θ: $R' \to K$ such that all $\theta \cdot \varrho'$ ($\varrho' \in R'$) have the same argument set, $r = \bigcup \cdot (\theta \cdot R')$ and, for every $\varrho' \in R'$, $\eta \cdot (\theta \cdot \varrho') \subseteqq \varrho'$.

Before formulating and proving a "homomorphism theorem" of abstract Galois endotheory, some special kinds of homomorphisms will be introduced.

1. Representative homomorphisms. Let D be a subsemigroup of $D(E)$, i.e., a semigroup of self-mappings of E. A surjection f: $E \to E'$ will be called a *representation* of D if $f \cdot x = f \cdot y$ implies $f \cdot (\delta \cdot x) = f \cdot (\delta \cdot y)$, for every $x, y \in E$ and $\delta \in D$.

When f is a representation of D and $e' \in E'$ then there is an $e \in E$ such that $e' = f \cdot e$ and $f \cdot (\delta \cdot e)$ does not depend on the particular choice of e. So δ^f: $e' = f \cdot e \to f \cdot (\delta \cdot e)$ is a self-mapping of E' such that $f\delta = \delta^f f$. Clearly, a surjection f: $E \to E'$ is a representation of D if and only if for each $\delta \in D$ there exists a δ^f such that the diagram

$$\text{(D)} \qquad \begin{array}{ccc} E & \xrightarrow{\delta} & E \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ E' & \xrightarrow{\delta^f} & E' \end{array}$$

commutes. We will write $D^f = \{\delta^f;\ \delta \in D\}$.

Proposition 1. *Let K be an abstract endofield with base set E, put $D = D(E/K)$, and let f: $E \to E'$ be a representation of D. Then the mapping (f): $r \to f \cdot r$ is a homomorphism of K onto an endofield K' where K' is the endofield determined by the property $D^f = D(E'/K')$.*

These kinds of homomorphisms will be called *representative*.

The proof requires the axiom of choice. By Proposition 1 of Section 3, f commutes with all operations required by the definition of homomorphisms between endofields. For any point P of E we have $(f) \cdot (D \cdot P) = f \cdot (D \cdot P) = fD \cdot P = \{f\delta \cdot P;\ \delta \in D\} = \{\delta^f f \cdot P;\ \delta \in D\} = D^f f \cdot P = D^f \cdot (f \cdot P)$. So the (f)-image of the D-orbit of P is the D^f-orbit of $f \cdot P$. Hence (f) satisfies (C) by Lemma 5. We have seen that (f) is a mapping of K into the abstract endofield K' defined by $D(E'/K') = D^f$. Now it has remained to show that this mapping is surjective, i.e., there is a point P: $X \to E$ such that $f \cdot P$ is a bijective point of E'. Take a bijective point $\tilde{P}'$: $\tilde{X}' \to E'$. As $f \cdot E = E'$, the axiom of choice yields the existence of a mapping h: $E' \to E$ such that $f \circ h = 1_{E'}$. So by putting $P = h \cdot \tilde{P}' = h \circ P'$ we have $f \cdot P = f \circ P = f \circ (h \circ \tilde{P}') = (f \circ h) \circ \tilde{P}' = 1_{E'} \circ \tilde{P}' = \tilde{P}'$, completing the proof.

2. *Norms and pseudo-norms.* Let K/k be an extension of abstract endofields with a base set E. For $r \in K$ the set of relations $\varrho \in k$ that includes r (as a subset) is not empty. The intersection of all these ϱ also belongs to k and it is the smallest relation in k that includes r. This relation will be called the *norm* of r in K/k (or, in other words, with respect to k), and will be denoted by $N_{K/k}(r)$. Yet, we need to consider a more general situation, too. Let $\bar{E}$ be a subset of E and let K and k be abstract endofields with respective base sets $\bar{E}$ and E. Put $\bar{D} = D(\bar{E}/K)$, $\Delta = D(E/k)$ and $\Delta_E = \{\delta \in \Delta;\ \delta \cdot \bar{E} \subseteq \bar{E}\}$. If $\bar{D}$ is a submonoid of $(\Delta_E | \bar{E}) = \{(\delta | \bar{E});\ \delta \in \Delta_E\}$ then K will be said to be a *pseudo-extension* of k, and K/k will be called a *pseudo-extension* of abstract endofields. As $\bar{E} \subseteq E$, the relations on $\bar{E}$ are relations on E as well. So, for each $r \in K$ there is a smallest relation in k that includes r, and it will still be denoted by $H_{K/k}(r)$ and called the *pseudo-norm* of r in K/k (or with respect to k). Clearly, $N_{K/k}(r) = \Delta \cdot r$. In particular, if $r = \bar{D} \cdot \bar{P}$ is the $\bar{D}$-orbit of some point $\bar{P}$ of $\bar{E}$ then $N_{K/k}(\bar{D} \cdot \bar{P}) = \Delta \cdot \bar{P}$. It is obvious that the mapping $N_{K/k}$: $r \to N_{K/k}(r)$ is a homomorphism with respect to the infinitary union, all projections, all contractions and all dilatations. But, generally, the pseudo-norm is not a homomorphism with respect to extensions and it is not a surjection of K onto k. (Note that the norm is always a surjection of K onto k since it is the identity mapping when restricted to k.) We shall study necessary and sufficient conditions for $N_{K/k}$ commuting with extensions or being surjective. As the pseudo-norm of a $\bar{D}$-orbit is a Δ-orbit, condition (C) is satisfied by $N_{K/k}$ in virtue of Lemma 5.

Lemma 7. *A pseudo-norm $N_{K/k}$ is a homomorphism with respect to all extensions if and only if $(\Delta | \bar{E}) = \Delta^{(E \to E)} \bar{D}$ where $\Delta^{(E \to E)} = \{\delta \in \Delta;\ \delta \cdot \bar{E} = E\}$. In particular, for a norm $N_{K/k}$, iff $\Delta = \Delta^{(s)} \cdot D$ where $D = D(E/K)$ and $\Delta^{(s)}$ is the monoid of all self-surjections of E.*

Proof. (The necessity part requires the axiom of choice.) As pseudo-norms commute with the infinitary unions, it suffices to prove the lemma only for $\bar{D}$-orbits. Let $\bar{P}$: $X \to \bar{E}$, $r = \bar{D} \cdot \bar{P}$, and let X' be a disjoint union $X' = X \dot{\cup} Y$. Then we have $N_{K/k}(r) = \Delta \cdot \bar{P}$, $\mathrm{ext}_{X'} \cdot r = \bar{D} \cdot \bar{P} \times \bar{E}^Y$, $\mathrm{ext}_{X'} \cdot N_{K/k}(r) = \Delta \cdot \bar{P} \times E^Y$ and $N_{K/k}(\mathrm{ext}_{X'} \cdot r) = \Delta \cdot (\bar{D} \cdot \bar{P} \times \bar{E}^Y)$. Denoting by ϱ this last relation, let us calculate it. According to the usual conventions, an X'-point P' will be written as an ordered pair (P, P^*) where $P = (P' | X)$ is an X-point and $P^* = (P' | Y)$ is a Y-point. Then

$$\varrho = \bigcup_{\delta \in \Delta} \delta \cdot (\bar{D} \cdot \bar{P} \times \bar{E}^Y) = \{(\delta \cdot (\bar{\delta} \cdot \bar{P}), \delta \cdot P^*);\ (\delta, \bar{\delta}, P^*) \in \Delta \times \bar{D} \times \bar{E}^Y\} =$$

$$= \bigcup_{\delta \in \Delta} \bigcup_{\bar{\delta} \in \bar{D}} (\{\delta \bar{\delta} \cdot \bar{P}\} \times (\delta \cdot \bar{E})^Y) = \bigcup_{P \in \Delta \cdot P} (\{P\} \times \bigcup_{\delta \in \theta(P)} (\delta \cdot \bar{E})^Y).$$

where $\theta(P)=\{\delta\in\Delta;\ (\exists\bar{\delta}\in\bar{D})(\delta\bar{\delta}\cdot\bar{P}=P)\}$ and, as $\bar{P}\cdot X\subseteq\bar{E}$, $\delta\bar{\delta}\cdot\bar{P}=(\delta\bar{\delta}|\bar{E})\cdot\bar{P}$. If $P\in\Delta\cdot\bar{P}=(\Delta|\bar{E})\cdot\bar{P}$ then there exists a $\hat{\delta}\colon \bar{E}\to E$, $\hat{\delta}\in(\Delta|\bar{E})$, such that $P=\hat{\delta}\cdot\bar{P}$. Further, if $\bar{P}\colon X\to\bar{E}$ is surjective then this $\hat{\delta}$ is unique and $\hat{\delta}\cdot\bar{P}=\delta\bar{\delta}\cdot\bar{P}$ implies $\hat{\delta}=\delta\bar{\delta}$. Hence, in this case, $\theta(P)=\theta(\hat{\delta})=\{\delta\in\Delta;\ (\exists\bar{\delta}\in\bar{D})(\delta\bar{\delta}=\hat{\delta})\}$. In the general case we have $\theta(\hat{\delta})\subseteq\theta(P)$. Since $\mathrm{ext}_{X'}\cdot N_{K/k}(r)=\Delta\cdot\bar{P}\times E^Y=\bigcup_{P\in\Delta\cdot P}(\{P\}\times E^Y)$, the equality $N_{K/k}(\mathrm{ext}_{X'}\cdot r)=\mathrm{ext}_{X'}\cdot N_{K/k}(r)$ holds for every $X'\supseteqq X$ and for every D-orbit $r=\bar{D}\cdot\bar{P}$ with argument set X if and only if for every $P\in\Delta\cdot\bar{P}$ and for every set Y we have $E^Y=\bigcup_{\delta\in\theta(\delta)}(\delta\cdot\bar{E})^Y$. When $\bar{P}\colon X\to\bar{E}$ is surjective, this condition turns into the following one: for every $\hat{\delta}\in(\Delta|\bar{E})$ and for every set Y we have $E^Y=\bigcup_{\delta\in\theta(\hat{\delta})}(\delta\cdot\bar{E})^Y$. Note that this later condition implies the former one for each $P\in\Delta\cdot\bar{P}$. So this is a condition we were looking for, i.e., a necessary and sufficient condition for $N_{K/k}$ commuting with all extensions. Further, this commutativity holds for all $r\in K$ if it holds for the $\bar{D}$-orbit of only one surjective point $\bar{P}$. This condition is certainly satisfied if, for each $\hat{\delta}\in(\Delta|\bar{E})$, there exists a $\delta\in\theta(\hat{\delta})$ such that $\delta\cdot\bar{E}=E$, i.e., if $\hat{\delta}=\delta\bar{\delta}\in\Delta^{(E\to E)}\bar{D}$, i.e., if $(\Delta|\bar{E})=\Delta^{(E\to E)}\bar{D}$. But, by Cantor's diagonal method and using the axiom of choice, we will prove that if $\delta\cdot\bar{E}\neq E$ holds for some fixed $\hat{\delta}\in(\Delta|\bar{E})$ with all $\delta\in\theta(\hat{\delta})$, i.e., if $(\Delta|\bar{E})\neq\Delta^{(E\to E)}\bar{D}$, then there exists a Y such that

$$E^Y\neq\bigcup_{\delta\in\theta(\hat{\delta})}(\delta\cdot\bar{E})^Y.$$

Indeed, if for all $\delta\in\theta(\hat{\delta})$ we have $\delta\cdot\bar{E}\neq E$, take a Y with card $Y\geqq\mathrm{card}\,\theta(\hat{\delta})$. Then there exists an injection $\psi\colon\theta(\hat{\delta})\to Y$. The set $(\delta\cdot\bar{E})^Y$ consists of all Y-points $Q\colon Y\to E$ satisfying $Q\cdot y\in\delta\cdot\bar{E}$ for any $y\in Y$. But if all $\delta\cdot\bar{E}$ differ from E then, by the axiom of choice, there is a Y-point Q of E such that $Q\cdot(\psi\cdot\delta)\in\delta\cdot\bar{E}$ for no $\delta\in\theta(\hat{\delta})$. So Q cannot belong to any $(\delta\cdot\bar{E})^Y$ and, consequently, does not belong to the union $\bigcup_{\delta\in\theta(\hat{\delta})}(\delta\cdot\bar{E})^Y$. Hence this union cannot be E^Y.

When the condition of this lemma is satisfied, the corresponding pseudo-norm or norm $N_{K/k}$ is said to be *regular*.

Lemma 8. *The pseudo-norm $N_{K/k}\colon K\to k$ is surjective if and only if there exist a subset $\bar{E}^*$ of $\bar{E}$ and $\delta,\delta'\in\Delta$ such that $(\delta|\bar{E}^*)\colon\bar{E}^*\to E$ is bijective and $\delta'(\delta|\bar{E}^*)=1_{\bar{E}^*}$.* (Note that in case $N_{K/k}$ is a norm, this condition is always satisfied by $\bar{E}^*=\bar{E}=E$ and $\delta=\delta'=1_E$.)

Proof. Let $\tilde{P}\colon\tilde{X}\to E$ be a bijective point of E, and assume that $N_{K/k}$ is surjective. Then there is an $r\in K$ such that $N_{K/k}(r)=\Delta\cdot\tilde{P}$. Further, by Lemma 2, this r can be chosen to be a $\bar{D}$-orbit $\bar{D}\cdot\bar{P}$. But then $N_{K/k}(\bar{D}\cdot\bar{P})=\Delta\cdot\bar{P}$ implies $\Delta\cdot\bar{P}=\Delta\cdot\tilde{P}$. As $\bar{P}\in\Delta\cdot\bar{P}$ and $\tilde{P}\in\Delta\cdot\tilde{P}$, this equality yields the existence of some δ

and δ' in Δ such that $\tilde{P}=\delta\cdot\bar{P}$ and $\bar{P}=\delta'\cdot\tilde{P}$. Since $\Delta\delta$ and $\Delta\delta'$ are subsets of Δ, the existence of these δ and δ' is, in fact, equivalent to the equation $\Delta\cdot\bar{P}=\Delta\cdot\tilde{P}$. The point $\bar{P}$ is a mapping of $\tilde{X}$ into $\bar{E}$, whence $\bar{E}^*=\bar{P}\cdot\tilde{X}\subset\bar{E}$. So $\delta\cdot\bar{E}^*=\delta\cdot(\bar{P}\cdot\tilde{X})= =(\delta\cdot\bar{P})\cdot\tilde{X}=\tilde{P}\cdot\tilde{X}=E$, and from the injectivity of $\tilde{P}=\delta\cdot\bar{P}=(\delta|\bar{P}\cdot\tilde{X})\cdot\bar{P}=(\delta|\bar{E}^*)\cdot\bar{P}$ we obtain that both $\bar{P}$ and $(\delta|\bar{E}^*)$ must be injective. That is, $(\delta|\bar{E}^*)$ is a bijection of $\bar{E}^*$ onto E. We have $\delta'(\delta|\bar{E}^*)\cdot\bar{P}=\delta'\cdot((\delta|\bar{E}^*)\cdot\bar{P})=\delta'\cdot(\delta\cdot\bar{P})=\delta'\cdot\tilde{P}=\bar{P}$. As $\bar{P}$ is injective and $\bar{E}^*=\bar{P}\cdot\tilde{X}$, we have $\delta'(\delta|\bar{E}^*)=1_{\bar{E}^*}$. Conversely, let $\bar{E}^*$, δ and δ' satisfy the conditions of the lemma. Then, as $\delta\in\Delta^{(\bar{E}^*\to E)}$, we have $(\delta|\bar{E}^*)\cdot\bar{E}^*=E$, and $\delta'(\delta|\bar{E}^*)=1_{\bar{E}^*}$ implies $\delta'\cdot E=\bar{E}^*\subseteqq\bar{E}$. So $\bar{P}=\delta'\cdot\tilde{P}$ is a point of $\bar{E}$, because $\bar{P}\cdot\tilde{X}=(\delta'\cdot\tilde{P})\cdot\tilde{X}=\delta'\cdot(\tilde{P}\cdot\tilde{X})=\delta'\cdot E=\bar{E}^*$. Now $\delta'(\delta|\bar{E}^*)=1_{\bar{E}^*}$ yields that both δ' and $(\delta|\bar{E}^*)$ are bijective and they are inverses of each other. So $(\delta|\bar{E}^*)\delta'=1_E$ and $\delta\cdot\bar{P}= =(\delta|\bar{E}^*)\cdot(\delta'\cdot\tilde{P})=(\delta|\bar{E}^*)\delta'\cdot\tilde{P}=1_E\cdot\tilde{P}=\tilde{P}$ and $N_{K/k}(\bar{D}\cdot\bar{P})=\Delta\cdot\tilde{P}$. Since $\Delta\cdot\tilde{P}$ generates k (cf. Remark 4 in Section 4), $N_{K/k}$ is surjective, indeed.

A pseudo-norm satisfying the conditions of Lemma 8 will be called a *quasi-norm* while the corresponding pseudo-extension will be called a *quasi-extension.* In particular, norms are always quasi-norms. We have seen that a pseudo-norm is a homomorphism of endofields iff it is a regular quasi-norm.

Remark 3. Let K/k be a pseudo-extension, and let E and $\bar{E}\subseteqq E$ be the base sets of k and K, respectively. Assume further that K/k is either a regular or a quasi-extension. Then card $\bar{E}=$card E.

To check this remark it is sufficient to show that card $\bar{E}\geqq$card E. If K/k is regular and $\Delta=D(E/k)$ then $\Delta^{(E\to E)}$ is not empty. Hence, by the axiom of choice, the assertion follows. In the other case, when K/k is a quasi-extension, there are a set $\bar{E}^*\subseteqq\bar{E}$ and a $\delta\in\Delta$ such that $(\delta|\bar{E}^*)$ is a bijection of $\bar{E}^*$ onto E and card $E=$card $\bar{E}^*\leqq$card $\bar{E}$.

Theorem. (Homomorphism theorem of abstract Galois endotheory.) *Let K and K' be abstract endofields with base sets E and E' and endomorphism monoids $D=D(E/K)$ and $D'=D(E'/K')$, respectively. Let η: $K\to K'$ be a homomorphism of K onto K'. Then there is a representation f: $E\to\bar{E}'\subseteqq E'$ of D such that $(f\cdot K)/K'$ is a regular quasi-extension and $\eta=N_{(f\cdot K)/K'}\circ(f)$.*

Proof. Let $\tilde{P}$: $\tilde{X}\to E$ be a bijective point. Then, by Lemma 5, η maps the D-orbit $D\cdot\tilde{P}$ onto the D'-orbit $D'\cdot P'$ of some $\tilde{X}$-point P': $\tilde{X}\to E'$. Let $f=P'\tilde{P}^{-1}$: $E\to E'$ and $\bar{E}'=P'\cdot\tilde{X}=f\cdot E$. Clearly, f is a surjection of E onto $\bar{E}'\subseteqq E'$ and $P'=f\cdot P$.

Let an arbitrary δ belong to D. Then $\delta\cdot\tilde{P}=[\varepsilon_{\delta\cdot P,\tilde{P}}]\cdot(\tilde{P}|\bar{X})$ where $\bar{X}=\tilde{P}^{-1}\cdot(\delta\cdot E)$. But the D-orbit of $\delta\cdot\tilde{P}$ is $D\cdot(\delta\cdot\tilde{P})=D\delta\cdot\tilde{P}\subseteqq D\cdot\tilde{P}$ as $D\delta\subseteqq D$. Since all mappings commute with projections and dilatations, we have $D\cdot(\delta\cdot\tilde{P})=D\cdot([\varepsilon_{\delta\cdot P,\tilde{P}}]\,\mathrm{pr}_{\bar{X}}\cdot\tilde{P})= =[\varepsilon_{\delta\cdot P,\tilde{P}}]\,\mathrm{pr}_{\bar{X}}\cdot(D\cdot\tilde{P})$. But η is a homomorphism with respect to the same operations. So

$$\eta\cdot(D\cdot(\delta\cdot\tilde{P}))=[\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}]\,\mathrm{pr}_{\tilde{X}}\cdot(\eta\cdot(D\cdot\tilde{P}))=$$
$$=[\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}]\,\mathrm{pr}_{\tilde{X}}\cdot(D'\cdot P')=D'\cdot([\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}]\cdot(P'|\bar{X})).$$

As the mapping f also commutes with projections and dilatations, we have

$$[\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}]\cdot(P'|\bar{X})=[\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}]\cdot(f\cdot\tilde{P}|\bar{X})=$$
$$=f\cdot([\varepsilon_{\delta\cdot\tilde{P},\tilde{P}}]\cdot(\tilde{P}|\bar{X}))=f\cdot(\delta\cdot\tilde{P})=f\delta\cdot\tilde{P},$$

whence $f\delta\cdot\tilde{P}\in\eta\cdot(D\cdot(\delta\cdot\tilde{P}))$. But, as $D\cdot(\delta\cdot\tilde{P})\subseteqq D\cdot\tilde{P}$ and η is a homomorphism with respect to $\cup$, Lemma 2 yields $\eta\cdot(D\cdot(\delta\cdot\tilde{P}))\subseteqq\eta\cdot(D\cdot\tilde{P})=D'\cdot P'=D'f\cdot\tilde{P}$. So there exists a $\delta'\in D'$ such that $f\delta\cdot\tilde{P}=\delta'f\cdot\tilde{P}=(\delta'|\bar{E}')f\cdot\tilde{P}$. The bijectivity of $\tilde{P}$ implies $f\delta=(\delta'|\bar{E}')f$. On the other hand, as $f\cdot E=\bar{E}'$ and $f\delta\cdot E=f\cdot(\delta\cdot E)\subseteqq\subseteqq f\cdot E=\bar{E}'$, we have $(\delta'|\bar{E}')\cdot\bar{E}'=(\delta'|\bar{E}')f\cdot E=f\delta\cdot E\subseteqq\bar{E}'$. That is, $(\delta'|\bar{E}')$ is a self-mapping of $\bar{E}'$. By this we have seen that, for each $\delta\in D$, there is a self-mapping $\bar{\delta}'$ of $\bar{E}'$ making the diagram (D) commutative, i.e., satisfying $f\delta=\bar{\delta}'f$. So f is a representation of D, and the preceeding $(\delta'|\bar{E}')$ is just δ^f. Further, as $\delta'\in D'$ and $\delta'\cdot\bar{E}'\subseteqq\bar{E}'$, we have that $(\delta'|\bar{E}')\in(D'_{\bar{E}'}|\bar{E}')$ and $D^f\subseteqq(D'_{\bar{E}'}|\bar{E}')$. Therefore $(f\cdot K)/K'$ is a pseudo-extension.

Consider a D-orbit $D\cdot P$. We have

$$D\cdot P=D\cdot([\varepsilon_{P,\tilde{P}}]\cdot(\tilde{P}|\tilde{X}_P))=[\varepsilon_{P,\tilde{P}}]\,\mathrm{pr}_{\tilde{X}_P}\cdot(D\cdot\tilde{P}).$$

So

$$\eta\cdot(D\cdot P)=[\varepsilon_{P,\tilde{P}}]\,\mathrm{pr}_{\tilde{X}_P}\cdot(D'\cdot P')=D'\cdot([\varepsilon_{P,\tilde{P}}]\,\mathrm{pr}_{\tilde{X}_P}\cdot(f\cdot\tilde{P}))=$$
$$=D'\cdot(f\cdot([\varepsilon_{P,\tilde{P}}]\,\mathrm{pr}_{\tilde{X}_P}\cdot\tilde{P}))=D'f\cdot P\supseteqq D^ff\cdot P=(f)\cdot(D\cdot P).$$

So, for any D-orbit $r=D\cdot P$, $\eta\cdot r$ is a D'-orbit containing the D^f-orbit $D^f\cdot(f\cdot P)= =(f)\cdot r$, whence it is the least relation in K' containing $(f)\cdot r$. Therefore $\eta\cdot r= =N_{(f\cdot K)/K'}((f)\cdot r)$; the same is true for every $r\in K$ since r is a union of D-orbits and both f and η commute with this union. We have seen that $\eta=N_{(f\cdot K)/K'}\circ(f)$.

It is easy to see that $N_{(f\cdot K)/K'}$ is a surjective mapping of $f\cdot K$ into K'; really, if $r'\in K'$ then there is an $r\in K$ such that $r'=\eta\cdot r=N_{(f\cdot K)/K'}((f)\cdot r)$ and $(f)\cdot r= =f\cdot r\in f\cdot K$. So $(f\cdot K)/K'$ is a quasiextension and $N_{(f\cdot K)/K'}$ is a quasi-norm. In order to show that it is also regular, let Y be arbitrary. Then

$$\eta\cdot(D\cdot\tilde{P}\times E^Y)=\eta\cdot(D\cdot\tilde{P})\times E'^Y=N_{(f\cdot K)/K'}(f\cdot(D\cdot\tilde{P}))\times E'^Y$$

and, as (f) commutes with dilatations,

$$\eta\cdot(D\cdot\tilde{P}\times E^Y)=N_{(f\cdot K)/K'}((f)\cdot(D\cdot\tilde{P}\times E^Y))=N_{(f\cdot K)/K'}(f\cdot(D\cdot\tilde{P})\times\bar{E}'^Y).$$

So $N_{(f\cdot K)/K'}(f\cdot(D\cdot\tilde{P})\times\bar{E}'^Y)=N_{(f\cdot K)/K'}(f\cdot(D\cdot\tilde{P}))\times\bar{E}'^Y$. But $f\cdot(D\cdot\tilde{P})=fD\cdot\tilde{P}= =D^ff\cdot\tilde{P}=D^f\cdot(f\cdot\tilde{P})$, and $f\cdot\tilde{P}\colon\tilde{X}\to\bar{E}'$ is a surjective point of $\bar{E}'$. So $N_{(f\cdot K)/K'}$ commutes with all extensions of the D^f-orbit of some surjective point, and we have seen in the proof of Lemma 7 that then the same is true for every relation $r\in f\cdot K$. Hence $N_{(f\cdot K)/K'}$ is regular. The proof of the homomorphism theorem of Galois endotheory is complete.

It would be interesting to see what the possible decompositions of a given homomorphism $\eta\colon K\to K'$, as products of a quasi-norm and a representative homomorphism, are. The formulate a result of this kind, let $\eta=N_{(f\cdot K)/K'}\circ(f)$ be one of these decompositions, and put $\bar{E}'=f\cdot E$. Then we have

Proposition 2. *Let*

$$\Delta'=\{\delta'\in D'=D(E'/K');\ (\exists\delta''\in D')((\delta''\delta'|\bar{E}')=(\delta''|\delta'\cdot\bar{E}')(\delta'|\bar{E}')=1_{E'})\}.$$

Then the set of all desired decompositions of η *is* $\{N_{(\delta' f\cdot K)/K'}\circ(\delta' f);\ \delta'\in\Delta'\}$.

Proof. Let P'' be some generating point of $\eta\cdot(D\cdot\tilde{P})=D'\cdot P'$, i.e., let P'' be a point with the property $D'\cdot P''=D'\cdot P'$. By the preceeding proof, if $f'=P''\tilde{P}^{-1}$ then $\eta=N_{(f'\cdot K)/K'}\circ(f')$. Now let $\eta=N_{(f'\cdot K)/K'}\circ(f')$ where f' is a representation of $D=D(E/K)$. Then $(f')\cdot(D\cdot\tilde{P})=D^{f'}\cdot(f'\cdot\tilde{P})$ and $D'\cdot P'=\eta\cdot(D\cdot\tilde{P})=$ $=N_{(f'\cdot K)/K'}(D^{f'}\cdot(f'\cdot\tilde{P}))=D'\cdot(f'\cdot\tilde{P})$. So $P''=f'\cdot\tilde{P}$ is a generating point of $\eta\cdot(D\cdot\tilde{P})$ and $f'=P''\tilde{P}^{-1}$. Therefore the considered decompositions correspond to different generating points of $\eta\cdot(D\cdot\tilde{P})$.

But if $P''\in\eta\cdot(D\cdot\tilde{P})=D'\cdot P'$ then there exists a $\delta'\in D'$ such that $P''=\delta'\cdot P'=$ $=\delta' f\cdot\tilde{P}$. Further, P'' is a generating point of $D'\cdot P'$ iff there exists a $\delta''\in D'$ such that $\delta''\cdot P''=P'$. As $P'\cdot\tilde{X}=\bar{E}'$, this means that $(\delta''\delta'|\bar{E}')=1_{E'}$. In this case $f'=P''\tilde{P}^{-1}=(\delta'\cdot P')\tilde{P}^{-1}=\delta'\circ(P'\tilde{P}^{-1})=\delta'\circ f=\delta' f$. Thus the proposition is proved.

Case of finite base sets. In the finite base set case $\operatorname{card}\bar{E}'=\operatorname{card}E'$ and $\operatorname{card}\bar{E}'\leqq\operatorname{card}\tilde{X}=\operatorname{card}\bar{E}$ imply that $\bar{E}'=E'$. So every quasi-norm is a norm. If $\eta\colon K\to K'$ is a homomorphism then, consequently, *we have* $\eta=N_{(f\cdot K)/K'}\circ(f)$ *where* $N_{(f\cdot K)/K'}$ *is a regular norm and* $f\colon E\to E'$ *is a representation of* D.

Proposition 3 (P. Lecomte). *If* $\eta=N_{(f\cdot K)/K'}\circ(f)$ *is a* $K\to K'$ *homomorphism such that* $f\colon E\to E'$ *is a representation of* $D=D(E/K)$ *(i.e.,* $N_{(f\cdot K)/K'}$ *is a regular norm) and if* η *is bijective then* f *is bijective, too, and* $f\cdot K=K'$. *That is, in this case* η *is a transportation of structures and, in particular, it is an isomorphism of* K *onto* K'.

Proof. If η is bijective then so is (f), too. But if $f\colon E\to E'$ is not bijective then there are $e_1, e_2\in E$, $e_1\neq e_2$, such that $f\cdot e_1=f\cdot e_2$. Let $\tilde{P}\colon\tilde{X}\to E$ be a bijective point and $x_1, x_2\in\tilde{X}$ such that $\tilde{P}\cdot x_1=e_1$ and $\tilde{P}\cdot x_2=e_2$. Consider the point $P\colon\tilde{X}\to E$ defined by $(P|\tilde{X}\setminus\{x_1,x_2\})=(\tilde{P}|\tilde{X}\setminus\{x_1,x_2\})$ and $P\cdot x_1=P\cdot x_2=e_1$. The D-orbits of P and $\tilde{P}$ are different, because $\tilde{P}\in D\cdot\tilde{P}$ is injective but no point in $D\cdot P$ can be injective. But if $e'=f\cdot e_1=f\cdot e_2$ then $(f)\cdot(D\cdot\tilde{P})=D^f\cdot(f\cdot\tilde{P})$ and $(f)\cdot(D\cdot P)=$ $=D^f\cdot(f\cdot P)$ coincide since for any $x\in\tilde{X}\setminus\{x_1,x_2\}$ we have $(f\cdot\tilde{P})\cdot x=f\cdot(\tilde{P}\cdot x)=$

$=f\cdot(P\cdot x)=(f\cdot P)\cdot x$ while $(f\cdot\tilde{P})\cdot x_i=f\cdot(\tilde{P}\cdot x_i)=f\cdot e_i=e'$ and $(f\cdot P)\cdot x_i= =f\cdot(P\cdot x_i)=f\cdot e_1=e'$, $i=1,2$. That is, when f is not injective then (f) is not either.

We have seen that (f): $K\to f\cdot K$ is a bijection, whence it is a surjection of K onto $f\cdot K$. But then $N_{(f\cdot K)/K'}$: $f\cdot K\to K'$ must be a bijection of $f\cdot K$ onto K'. If $r\in f\cdot K$ then $N_{(f\cdot K)/K'}\cdot r=N_{(f\cdot K)/K'}(N_{(f\cdot K)/K'}\cdot r)$ and the injectivity of $N_{(f\cdot K)/K'}$ yields $r=N_{(f\cdot K)/K'}\cdot r$. Thus $N_{(f\cdot K)/K'}$ is the identity mapping of K and $K'= =N_{(f\cdot K)/K'}\cdot K=K$. So $\eta=(f)$ is a transportation of structures. (Hence η is an isomorphism and so is η^{-1}.)

Consequence. *If K is an abstract endofield with finite base set then every bijective homomorphism of K is a transportation of structure (whence it is an isomorphism).*

Indeed, every quasi-norm is a norm in this case.

To close this paragraph we mention some open problems. Given an arbitrary bijective homomorphism η: $K\to K'$ of abstract endofields, is it always true that

(α) it is a transportation of structure?

(β) it is an isomorphism?

(γ) η^{-1} is a homomorphism of K' onto K?

and

(δ) the condition (γ) implies (α) and (β)? (In other words, can a regular quasi-norm be injective without being a representative homomorphism?)

6. Abstract Galois set theory

Let k be an abstract endofield on E. For $e\in E$ the relation $(x;e)=\{\{x\to e\}\}$ is independent of the particular choice of x up to restricted floating equivalence; and it will be identified with e if considered modulo this equivalence. So the endo-extension of k generated by $(x;A)=\{(x;a);\ a\in A\}$ does not depend on the choice of x; it will be denoted by $k(A)$ and called the *set-extension of k generated by A* (or *by the adjunction of A*). Extensions of the form $k(A)/k$, where $A\subseteqq E$, are called set extensions; their study is called *abstract Galois set theory*. One of the main problems in this theory is to describe the set $\bar{A}_k=\{\bar{a}\in E;\ (x;\bar{a})\in k(A)\}$ in terms of k and A. This set $\bar{A}_k$ will be called the *rationality domain* of $k(A)$. Clearly, $\bar{A}_k$ is the set of all $e\in E$ preserved by every $\delta\in D(E/k(A))$ that fixes the points of A. Another problem, which has been studied only in some particular cases and will not be considered here, is to characterize the monoids of the form $D(E/k(A))$ or groups of the form $G(E/k(A))$ where $A\subseteqq E$.

Theorem 1. *Let A be a subset of E. The set-extension $k(A)$ is the class of all relations that are (infinitary) unions of relations of the form*

$$(\varrho)\qquad \mathrm{pr}_X\cdot\Big(r\cap\Big(\bigcap_{x\in X_0}\mathrm{ext}_X\cdot(x;\ \theta\cdot x)\Big)\Big)$$

where $\bar{X}$ is an argument set, $\bar{X}$ and $\bar{X}_0$ are subsets of X, $X=\bar{X}\cup X_0$, r is an X-relation in k, and θ: $X_0\to A$ is a mapping of X_0 into A.

Proof. Clearly, every relation of the considered form belongs to $k(A)$ as we have only used direct fundamental operations to obtain it from $r\in k\subseteqq k(A)$ and from certain $(x; a)$, $a\in A$. Further, every $r\in k$ is of this form (take $\bar{X}_0=\emptyset$ and $\bar{X}=X$), and so is every $(x; a)$, $a\in A$ (take $X=\{x\}=\bar{X}=X_0$ and θ: $x\to a$). So, to prove the theorem, it suffices to show that the considered class of relations is closed with respect to all direct fundamental operations. But before doing so let us make some remarks.

Remark 1. Let $X\supseteqq X'\supseteqq X_0$, let r be an X-relation on E, and let r_0 be an X_0-relation on E. Then $\mathrm{pr}_{X'}\cdot(r\cap \mathrm{ext}_X\cdot r_0)=\mathrm{pr}_{X'}\cdot r\cap \mathrm{ext}_{X'}\cdot r_0$.

Indeed, let P': $X'\to E$ be an X'-point. We have $P'\in\mathrm{pr}_{X'}\cdot(r\cap\mathrm{ext}_X\cdot r_0)$ iff there exists an X-point $P\in r$ such that $P'=(P|X')$ and $(P|X_0)\in r_0$. But this means that $P'\in\mathrm{pr}_{X'}\cdot r$ and $(P|X_0)=((P|X')|X_0)=(P'|X_0)$. So the additional condition $(P|X_0)\in r_0$ is equivalent to $(P'|X_0)\in r_0$, proving the remark.

Remark 2. A relation of the form (ϱ) but hurting the condition $X=\bar{X}\cup X_0$ can be represented as a relation fully being of the form (ϱ).

Indeed, as $\mathrm{ext}_X\cdot(x;\theta\cdot x)=\mathrm{ext}_X^{X_0}\,\mathrm{ext}_{X_0}\cdot(x;\theta\cdot x)$ and

$$\bigcap_{x\in X_0}\mathrm{ext}_X\cdot(x;\ \theta\cdot x)=\mathrm{ext}_X^{X_0}\cdot\bigcap_{x\in X_0}\mathrm{ext}_{X_0}\cdot(x;\ \theta\cdot x),$$

the relation (ϱ) can also be written as $\mathrm{pr}_{\bar{X}}\cdot(r\cap\mathrm{ext}_X\cdot r_0)$ with $r_0=\bigcap_{x\in X_0}\cdot(x;\theta\cdot x)$. Suppose $X\neq\bar{X}\cup X_0$ and put $X'=\bar{X}\cup X_0\supseteqq X_0$. Then, by Remark 1,

$$\mathrm{pr}_{\bar{X}}\cdot(r\cap\mathrm{ext}_X\cdot r_0)=\mathrm{pr}_{\bar{X}}^{X'}\,\mathrm{pr}_{X'}\cdot(r\cap\mathrm{ext}_X\cdot r_0)=$$

$$=\mathrm{pr}_{\bar{X}}^{X'}\cdot(\mathrm{pr}_{X'}\cdot r\cap\mathrm{ext}_{X'}\cdot r_0)=\mathrm{pr}_{\bar{X}}\cdot\big(\mathrm{pr}_{X'}\cdot r\cap\big(\bigcap_{x\in X_0}\mathrm{ext}_{X'}\cdot(x;\ \theta\cdot x)\big)\big).$$

Since $\mathrm{pr}_{X'}\cdot r\in k$, this last expression is also of the form (ϱ) with $X'=\bar{X}\cup X_0$ instead of X.

Remark 3. Let r be an X-relation, and let $\bar{X}\subseteqq X$, $X\cap Y=\emptyset$. Then $\mathrm{ext}_{\bar{X}\cup Y}\,\mathrm{pr}_{\bar{X}}\cdot r=\mathrm{pr}_{\bar{X}\cup Y}\,\mathrm{ext}_{X\cup Y}\cdot r$.

Indeed, as $\bar{X}\cap Y=\emptyset$, an $(\bar{X}\cup Y)$-point $\bar{P}^*$ can be represented by a uniquely determined pair $(\bar{P}, Q)$ where $\bar{P}$ is an $\bar{X}$-point, Q is a Y-point, and $\bar{P}^*=(\bar{P}, Q)\in\mathrm{ext}_{\bar{X}\cup Y}\,\mathrm{pr}_{\bar{X}}\cdot r$ is equivalent to $\bar{P}\in\mathrm{pr}_{\bar{X}}\cdot r$. But then $(\bar{P}, Q)=((P, Q)|\bar{X}\cup Y)$, and $P\in r$ is equivalent to $P^*=(P, Q)\in\mathrm{ext}_{X\cup Y}\cdot r$. So $\bar{P}^*\in\mathrm{ext}_{\bar{X}\cup Y}\,\mathrm{pr}_{\bar{X}}\cdot r$ is equivalent to the existence of an $(X\cup Y)$-point P^* such that $\bar{P}^*=(P^*|\bar{X}\cup Y)$ and $P^*\in\mathrm{ext}_{X\cup Y}\cdot r$, i.e., to $\bar{P}^*\in\mathrm{pr}_{\bar{X}\cup Y}\,\mathrm{ext}_{X\cup Y}\cdot r$. This proves the remark.

Now we can start to prove the theorem. The closedness with respect to the infinitary union needs no argument. As infinitary intersections distribute over infinitary unions and the rest of the direct fundamental operations commute with the infinitary union, it will be sufficient to prove that these direct fundamental operations applied to relations of the form (ϱ) yield relations of the same form.

(α) The case of the infinitary intersection. Let us have a set R of relations with a common argument set $\bar{X}$ and assume that each $\varrho\in R$ is equal to

$$\mathrm{pr}_{\bar{X}}\cdot(r(\varrho)\cap(\bigcap_{x\in X_0(\varrho)}\mathrm{ext}_{X(\varrho)}\cdot(x;\theta_\varrho\cdot x)))$$

where $r(\varrho)\in k$ is an $X(\varrho)$-relation, $X_0(\varrho)\subseteq X(\varrho)=\bar{X}\cup X_0(\varrho)$ and θ_ϱ is a mapping of $X_0(\varrho)$ into A. We will write $\varrho^*=r(\varrho)\cap(\bigcap_{x\in X_0(\varrho)}\cdot(\mathrm{ext}_{X(\varrho)}\cdot(x;\theta_\varrho\cdot x)))$. By Lemma 1 of Section 2 we have

$$\cap\cdot R=\bigcap_{\varrho\in R}\mathrm{pr}_{\bar{X}}\cdot\varrho^*=\mathrm{pr}_{\bar{X}}\cdot(\cap_f^{(\bar{X})}\cdot R^*)$$

where $\cap_f^{(\bar{X})}$ denotes the semi-free intersection of anchor $\bar{X}$ and $R^*=\{\varrho^*;\ \varrho\in R\}$. Let us study this semi-free intersection. Without changing ϱ, let us float the arguments in $X(\varrho)\setminus\bar{X}$ so that the sets $Y(\varrho)=X(\varrho)\setminus\bar{X}$ become pairwise disjoint; we can assume that this has already been done. Then $\cap_f^{(\bar{X})}$ turns, up to canonical identification, into the ordinary intersection. On the other hand, a floatage (φ) of ϱ^* does not affect the form of this relation; really, we have

$$(\varphi)\cdot\varrho^*=(\varphi)\cdot r(\varrho)\cap(\bigcap_{x\in X_0(\varrho)}\mathrm{ext}_{\varphi\cdot X(\varrho)}\cdot(\varphi\cdot x;\ \theta_\varrho\cdot x))=$$
$$=(\varphi)\cdot r(\varrho)\cap(\bigcap_{y\in\varphi\cdot X_0(\varrho)}\mathrm{ext}_{\varphi\cdot X(\varrho)}\cdot(y,\ \theta_\varrho\varphi^{-1}\cdot y)).$$

As $x\in X(\varrho)\setminus\bar{X}$ are the only floating arguments, the previous floatage preserves $\bar{X}$ and $\theta_\varrho\varphi^{-1}\cdot x=\theta_g\cdot x$ holds for every $x\in\bar{X}$. Suppose that this preliminary floatage has already been done and let us return to the previous notations. Let $Y(\varrho)=X(\varrho)\setminus\bar{X}$, $\bar{X}_0(\varrho)=\bar{X}_0(\varrho)\cap\bar{X}$, $Y=\bigcup_{\varrho\in R}Y(\varrho)$, $\bar{X}_0=\bigcup_{\varrho\in R}\bar{X}_0(\varrho)$, $X_0=\bar{X}_0\dot{\cup}Y$ and $X=\bigcup_{\varrho\in R}X(\varrho)=$ $=\bar{X}\dot{\cup}Y$ where $\dot{\cup}$ stands for the disjoint union. We have

$$\cap_f^{(\bar{X})}\cdot R^*=\bigcap_{\varrho\in R}\mathrm{ext}_X\cdot\varrho^*=\bigcap_{\varrho\in R}(\mathrm{ext}_X\cdot r(\varrho)\cap(\bigcap_{x\in X_0(\varrho)}\mathrm{ext}_X\cdot(x;\ \theta_\varrho\cdot x)))=$$
$$=(\bigcap_{\varrho\in R}\mathrm{ext}_X\cdot r(\varrho))\cap(\bigcap_{\varrho\in R}\bigcap_{x\in X_0(\varrho)}\mathrm{ext}_X\cdot(x;\ \theta_\varrho\cdot x)).$$

For $x\in\bar{X}_0$, let $R(x)=\{\varrho\in R;\ x\in\bar{X}_0(\varrho)\}$. Then the preceding expression turns into

$$(\bigcap_{\varrho\in R}\mathrm{ext}_X\cdot r(\varrho))\cap(\bigcap_{x\in\bar{X}_0}\bigcap_{\varrho\in R(x)}\mathrm{ext}_X\cdot(x;\ \theta_\varrho\cdot x))\cap(\bigcap_{\varrho\in R}\bigcap_{x\in Y(\varrho)}\mathrm{ext}_X\cdot(x;\ \theta_\varrho\cdot x)).$$

If this relation is empty then it belongs to k, and so does its $\bar{X}$-projection $\bigcap \cdot R$. If $\bigcap \cdot R \neq \emptyset$ then, for every $x \in \bar{X}_0$, $\bigcap_{\varrho \in R(x)} \mathrm{ext}_X^{\{x\}} \cdot (x; \theta_\varrho \cdot x) = \mathrm{ext}_X \cdot \bigcap_{\varrho \in R(x)} (x; \theta_\varrho \cdot x) \neq \emptyset$ and $\bigcap_{\varrho \in R(x)} (x; \theta_\varrho \cdot x) \neq \emptyset$. This means that $\theta_\varrho \cdot x$ does not depend on $\varrho \in R(x)$, so it will be denoted by $\theta \cdot x$. As Y is the disjoint union of $Y(\varrho)$, $\varrho \in R$, each $y \in Y$ belongs to exactly one $Y(\varrho)$; let $\theta \cdot y$ stand for the corresponding $\theta_\varrho \cdot y$. So θ: $x \to \theta \cdot x$ is a mapping of $X_0 = \bar{X}_0 \cup Y$ into A, and $\bigcap_{\varrho \in R(x)} (x; \theta_\varrho \cdot x) = (x; \theta \cdot x)$ for $x \in \bar{X}_0$ while $(x; \theta_\varrho \cdot x) = (x; \theta \cdot x)$ for $x \in Y$. Therefore, in case $\bigcap \cdot R \neq \emptyset$, we have

$$\bigcap_f^{(\bar{X})} \cdot R^* = \Big(\bigcap_{\varrho \in R} \mathrm{ext}_X \cdot r(\varrho)\Big) \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x;\, \theta \cdot x)\Big).$$

Hence, putting $r = \bigcap_{\varrho \in R} \mathrm{ext}_X \cdot r(\varrho) \in k$, we have $\bigcap \cdot R = \mathrm{pr}_{\bar{X}} \cdot \big(r \cap \big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x;\, \theta \cdot x)\big)\big)$. Consequently, $\bigcap \cdot R$ is of the form (ϱ). Besides, we have $X_0 \cup \bar{X} \supseteq Y \cup \bar{X} = X$, implying $X = X_0 \cup \bar{X}$.

(β) Projections. For $\hat{X} \subseteq \bar{X}$,

$$\mathrm{pr}_{\hat{X}} \cdot \Big(\mathrm{pr}_{\bar{X}} \cdot \Big(r \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x;\, \theta \cdot x)\Big)\Big)\Big) = \mathrm{pr}_{\hat{X}} \cdot \Big(r \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x;\, \theta \cdot x)\Big)\Big),$$

which is of the form (ϱ) again except that $\hat{X} \cup X_0$ may differ from X. But this is not essential by Remark 2 of this section.

(γ) Extensions. If $\bar{X}' \supseteq \bar{X}$ then, by Remark 3 of this section,

$$\mathrm{ext}_{\bar{X}'}^{\bar{X}} \mathrm{pr}_{\bar{X}} \cdot \Big(r \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x; \theta \cdot x)\Big)\Big) = \mathrm{pr}_{\bar{X}'} \cdot \Big(\mathrm{ext}_{X \cup \bar{X}'} \cdot \Big(r \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x; \theta \cdot x)\Big)\Big)\Big) =$$

$$= \mathrm{pr}_{\bar{X}'} \cdot \Big(\mathrm{ext}_{X \cup \bar{X}'} \cdot r \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_{X \cup \bar{X}'} \cdot (x;\, \theta \cdot x)\Big)\Big).$$

(ϱ) Contractions. Let $\varrho = \mathrm{pr}_{\bar{X}} \cdot \varrho^*$, where $\varrho^* = r \cap \big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x; \theta \cdot x)\big)$, and let the contraction $(\bar{\varphi}) = (\bar{\varphi}\colon \bar{X} \to \bar{Y})$ be applicable to ϱ. Let $T(\bar{\varphi})$ be the type of $\bar{\varphi}$, and let φ: $X \to Y = \bar{Y} \cup (X \setminus \bar{X})$ be a surjection such that $(\varphi|\bar{X}) = \bar{\varphi}$ and $(\varphi|(X \setminus \bar{X}))$ is the identity. ($\bar{Y}$ and $X \setminus \bar{X}$ are assumed to be disjoint as otherwise we may perform a floatage of the arguments in $X \setminus \bar{X}$.) Now $T(\bar{\varphi})$ and $T(\varphi)$ coincide on $\bar{X}$, and $T(\varphi)$ induces the discrete (i.e. the smallest) equivalence on $X \setminus \bar{X}$. As $\mathrm{pr}_{\bar{X}} \cdot \varrho^*$ is compatible with $\bar{\varphi}$, ϱ^* is compatible with $T(\varphi)$. So

$$\varrho^* = \big(\varrho^* \cap I_{T(\varphi)}(E)\big) = \big(r \cap I_{T(\varphi)}(E)\big) \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_X \cdot (x, \theta \cdot x)\Big),$$

and $r' = r \cap I_{T(\varphi)}(E)$ is also an X-relation of k. If $\varrho^* \neq \emptyset$ then we have $\theta \cdot x_1 = \theta \cdot x_2$ for $x_1, x_2 \in X_0 \setminus \bar{X}$, since otherwise $I_{T(\varphi)}(E) \cap \mathrm{ext}_X \cdot (x_1; \theta \cdot x_1) \cap \mathrm{ext}_X \cdot (x_2; \theta \cdot x_2)$ would be empty. As (φ) commutes with $\bigcap$ and $(\varphi) \cdot I_{T(\varphi)}(E) = E^Y$, we have

$$(\bar{\varphi}) \cdot \varrho = \mathrm{pr}_{\bar{\varphi}, \bar{X}} \cdot \big((\varphi) \cdot \varrho^*\big) = \mathrm{pr}_{\bar{Y}} \cdot \Big((\varphi) \cdot r' \cap \Big(\bigcap_{x \in X_0} \mathrm{ext}_Y \cdot (\varphi \cdot x;\, \theta \cdot x)\Big)\Big).$$

This means that $(\bar{\varphi})\cdot\varrho$ is of the form (ϱ), where $X, \bar{X}$ and X_0 are replaced by $Y=\varphi\cdot X=\bar{\varphi}\cdot\bar{X}\cup(X\setminus\bar{X})$, $\bar{Y}=\bar{\varphi}\cdot\bar{X}$ and $Y_0=\varphi\cdot X_0$, r is replaced by $(\varphi)\cdot r'=$ $=(\varphi)\cdot(r\cap I_{T(\varphi)}(E))$, and the X_0-point $\theta\colon X_0\to A$ is replaced by $(\varphi|X_0)\cdot\theta$, which is well-defined as θ is compatible with $(\varphi|X_0)$.

(ε) Dilatations. We preserve the meanings of X, $\bar{X}$, X_0, ϱ, ϱ^*, r and θ, but consider a surjection $\bar{\psi}\colon \bar{Y}\to\bar{X}$ instead of $\bar{\varphi}\colon \bar{X}\to\bar{Y}$. We suppose that, by some preliminary semi-free floatage of anchor $\bar{X}$, X is already transformed so that $(X\setminus\bar{X})\cap\bar{Y}=\emptyset$. Let $Y=\bar{Y}\dot{\cup}(X\setminus\bar{X})$, and let $\psi\colon Y\to X$ be the surjection for which $(\psi|\bar{X})=\bar{\psi}$ and $(\psi|(X\setminus\bar{X}))$ is the identity. Obviously, we have $[\bar{\psi}]\cdot\varrho=\mathrm{pr}_{\bar{Y}}\cdot([\psi]\cdot\varrho^*)$. But

$$[\psi]\cdot\varrho^*=[\psi]\cdot r\cap\Big(\bigcap_{x\in X_0}[\psi]\cdot\mathrm{ext}_X\cdot(x;\,\theta\cdot x)\Big),$$

and $[\psi]\cdot r\in k$. As it is easy to see,

$$[\psi]\cdot\mathrm{ext}_X\cdot(x;\,\theta\cdot x)=[\psi]\cdot\mathrm{ext}_X\cdot\{\{x\to\theta\cdot x\}\}=$$
$$=I_{T(\psi)}(E)\cap\Big(\bigcap_{y\in\psi^{-1}\cdot x}\mathrm{ext}_Y\cdot\{\{y\to\theta\cdot x=\theta\psi\cdot y\}\}=$$
$$=I_{T(\psi)}(E)\cap\Big(\bigcap_{y\in\psi^{-1}\cdot x}\mathrm{ext}_Y\cdot(y;\,\theta\psi\cdot y)\Big).$$

So, if $Y_0=\psi^{-1}\cdot X_0$ and $\hat{r}=[\psi]\cdot r\cap I_{T(\psi)}(E)$, we have

$$[\psi]\cdot\varrho^*=[\psi]\cdot r\cap I_{T(\psi)}(E)\cap\Big(\bigcap_{x\in X_0}\bigcap_{y\in\psi^{-1}\cdot x}\mathrm{ext}_Y\cdot(y;\,\theta\psi\cdot y)\Big)=$$
$$=\hat{r}\cap\Big(\bigcap_{y\in Y_0}\mathrm{ext}_Y(y;\,\theta\psi\cdot y)\Big).$$

This means that $[\bar{\psi}]\cdot\varrho$ is still of the form (ϱ), where X, X_0, $\bar{X}$, r and θ are replaced by $Y=\psi^{-1}\cdot X$, $Y_0=\psi^{-1}\cdot X_0$, $\bar{Y}=\psi^{-1}\cdot\bar{X}$, $\hat{r}=[\psi]\cdot r\cap I_{T(\psi)}(E)$ and $\theta\psi\colon Y\to A$. The theorem is proved.

Remark 4. The relation $\varrho^*=r\cap\big(\bigcap_{x\in X_0}\mathrm{ext}_X\cdot(x;\theta\cdot x)\big)$ is the set of all points $P\in r$ which extend the X_0-point $\theta\colon X_0\to A$ on X.

Remark 5. If k is an abstract field then so is $k(A)$. Therefore, in this case, the class of infinitary unions of relations of the form (ϱ) is also closed with respect to the negation $\neg$.

Really, every $\delta\in D(E/k)=G(E/k)$ is a permutation on E. But $\delta\cdot(x;a)\subseteqq(x;a)$ holds iff $\delta\cdot a=a$, which implies $\delta\cdot(x;a)=(x;a)$. Therefore every $\delta\in D(E/k(A))$ not only stabilizes but preserves every $(x;a)$, $a\in A$. Thus $D(E/k(A))=G(E/k(A))$ and $k(A)$ is an abstract field. The rest of the remark can be proved directly; note that even in case r is of the form (ϱ), $\neg r$ is an infinitary union of relations of the form (ϱ) in general.

Operations generated by relations. Let D be a subset of E^X. A mapping $\omega\colon D\to E$ will be called an *X-operation* (or *partial X-composition*) on E. The set D will be called the *(definition) domain* of ω. When $D=E^X$, ω is said to be a *complete* X-operation on E; the terms *X-function* of E and *X-polymapping* of E are also used.

Let $X'=X\dot{\cup}\{y\}$ and let $r\subseteqq E^{X'}$ be an X'-relation on E. The relation r and the argument y will define an X-operation $\omega_r^{(y)}\colon D_r^{(y)}\to E$ in the following way. An X'-point P' will often be denote by $(P, \{y\to e\})$ where $P=(P'|X)$ and $\{y\to e\}= =(P'|\{y\})$. If $P'\in r$ then $e=(P'|\{y\})\cdot y$ is called a *prolongation* of $P=(P'|X)$ in r. Clearly, $P\in E^X$ has prolongations in r iff $P\in \mathrm{pr}_X\cdot r$. Let $D_r^{(y)}$ be the set of all $P\in E^X$ that have exactly one prolongation in r. For $P\in D_r^{(y)}$ let $\omega_r^{(y)}\cdot P$ be the unique prolongation of P in r.

It is not hard to express $D_r^{(y)}$ from r by means of fundamental operations. Let y' be an argument, not in X', and let $\varphi\colon X'\to X\cup\{y'\}$ be the floatage for which $(\varphi|X)$ is the identity and $\varphi\cdot y=y'$. Put $X''=X\cup\{y,y'\}=X'\cup\{y'\}$, and let us consider the set of all points $P''\in \mathrm{ext}_{X''}\cdot r\cap \mathrm{ext}_{X''}\cdot((\varphi)\cdot r)$ such that $(P''|X)=P$. This is clearly the set of all points of the form $(P, \{y\to e\}, \{y'\to e'\})$, where e and e' are arbitrary prolongations of P in r. An X-point $P\in\mathrm{pr}_X\cdot r$ has several distinct prolongations in r iff

$$\mathrm{ext}_{X''}\cdot r\cap \mathrm{ext}_{X''}\cdot((\varphi)\cdot r) \quad\text{and}\quad \{P\}\times(\neg D_{y,y'}(E)) = \mathrm{ext}_{X''}\cdot\{P\}\cap(\neg \mathrm{ext}_{X''}\cdot D_{y,y'}(E))$$

are not disjoint. So the set of all these points is $\mathrm{pr}_X\cdot(\mathrm{ext}_{X''}\cdot r\cap\mathrm{ext}_{X''}\cdot((\varphi)\cdot r)\cap \cap(\neg\mathrm{ext}_{X''}\cdot D_{y,y'}(E)))$. Therefore we have

$$D_r^{(y)} = (\mathrm{pr}_X\cdot r)\cap(\neg\mathrm{pr}_X\cdot(\mathrm{ext}_{X''}\cdot r\cap\mathrm{ext}_{X''}\cdot((\varphi)\cdot r)\cap(\neg\mathrm{ext}_{X''}\cdot D_{y,y'}(E)))).$$

In case X is empty, i.e. $X'=\{y\}$, $D_r^{(y)}$ is non-empty iff the unique "empty" point $P_\varnothing$ has a unique prolongation e in r, i.e., $r=(y;e)$. Then $\omega_r^{(y)}\cdot P_\varnothing=e$, and $\omega_r^{(y)}$ is, in fact the adjunction of the element $e\in E$.

We say that $A\subseteqq E$ is closed with respect to an X-operation $\omega\colon D\to E$ if $\omega\cdot P\in A$ holds whenever $P\colon X\to A$ and $P\in D$.

Theorem 2. *The rationality domain $\bar{A}_k$ of $k(A)$ is the closure of A with respect to all operations $\omega_r^{(y)}$ such that $r\in k$ and y belongs to the argument set of r.*

Proof. Let X_r be the argument set of a relation $r\in k$, and let $y\in X_r$. Put $X=:X_r\setminus\{y\}$, and let $P\colon X\to\bar{A}_k$ be a point belonging to $D_r^{(y)}$. If $e=e(P)=\omega_r^{(y)}\cdot P$ then

$$\mathrm{pr}_{\{y\}}\cdot((\{P\}\times E^{\{y\}})\cap r) = \{\{y\to e\}\} = (y;\ e).$$

As $\{P\}\times E^{\{y\}} = \bigcap_{x\in E}\mathrm{ext}_{X_r}\cdot(x;P\cdot x)$ and $P\cdot x\in\bar{A}_k$, we have $(x;P\cdot x)\in k(A)$. Since $(y;e)$ is obtained from $(x;P\cdot x)\in k(A)$, $x\in X$, and from $r\in k(A)$ by direct fundamental operations, $(y;e)$ belongs to $k(A)$. Therefore $\omega_r^{(y)}\cdot P=e(P)\in\bar{A}_k$, and $\bar{A}_k$ is closed with respect to all the mentioned operations $\omega_r^{(y)}$.

Now suppose that $e\in\bar{A}_k$, i.e. $(y;e)\in k(A)$. By the previous theorem, $(y;e)$ is the infinitary union of an appropriate set of $\{y\}$-relations of the form

$$\varrho = \mathrm{pr}_{\{y\}}\cdot(r\cap(\bigcap_{x\in X_0}\mathrm{ext}_X\cdot(x;\ \theta\cdot x)))$$

where $X=\{y\}\cup X_0$, r is an X-relation in k and $\theta\colon X_0\to A$ is a mapping of X_0 into $A\subseteqq\bar{A}_k$. As $(y;e)$ is irreducible, it must be equal to some of these relations. So we assume that $(y;e)$ is the above-mentioned ϱ. If $y\in X_0$ then $e=\varrho\cdot y==\theta\cdot y\in A$. If $y\notin X_0$ then $\varrho^*=r\cap(\bigcap_{x\in X_0}\mathrm{ext}_X\cdot(x;\theta\cdot x))$ is the set of X-points $P=(P_0=(P|X_0),(P|\{y\})=\{\{y\to e'\}\})$ such that $P_0=\theta$ and e' is a prolongation of θ in r. Therefore ϱ is the set of all $\{y\}$-points $\{\{y\to e'\}\}$ such that e' is a prolongation of θ in r. But, by the assumption, $\varrho=(y;e)$, whence e is the only prolongation of θ in r, $\theta\in D_r^{(y)}$ and $e=\omega_r^{(y)}\cdot\theta$. As θ is a point of A, e belongs to the closure of A with respect to $\omega_r^{(y)}$. Therefore $\bar{A}_k$ is included in the closure of A with respect to all $\omega_r^{(y)}$, which proves the theorem.

Remark 6. If $\omega_r^{(y)}$ is an $\emptyset$-operation with $D_r^{(y)}\neq\emptyset$ and $r\in k$ then $D_r^{(y)}=\{P_\emptyset\}$ and $r=(y;\omega_r^{(y)}\cdot P_\emptyset)$. In this case $e=\omega_r^{(y)}\cdot P_\emptyset$ belongs to the rationality domain $\bar{\emptyset}_k$ of k, and this operation is the mere adjunction of $e\in E$ belonging to this domain, i.e. preserved by all $\delta\in D(E/k)$. Therefore $\bar{A}_k$ can also be characterized as the closure of $A\cup\bar{\emptyset}_k$ with respect to all X-operations $\omega_r^{(y)}$ such that $X\neq\emptyset$, $r\in k$ and $y\in X_r$.

Remark 7. If an X-relation $\varrho\in k$ is the infinitary union $\bigcup\cdot R$ of a set R of relations in k and $y\in X$ then for each $\bar{P}\in D_\varrho^{(y)}$ there exists a relation $r\in R$ such that $\bar{P}\in D_r^{(y)}$ and $\omega_r^{(y)}\cdot\bar{P}=\omega_\varrho^{(y)}\cdot\bar{P}$. Further, this r can be chosen so that r is semi-regular, $r\subseteqq\varrho$ and $P=(\bar{P},\{y\to\omega_\varrho^{(y)}\cdot\bar{P}\})$ belongs to $t(r)$, the head of r. Moreover, the $D(E/k)$-orbit $D(E/k)\cdot P$ of P is such a semi-regular relation r.

Indeed, if $e=\omega_\varrho^{(y)}\cdot\bar{P}$ then $P=(\bar{P},\{y\to e\})\in\varrho$ and e is the only prolongation of $\bar{P}$ in ϱ. Since $\varrho=\bigcup\cdot R$, there exists an $r\in R$ such that $P\in r$ and every prolongation of $\bar{P}$ in r is a prolongation of $\bar{P}$ in ϱ. So e is the unique prolongation of $\bar{P}$ in r, which implies $\bar{P}\in D_r^{(y)}$ and $e=\omega_r^{(y)}\cdot\bar{P}$. Since, by Lemma 2 of Section 2, each relation $\varrho\in k$ can be decomposed into a set R of semi-regular relations belonging to k such that any $P\in\varrho$ belongs to $t(r)$ for some $r\in R$, the rest of Remark 7 follows.

Remark 8. Let $r\in K$ be a semi-regular X-relation, let $y\in X$, and assume that $P=(\bar{P},\{y\to\omega_r^{(y)}\cdot\bar{P}\})\in t(r)$ for some $\bar{P}\in D_r^{(y)}$. Then there exists a semi-regular relation $r'\in k$ such that $T(r')$, the type of r', is the discrete equivalence on X', the argument set of r', and there are $y'\in X'$ and a point $\bar{P}'\in D_{r'}^{(y')}$ with $\bar{P}'\cdot(X'\setminus\{y'\})\subseteqq\subseteqq\bar{P}\cdot(X\setminus\{y\})$, $\omega_r^{(y)}\cdot\bar{P}=\omega_{r'}^{(y')}\cdot\bar{P}'$ and $P'=(\bar{P}',\{y'\to\omega_{r'}^{(y')}\cdot\bar{P}'\})\in t(r')$.

Indeed, let X be the argument set of r, and put $e=\omega_r^{(y)}\cdot\bar{P}$. Let φ be the canonical mapping of X onto $X'=X/T(r)$. Then $r'=(\varphi)\cdot r$ is well-defined and $T(r')$

is the discrete equivalence on X'. As $T(P)=T(r)$, we have $T((\varphi)\cdot P)=T(r')$. Hence $(\varphi)\cdot P$ is an injective point and belongs to $t(r')$. If $y'=\varphi\cdot y$ then the value $P'\cdot y'$ of the point $P'=(\varphi)\cdot P$ at y' is equal to $P\cdot y=e$. Put $\bar{X}=X\setminus\{y\}$, $\bar{X}'=X'\setminus\{y'\}$ and $\bar{P}'=(P'|\bar{X}')$. It is clear that $\bar{P}'\cdot\bar{X}'\subseteqq\bar{P}\cdot\bar{X}$ (in particular, if $\bar{P}$ is a point of $\bar{A}_k$ then so is $\bar{P}'$), and e is the only prolongation of $\bar{P}'$ in r'. Therefore $\bar{P}'\in D_{r}^{(y')}$ and $\omega_{r}^{(y')}\cdot\bar{P}'=e=\omega_{r}^{(y)}\cdot\bar{P}$.

Remark 9. It follows from the preceding remarks that a subset of E is closed with respect to all $\omega_r^{(y)}$, $r\in k$, iff it is closed with respect to the operations $\omega_r^{(y)}$ such that r has an injective point. Moreover, if X^0 is an argument set with $\operatorname{card} X^0\geqq\operatorname{card} E$ then a subset of E is closed with respect to all $\omega_r^{(y)}$ iff it is closed with respect to those that are determined by relations of k under X^0. In particular, $\bar{A}_k$ is the closure of $A\subseteqq E$ with respect to this last variety of operations.

Indeed, if an X_r-relation r has an injective point then $\operatorname{card} X_r\leqq\operatorname{card} E$. On the other hand, if (φ) is a floatage then $D_{(\varphi)\cdot r}^{(\varphi\cdot y)}=(\varphi|(X_r\setminus\{y\}))\cdot D_r^{(y)}$ and $\omega_{(\varphi)\cdot r}^{(\varphi\cdot y)}\cdot((\varphi|(X_r\setminus\{y\})\cdot\bar{P}))=\omega_r^{(y)}\cdot\bar{P}$ where $\bar{P}\in D_r^{(y)}$.

Let $S=(E,R)$ be a structure, and let $k=K_e(S)$ be the corresponding abstract endofield. We have seen that $\bar{A}_k$, the closure of $A\subseteqq E$, is the closure of A with respect to the operations $\omega_r^{(y)}$ such that the r are relations in k under a fixed X^0 with $\operatorname{card} X^0\geqq\operatorname{card} E$. That is, these r belong to $\bar{R}^{(X^0)}=R_{\mathrm{df}}^{(X^0)}$ and to k. Now the question is whether a sufficiently wide class of structures can be defined such that the "huge set" $R_{\mathrm{df}}^{(X^0)}$ can be replaced by the (much smaller) set R in case of these structures. The answer is positive; an appropriate class, the class of the so-called *eliminative structures*, can be defined. I will not speak about these structures in the present paper — it will be done in some other publication, which will contain the necessary proofs. However, the structure (E,R) of classical Galois theory is eliminative, and from this fact, accepted here without proof, we are going to deduce the fundamental theorem of classical Galois theory. In other words, let E be a normal algebraic or an algebraically closed field extension of some basic field k, let $R=\{(f=0); f\in k[x_1,x_2,\dots,x_n,\dots]\}$, and let A be a subset of E; we take it for granted that the rationality domain of the abstract set extension $(k)(A)$, obtained by adjoining A to the abstract endofield (or field) (k) defined by the structure (E,R), is the closure of A with respect to all operations defined by the relations $(f=0)$, $f\in k[x_1,x_2,\dots,x_n,\dots]$, of this structure. In order to deduce the fundamental theorem of classical Galois theory, first we introduce some constructions that yield operations from operations.

(1) Let X be an argument set and let U be a set of X-operations $\omega\colon D_\omega\to E$. An X-operation $\omega^*\colon D_{\omega^*}\to E$ is called a *mosaic* of the operations $\omega\in U$ if there

exists a partition $\{d_\omega;\ \omega\in U\}$ of D_{ω^*}, i.e. $\omega\neq\omega'\Rightarrow d_\omega\cap d_{\omega'}=\emptyset$ and $\bigcup_{\omega\in U} d_\omega=D_{\omega^*}$, such that $d_\omega\subseteqq D_\omega$ and $(\omega^*|d_\omega)=(\omega|d_\omega)$ hold for every $\omega\in U$.

Lemma 1. *If $A\subseteqq E$ is closed with respect to all $\omega\in U$ then it is closed with respect to every mosaic ω^* of $\omega\in U$.*

Proof. Let $P\colon X\to A$ be a point in D_{ω^*}. Then there exists one and only one $\omega\in U$ such that $P\in d_\omega\subseteqq D_\omega$ and $\omega^*\cdot P=\omega\cdot P$. Now the assertion follows from $\omega\cdot P\in A$.

(2) Let $\Omega\colon D\to E$ be an X-operation, let Y be an argument set, and for each $x\in X$ let $\omega^x\colon d_x\to E$ be a Y-operation. For $Q\in\bigcap_{x\in X} d_x$, let $\omega\cdot Q$ be the X-point $\{x\to\omega^x\cdot Q;\ x\in X\}$. Let d be the set of all $Q\in\bigcap_{x\in X} d_x$ such that $\omega\cdot Q\in D$. Then a Y-operation, denoted by $\Omega(\{x\to\omega^x\})$, can be defined in the following way. The domain of $\Omega(\{x\to\omega^x\})$ is d, and for every $Q\in d$ we put $\Omega(\{x\to\omega^x\})\cdot Q=\Omega\cdot(\omega\cdot Q)$. This Y-operation will be called the *superposition* of Ω and the "operation point" $\omega\colon x\to\omega^x,\ x\in X$.

Lemma 2. *If $A\subseteqq E$ is closed with respect to Ω and to all ω^x, $x\in X$, then it is also closed with respect to $\Omega(\{x\to\omega^x\})$.*

Proof. Let $P\colon Y\to A$ belong to d. Then, for every $x\in X$, $Q\in d_x$ and $\omega^x\cdot Q\in A$. So $\omega\cdot Q\colon x\to\omega^x\cdot Q$ is an X-point of A. On the other hand, $\omega\cdot Q\in D$. Therefore $\Omega(\{x\to\omega^x\})\cdot Q=\Omega\cdot(\omega\cdot Q)\in A$, proving the lemma.

For a set U of operations and a subset B of U, B will be called a *basis* of U if each $\omega\in U$ can be obtained from the operations of B by a combination of mosaics and superpositions represented by a tree of finite height. Clearly, a subset A of E is closed with respect to all $\omega\in B$ iff it is closed with respect to all $\omega\in U$.

To conclude the paper, we determine a simple basis of operations $\omega^{(y)}_{f=0}$ defined by the relations $(f=0)$, $f\in k[x_1, x_2, \ldots, x_n, \ldots]$, of classical Galois theory. Clearly, floatages do not change, up to floatages of arguments, the operations defined by a relation. So we can consider only the polynomials $f(x_1, x_2, \ldots, x_s, y)\in k[x_1, \ldots, x_s, y]$ (where s can be arbitrary) and the corresponding operations $\omega^{(y)}_{f=0}$. Let n be the degree of such an f for y, i.e.,

$$f(x_1, \ldots, x_s, y)=\sum_{0\leqq i\leqq n} f_i(x_1, \ldots, x_s)y^i.$$

Let $P\colon X_s=\{x_1, \ldots, x_s\}\to E$ be an X_s point of E, which will be represented by the system $(\xi_1, \ldots, \xi_s)$ of its values $\xi_i=P\cdot x_i\in E$, $i=1, \ldots, s$. The point P has exactly one prolongation in $r=(f=0)$ iff the polynomial $f(\xi_1, \ldots, \xi_s, y)\in E[y]$ has exactly one root, i.e., for some j, $1 \leqq j \leqq n$, $f(\xi_1, \ldots, \xi_s, y) = a(y-\eta)^j$, $a, \eta \in E$ and $a \neq 0$, i.e.

$$(\mathrm{C}_j) \qquad \begin{cases} f_i(\xi_1, \ldots, \xi_s) = 0 \quad \text{if} \quad j < i \\ f_j(\xi_1, \ldots, \xi_s) = a \neq 0 \\ f_i(\xi_1, \ldots, \xi_s) = (-1)^{j-i} \binom{j}{j-i} a\eta^{j-i} \quad \text{if} \quad i < j. \end{cases}$$

If the characteristic of k is 0 then $f_{j-1}(\xi_1, \ldots, \xi_s) = -ja\eta$; i.e., (C_j) implies $jf_j(\xi_1, \ldots, \xi_s)\eta + f_{j-1}(\xi_1, \ldots, \xi_s) = 0$ and $f_j(\xi_1, \ldots, \xi_s) \neq 0$. Therefore, if we consider the polynomial

$$g_j(x_1, \ldots, x_s, y) = f_{j-1}(x_1, \ldots, x_s) + jf_j(x_1, \ldots, x_s)\, y$$

then the operation $\omega^{(y)}_{g_j=0}$ is defined on $D_j = D^{(y)}_{g_j=0} = \neg\,(f_j=0)$. Clearly, D_j includes the set d_j of all points $P = (\xi_1, \ldots, \xi_s) \in E^{X_s}$ that satisfy (C_j). In other words, d_j, the set of all P for which $f(\xi_1, \ldots, \xi_s, y)$ is of the form $a(y-\eta)^j$ for some $a, \eta \in E$, $a \neq 0$, is included in D_j. For $P \in d_j$ we clearly have

$$(\alpha) \qquad \omega^{(y)}_{f=0} \cdot P = \eta = \omega^{(y)}_{g_j=0} \cdot P = -f_{j-1}(\xi_1, \ldots, \xi_s)(jf_j(\xi_1, \ldots, \xi_s))^{-1}.$$

Now consider the case when the characteristic p of k is different from 0. Let $j = hp^{s(j)}$ where h is not divisible by the prime number p. We have, in k, $\binom{j}{i} = 0$ if $0 < i < p^{s(j)}$ and $\binom{j}{i} = h$ if $i = p^{s(j)}$. In particular, we have $f_{j-p^{s(j)}}(\xi_1, \ldots, \xi_s) = -ha\eta^{p^{s(j)}}$. Let

$$g_{j,p}(x_1, \ldots, x_s, y) = hf_j(x_1, \ldots, x_s)\, y^{p^{s(j)}} + f_{j-p^{s(j)}}(x_1, \ldots, x_s).$$

Then (C_j) implies that $hf_j(\xi_1, \ldots, \xi_s) \neq 0$ and η is the unique root of $g_{j,p}(\xi_1, \ldots, \xi_s, y)$. Therefore $\omega^{(y)}_{g_{j,p}=0}$ is defined on the same $D_j = \neg\,(f_j=0)$, $d_j \subseteqq D_j$ and, for every $P \in d_j$, we have

$$(\beta) \quad \omega^{(y)}_{f=0} \cdot P = \eta = \omega^{(y)}_{g_{j,p}} = 0 \cdot P = (-f_{j-p^{s(j)}}(\xi_1, \ldots, \xi_s)(hf_j(\xi_1, \ldots, \xi_s))^{-1})^{p^{-s(j)}}.$$

In both cases, $(d_1, d_2, \ldots, d_n)$ is a partition of $D^{(y)}_{f=0}$. Therefore, $\omega^{(y)}_{f=0}$ is a mosaic of the operations $\omega_1, \omega_2, \ldots, \omega_n$ where these ω_j are defined on the respective sets $D_j = \neg\,(f_j=0)$ in the following way: for $P = (\xi_1, \ldots, \xi_s) \in D_j$ we put

$$\omega_j \cdot P = -f_{j-1}(\xi_1, \ldots, \xi_s)(jf_j(\xi_1, \ldots, \xi_s))^{-1}$$

when k is of zero characteristic, and we put

$$\omega_j \cdot P = (-f_{j-p^{s(j)}}(\xi_1, \ldots, \xi_s)(hf_j(\xi_1, \ldots, \xi_s))^{-1})^{p^{-s(j)}}$$

when the characteristic of k is $p \neq 0$ (cf. (α) and (β)). It is clear that the ω_j are superpositions of the operations $(x_1, x_2) \to x_1 + x_2$, $(x_1, x_2) \to x_1 x_2$, $x_1 \to x_1^{-1}$ (defined on $E \setminus \{0\}$), $x_1 \to \sqrt[p]{x_1}$ if $p \neq 0$ (defined on $\{x^p;\, x \in E\}$), and the adjunctions $P_\emptyset \to \alpha$, $\alpha \in k$.

Therefore these operations form a basis of $\{\omega^{(\nu)}_{f=0};\ f\in k[x_1, \ldots, x_s, \ldots]\}$. If the afore-mentioned theorem about eliminative structures is proved and it is shown that the considered structure is eliminative then it follows that the rationality domain of $(k)(A)$, i.e. the set of all $e\in E$ that are preserved by any automorphism of E/k preserving every $a\in A$, is the closure of $A\cup k = A\cup\overline{\emptyset}$ with respect to addition, multiplication, inversion and, if $p\neq 0$, forming p-th roots. This is one of the classical formulations of the first Galois theorem.

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