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Rare bases for finite intervals of integers

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In this paper we discuss the following finite problem for additive bases: What is the least possible number of elements of a set B, for which all integers in the interval $[1, n]$ can be represented as the sum of two elements of B. (B can be called a basis of order 2 for the interval $[1, n]$). Let us denote this minimal number by c_n.

Clearly, $c_n \geqq \sqrt{2} \cdot \sqrt{n}$ holds, since if there are k elements in B, then we can form at most $\binom{k+1}{2} \sim \frac{k^2}{2}$ sums which have to give at least n different values, hence $n \leqq \frac{k^2}{2}$, i.e. $k \geqq \sqrt{2} \cdot \sqrt{n}$.

On the other hand, a simple construction shows $c_n \leqq 2 \cdot \sqrt{n}$. Let B be the union of two arithmetical progressions; $0, 1, 2, \ldots, \lceil\sqrt{n}\rceil$ and $2 \cdot \lceil\sqrt{n}\rceil,\ 3 \cdot \lceil\sqrt{n}\rceil, \ldots, \lceil\sqrt{n}\rceil \cdot \lceil\sqrt{n}\rceil$, where $\lceil a \rceil$ means the least integer $s \geqq a$. These approximately $2 \cdot \sqrt{n}$ elements form a basis of order 2 for the interval $[1, n]$.

Rohrbach conjectured in 1937 that $c_n = 2\sqrt{n} + O(1)$. This was disproved in 1976 by HÄMMERER and HOFMEISTER [2], they showed $c_n \leqq \sqrt{3.6} \cdot \sqrt{n}$. (For further references and related problems see [1], 47.)

The aim of this paper is to improve the result of Hämmerer and Hofmeister:

Theorem. $c_n \leqq \sqrt{3.5} \cdot \sqrt{n} + o(\sqrt{n})$.

The method of proof is different from that of [2], it is completely elementary, and is based on similar simple ideas as the one which gave the obvious upper bound. Some further refinements might yield even better upper bounds for c_n.

Proof. For a clearer exposition of the construction we show first $c_n \leqq \sqrt{3.6} \cdot \sqrt{n} + o(\sqrt{n})$. B will consists of the union of 4 arithmetical progressions:

I. $\alpha_0 = 0,\ \alpha_1 = 1, \ldots, \alpha_t = t$; here the difference of the consecutive terms is $d_1 = 1$.

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II. $\beta_0 = \alpha_t = t, \quad \beta_1 = 2t+1, \ldots, \beta_{3t} = 3t(t+1)+t = 3t^2+4t; \quad d_2 = t+1.$

III. $\gamma_1 = 3t^2+5t+1, \ldots, \gamma_{t+1} = 3t^2+6t+1; \quad d_3 = 1.$

IV. $\delta_1 = 6t^2+12t+3, \ldots, \delta_{t+1} = 7t^2+12t+3; \quad d_4 = t.$

The following inclusions (mostly in form of equalities) are all obvious, except the last but one, which we shall verify below:

$$\{\alpha_i+\alpha_j\} \supseteqq [1, 2t], \quad \{\alpha_i+\beta_j\} \supseteqq [2t+1, 3t^2+5t],$$

$$\{\alpha_i+\gamma_j\} \supseteqq [3t^2+5t+1, 3t^2+7t+1], \quad \{\beta_i+\gamma_j\} \supseteqq [3t^2+7t+2, 6t^2+10t+1],$$

$$\{\gamma_i+\gamma_j\} \supseteqq [6t^2+10t+2, 6t^2+12t+2], \quad \{\alpha_i+\delta_j\} \supseteqq [6t^2+12t+3, 7t^2+13t+3],$$

$$\{\beta_i+\delta_j\} \supseteqq [7t^2+13t+4, 9t^2+17t+3], \quad \{\gamma_i+\delta_j\} \supseteqq [9t^2+17t+4, 10t^2+18t+4].$$

To verify the last but one inclusion we use the following two equalities, which are straightforward from the construction of II and IV:

$$\beta_{i+1}+\delta_{j-1} = \beta_i+\delta_j+1 \quad \text{and} \quad \beta_{i-t+1}+\delta_{j+t} = \beta_i+\delta_j+1.$$

Hence we obtain the consecutive elements of the interval $[7t^2+137+3, 9t^2+17t+3]$ by the following sums:

$$\begin{array}{llll}
\beta_0+\delta_{t+1}, & \beta_1+\delta_t, & \beta_2+\delta_{t-1}, \ldots, & \beta_t+\delta_1, \\
\beta_1+\delta_{t+1}, & \beta_2+\delta_t, & \ldots, & \beta_{t+1}+\delta_1, \\
\beta_2+\delta_{t+1}, & \ldots & & \\
\vdots & & & \\
\beta_{2t}+\delta_{t+1}, & \beta_{2t+1}+\delta_t, & \ldots, & \beta_{3t}+\delta_1, \\
\beta_{2t+1}+\delta_{t+1}, & \beta_{2t+2}+\delta_t, & \ldots, \quad \beta_{3t}+\delta_2. &
\end{array}$$

Summarizing our construction, B contains $k=6t+3$ elements and is a basis for the interval $[1, n]$, where $n=10t^2+18t+4$. This proves $c_n \leqq \sqrt{3.6}\cdot\sqrt{n}+o(\sqrt{n})$ (for all n).

To obtain $c_n \leqq \sqrt{3.5}\cdot\sqrt{n}+o(\sqrt{n})$ we have to add just another arithmetical progression to B:

V. $\varepsilon_1 = 10t^2+18t+5, \ldots, \varepsilon_{t+1} = 11t^2+18t+5; \quad d_5 = t.$

Now

$$\{\alpha_i+\varepsilon_j\} \supseteqq [10t^2+18t+5, 11t^2+19t+5],$$

$$\{\beta_i+\varepsilon_j\} \supseteqq [11t^2+19t+6, 13t^2+23t+5],$$

$$\{\gamma_i+\varepsilon_j\} \supseteqq [13t^2+23t+6, 14t^2+24t+6].$$

Here the first and last inclusions are obvious, and the second one follows exactly the same way as the one for $\{\beta_i+\delta_j\}$.

Hence we have a basis for $[1, n]$ with $n=14t^2+24t+6$, and it consists of $k=7t+4$ elements, i.e. $k\sim\sqrt{3.5}\cdot\sqrt{n}$.

References

[1] P. Erdős—R. L. Graham, *Old and New Problems and Results in Combinatorial Number Theory,* L'Enseignement Mathématique (Genève, 1980).

[2] N. Hämmerer—G. Hofmeister, Zu einer Vermutung von Rohrbach, *J. Reine und Angew. Math.,* **286/287** (1976), 239—246.

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