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Wreath product decomposition of categories. I*)

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1. Introduction. In this paper I prove a theorem (Theorem 4.1) giving sufficient conditions for decomposing a functor F: $\mathbf{C} \to \mathbf{Cat}$ into the wreath product of two functors, given a natural transformation λ: $F \to G$. When the functors are discrete (set-valued) the sufficient conditions always hold.

The theorem is a double generalization of the theorem about embedding a group into a wreath product due to KALOUJNINE—KRASNER ([7], stated also in WELLS [13]). To be precise, it generalizes the one-step version of that theorem, although for any action — not just for the regular representation as it is commonly stated in group theory texts.

The generalization is double in the sense that the group is generalized to a category and the action not merely to a set-valued functor (which already gives a new theorem) but to a **Cat**-valued one. The theorem provides a decomposition of *any* Set-valued functor with given quotient, and any **Cat**-valued one provided the fibers of the quotient are split opfibrations. Since the wreath product itself is a split fibration, this brings the theory of fibrations into the picture in two different ways.

Some applications are given in Section 6. One, Proposition 6.4, provides a generalization of a technique used in some proofs of the Krohn—Rhodes Theorem (see KROHN—RHODES [10], WELLS [13]). (A generalization of another of the techniques to **Cat**-valued functors is in WELLS [17].)

My hope is that both techniques might be useful in developing a theory of *state-transition systems* with *structured, typed states*. Any functor F: $\mathbf{C} \to \mathbf{Cat}$ can be thought of as such a system. The objects of **C** are the types of states. For each object c, the objects of Fc are the states of type c. The transitions are the functors Ff: $Fc \to Fd$ for f: $c \to d$ in **C**. The structure on the states of type c is the category structure on Fc (thus having a poset or monoid or group structure as possible special cases).

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Perhaps the theorem of the present paper will also be useful in developing a theory of varieties for categories, in the way the embedding into a wreath product has proved useful in group theory (Neumann [12]).

Categorical fibrations and opfibrations are discussed in Section 2, and the wreath product with categorical action in Section 3. The decomposition theorem is stated in Section 4 and proved in Section 5. Some applications are given in Section 6.

Throughout this paper, a set is identified with the category which has the elements of the set as objects and no non-identity arrows. Such a category is called *discrete.*

These results were obtained in part while I was a guest of the Forschungsinstitut für Math., E.T.H. Zürich, for whose support I am grateful. An earlier version, containing errors, called *Wreath product decomposition of categories and functors*, was distributed but never published.

2. Fibrations. In this section, I outline that part of the theory of split fibrations and opfibrations needed for the main theorems. The material is not new, and is scattered through Grothendieck [5], Giraud [1], Gray [2], [3], [4].

Given a functor P: $\mathbf{E}\to\mathbf{C}$ there is an induced functor S from the arrow category $\mathrm{Ar}\,\mathbf{E}$ to the comma category $(\mathbf{C}, P)$ which takes u: $e'\to e$ to (Pu, e). A right adjoint right inverse R for S is called a *cleavage*, and a *left* adjoint right inverse R° to the functor S°: $\mathrm{Ar}\,\mathbf{E}\to(P, \mathbf{C})$ which takes u: $e'\to e$ to (e', Pu) is an *opcleavage.* P, together with a cleavage R, is a *fibration* of $\mathbf{C}$. If R° is an opcleavage, (P, R°) is an *opfibration* of $\mathbf{C}$. Neither a cleavage nor an opcleavage necessarily exists for any given functor P.

Assume $(P$: $\mathbf{E}\to\mathbf{C}, R)$ is a fibration. Let f: $b\to c$ in $\mathbf{C}$ and u: $e'\to e$ lie over c (i.e. $Pu=1_c$). Define $\Phi f.\, e''=\mathrm{dom}\, R(f, e'')$ for any object e'' over c, and $\Phi f.\, u$ by requiring $R(1_b, u)=(\Phi f.\, u, u)$ (the second component is necessarily u). Similarly for an opfibration (P, R°), let $\Phi^\circ f.\, e''=\mathrm{cod}\, R^\circ(e'', f)$ for e'' over b, and $R^\circ(u, 1_c)=$ $=(u, \Phi^\circ f.\, u)$. One then has the commutative squares

(2.1)
$$\begin{array}{ccc} \Phi f.\, e' & \xrightarrow{R(f,e')} & e' \\ {\scriptstyle \Phi f.\, u}\downarrow & & \downarrow{\scriptstyle u} \\ \Phi f.\, e & \xrightarrow[R(f,e)]{} & e \end{array} \qquad \begin{array}{ccc} e' & \xrightarrow{R^\circ(e',f)} & \Phi^\circ f.\, e' \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \Phi^\circ f.\, u} \\ e & \xrightarrow[R^\circ(e,f)]{} & \Phi^\circ f.\, e \end{array}$$

By setting $\Phi c=\Phi^\circ c=P^{-1}c$ (the full subcategory of $\mathbf{E}$ lying over 1_c) one has Φ, Φ° both defined on objects and arrows of c. They may not be functors. If they are, they are functors to $\mathbf{Cat}$ and $R(f, —)$ and $R^\circ(—, f)$ are natural transformations for each f. If $P^{-1}c$ is a set (no non-trivial arrows) the fibration or opfibration is called *discrete.*

A fibration (P, R) is *split* if

a) Φ is a functor, and

b) if $f\colon c'\to c$, $g\colon c\to c''$ in **C** and $Pe''=c''$, $Pe=c$, then

$$R(f, \Phi g.e'')\circ R(g, e'') = R(g\circ f, e''). \tag{2.2}$$

Then Φ is a *splitting*, and I shall refer to the split fibration as $(P\colon \mathbf{E}\to\mathbf{C}, R, \Phi)$.

A split opfibration $(P, R^{\circ}, \Phi^{\circ})$ requires

a)° Φ° is a functor, and

b)° if $f\colon c'\to c$, $g\colon c\to c''$ in **C**, $Pe'=c'$, $Pe=c$, then

$$R^{\circ}(e', g\circ f) = R'(\Phi^{\circ}f.e', g)\circ R^{\circ}(e', f). \tag{2.2°}$$

It is easy to see that $(P\colon \mathbf{E}\to\mathbf{C}, R, \Phi)$ is a split fibration if and only if $(P^{op}\colon \mathbf{E}^{op}\to\mathbf{C}^{op}, R^{op}, \Phi^{op})$ is a split opfibration.

A morphism of split fibrations is a pair $(U, V)\colon (P\colon \mathbf{E}\to\mathbf{C}, R, \Phi)\to(P'\colon \mathbf{E}'\to\mathbf{C}', R', \varphi')$ where $U\colon \mathbf{C}\to\mathbf{C}'$ and $\mathbf{V}\colon \mathbf{E}\to\mathbf{E}'$ are functors for which

$$\begin{array}{ccc} \mathbf{E} & \xrightarrow{V} & \mathbf{E}' \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle P'} \\ \mathbf{C} & \xrightarrow[U]{} & \mathbf{C}' \end{array} \tag{2.3}$$

commutes and for $f\colon b\to c$ in **C**, e an object of Φc,

$$V\big(R(f, e)\big) = R'(Uf, Ve). \tag{2.4}$$

Composition of morphisms is componentwise, giving a category **F** of split fibrations.

Morphisms of opfibrations are defined similarly. (2.3)° is the same as (2.3) and (2.4) becomes

$$V\big(R^{\circ}(e, f)\big) = R'(Ve, Uf) \tag{2.4°}$$

where e is an object of $\Phi^{\circ}b$. The resulting category is denoted $\mathbf{F}^{\circ}$.

It follows from (2.4) that

$$V(\Phi f.e) = \Phi'(Uf).Ve, \tag{2.5}$$

i.e. V respects fibers. A similar statement holds for morphisms of opfibrations.

Now I define another category **Scat** which will turn out to be equivalent to both **F** and $\mathbf{F}^{\circ}$. The objects of **Scat** are all **Cat**-valued functors from all categories. An arrow $(K, \lambda)\colon F\to G$ has $K\colon \operatorname{dom} F\to\operatorname{dom} G$ a functor and $\lambda\colon F\to G\circ K$ a natural transformation. Composition is given by

$$(L, \mu)\circ(K, \lambda) = (L\circ K, \mu K\circ\lambda). \tag{2.6}$$

All functor categories **Func** (**C**, **Cat**) are subcategories of **Scat**, and so is the comma category (**Cat**, **Cat**), where the second "**Cat**" is an object in the first. **Scat** is the category called $\mathrm{Cat}_0\circ\mathrm{Cat}_0$ by KELLY [8, § 7].

Given any functor F: $\mathbf{C}^{\mathrm{op}} \to \mathbf{Cat}$, let $SD(F)$ be the category defined this way: an object of $SD(F)$ is a pair (c, x) with c an object of $\mathbf{C}$ and x an object of Fc. An arrow (f, u): $(c, x) \to (c', x')$ has f: $c' \to c$ in $\mathbf{C}$ and u: $x \to Ff.x'$ in Fc. If (g, v): $(c', x') \to (c'', x'')$, then

(2.7) $$(g, v) \circ (f, u) = (f \circ g, (Ff.v) \circ u).$$

Likewise, given F: $\mathbf{C} \to \mathbf{Cat}$ define $SD^\circ(F)$ the same way except that for (f, u): $(c, x) \to (c', x')$, f: $c \to c'$ and u: $Ff.x \to x'$, and

(2.7)° $$(g, v) \circ (f, u) = (g \circ f, V \circ (Fg.u)).$$

There are then functors $SN(F)$: $SD(F) \to \mathbf{C}^{\mathrm{op}}$ and $SN^\circ(F)$: $SD(F) \to \mathbf{C}$ taking (f, u) to f.

There are then functors $R_F(R_F)$ and $\bar{F}(\bar{F}^\circ)$ for which $(SN(F), R_F, \bar{F})$ (resp. $(SN^\circ(F), R_F^\circ, \bar{F}^\circ)$) is a split fibration (split opfibration). The definitions are, for (f, u): $(c, x) \to (c', x')$ in $SD(F)$,

(2.8) $$R_F(f, (c', x')) = (f, 1_{Ff.x'}): (c, Ff.x') \to (c', x')$$

and for (f, u): $(c, x) \to (c', x')$ in $SD^\circ(F)$,

(2.8)° $$R_F^\circ((c, x), f) = (1_{Ff.x}, f): (c, x) \to (c', Ff.x).$$

As for $\bar{F}$ and $\bar{F}^\circ$ the definitions are determined by R_F. In particular (because it is used later), for F: $\mathbf{C} \to \mathbf{Cat}$, u an arrow in Fc,

(2.9)° $$\bar{F}^\circ f.(1_c, u) = (1_{c'}, Ff.u).$$

These constructions make SN: $\mathbf{Scat} \to \mathbf{F}$ and SN°: $\mathbf{Scat} \to \mathbf{F}^\circ$ into the object maps of functors.

I will continue the development only for opfibrations, since the constructions for fibrations are not needed. Let F: $\mathbf{C} \to \mathbf{Cat}$, G: $\mathbf{D} \to \mathbf{Cat}$, (K, λ): $F \to G$ in $\mathbf{Scat}$. Let (f, u): $(c, x) \to (c', x')$ in $SD^\circ(F)$. Then define

(2.10)° $$SD^\circ(K, \lambda)(f, u) = (Kf, \lambda c'.u)$$

and

(2.11)° $$SN^\circ(K, \lambda) = (K, SD^\circ(K, \lambda)).$$

Thus SD°: $\mathbf{Scat} \to \mathbf{Cat}$ and SN°: $\mathbf{Scat} \to \mathbf{F}^\circ$ are functors.

SN° is an equivalence of categories. Define the functor Λ°: $\mathbf{F}^\circ \to \mathbf{Scat}$ as follows.

(2.12)° $$\Lambda^\circ(P: \mathbf{E} \to \mathbf{C}, R^\circ, \Phi^\circ) = \Phi^\circ.$$

(2.13)° $$\Lambda^\circ(U, V) = (U, \alpha_V), \quad \text{where}$$

(2.14)° $$\alpha_V.c = V \mid \Phi^\circ c$$

for c an object of $\mathbf{C}$.

There is a natural isomorphism $\varepsilon\colon \mathrm{id}_{\mathbf{Scat}}\to \Lambda^\circ \circ SN^\circ$, whose component at $F\colon \mathbf{C}\to\mathbf{Cat}$ is

(2.15)° $$\varepsilon F = (1_{\mathbf{C}}, \bar{\varepsilon}F)\colon F \to \bar{F}^\circ$$

(see (2.9)), where for an object c of $\mathbf{C}$, $\bar{\varepsilon}F.c\colon Fc\to \bar{F}^\circ c$ takes an object x to (c, x) and an arrow u over 1_c to $(1_c, u)$.

There is also a natural isomorphism $\eta\colon \mathrm{id}_{\mathbf{F}^\circ}\to SN^\circ\circ\Lambda^\circ$, defined as follows. Given a split opfibration $(P\colon \mathbf{E}\to\mathbf{C}, R^\circ, \Phi^\circ)$, let $I\colon \mathbf{E}\to SD^\circ(\Phi^\circ)$ take an arrow u to (Pu, u). Then the component of η at (P, R°, Φ°) is $(\mathrm{id}_{\mathbf{C}}, I)\colon (P, R^\circ, \Phi^\circ)\to(SN^\circ(\Phi^\circ), R_{\Phi^\circ}, \bar{\Phi}^\circ)$. Thus SN° and Λ° are equivalences.

This Lemma is needed later:

Lemma 2.1. *Let $(U, V), (U, W)\colon (P\colon \mathbf{E}\to\mathbf{C}, R^\circ, \Phi^\circ)\to(P'\colon \mathbf{E}'\to\mathbf{C}', R^{\circ\prime}\ \Phi^{\circ\prime})$ be morphisms of split opfibrations for which for every object c of $\mathbf{C}$, $V|Gc=W|Gc$. Then $V=W$.*

Proof. Let $m\colon e\to e_0$ in E lie over $f\colon b\to c$. It is enough to show that $Vm=Wm$. Since R° is left adjoint to S°, there is a unique morphism of $\mathrm{Ar}\,\mathbf{E}$ from $R^\circ(e,f)$ to m corresponding to the identity arrow in $(P, \mathbf{C})$ from (e,f) to $(e,f)=S^\circ m$. Since R° is left inverse to S°, this arrow must be of the form $(1_e, k)$ where $k\colon \Phi^\circ f.e\to e_0$ and k is in $\Phi^\circ c$. Then by definition of morphism in $\mathrm{Ar}\,\mathbf{E}$, $m=k\circ R^\circ(e,f)$. Hence by (2.4)°,

$$Vm = Vk\circ VR^\circ(e,f) = Wk\circ R^{\circ\prime}(Uf, Ve) = Wk\circ R^{\circ\prime}(Uf, We) = Wk\circ WR^{\circ\prime}(f, e) = Wm$$

since k is in $\Phi^\circ C$ and e is in $\Phi^\circ b$.

3. The wreath product of categories. Given categories $\mathbf{B}$ and $\mathbf{C}$ and a functor $G\colon \mathbf{C}\to\mathbf{Cat}$, let $G_{\mathbf{B}}=\mathbf{Func}\,(G(-), \mathbf{B})\colon \mathbf{C}^{\mathrm{op}}\to\mathbf{Cat}$. The *wreath product* of $\mathbf{B}$ by $\mathbf{C}$ with action G, denoted $\mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}$, is $SD(G_{\mathbf{B}})$. Thus via $SN(G_{\mathbf{B}})$ it is a split fibration of $\mathbf{C}$ in a canonical way. Note that $\mathbf{Scat}=\mathbf{Cat}\,\mathrm{wr}^I\,\mathbf{Cat}$ with I being the identity functor.

The concept is due to Kelly [8, § 5], who denotes $\mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}$ by $[\mathbf{C}, G]\circ B$ and calls it the *composite*. His definition is more general than mine, since for him $\mathbf{B}$ can be any object in a 2-category.

$\mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}$ is natural in both variables in the sense that functors $U\colon \mathbf{B}\to\mathbf{B}'$ and $V\colon \mathbf{C}'\to\mathbf{C}$ induce a functor $SD\,(\mathbf{Func}\,(G(-), U), V)\colon \mathbf{B}\,\mathrm{wr}^{GV}\mathbf{C}'\to\mathbf{B}'\,\mathrm{wr}^G\,\mathbf{C}$ which is natural in both variables.

More important, a functor $F\colon \mathbf{B}\to\mathbf{Cat}$ induces a functor $F\,\mathrm{wr}\,G\colon \mathbf{B}\,\mathrm{wr}^G\mathbf{C}\to\mathbf{Cat}$ which generalizes the concept of the wreath product of two actions. Given F, define $\bar{F}\colon \mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}\to\mathbf{Scat}$ as follows. For an object (c, P) of $\mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}$ (whence $P\colon GC\to\mathbf{B}$ is a functor), set $\bar{F}(c, P)=F\circ P$. For an arrow $(f, \lambda)\colon (c, P)\to(d, Q)$ (whence $f\colon c\to d$ in $\mathbf{C}$ and $\lambda\colon P\to Q\circ Gf$), set $\bar{F}(f, \lambda)=(Gf, F\lambda)$. Then set $F\,\mathrm{wr}\,G=SD^\circ\circ\bar{F}\colon \mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}\to\mathbf{Cat}$.

KELLY [8, §7] shows that wreathing for categories and for functors is associative up to a 2-natural isomorphism.

If **B** and **C** are groups regarded as categories and G is discrete (**Set**-valued) then $\mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}$ is the usual wreath product of groups. If G is not discrete then $\mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}$ is a groupoid. If **B** is a set regarded as a discrete category, **C** is a monoid acting on **B** and G is the action, then $\mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}$ is a directed graph with objects which are functions $f\colon \mathbf{B}\to\mathbf{B}$ and edges $f\to fg^{-1}$ where g is an invertible element of **C**. When **B** and **C** are groupoids, $\mathbf{B}\,\mathrm{wr}^G\,\mathbf{C}$ has as a special case the untwisted version of the wreath product due to HOUGHTON [6]. Here the functor G is discrete; its value at an object c of **C** is the total sieve on c (the set of all arrows into c).

4. Coordinate systems. In the Kaloujnine—Krasner setup a group action is decomposed along a quotient action. The second coordinate is the quotient, and the first coordinate (the one with the most dependencies) is the action on a fiber. One can get away with this because the fibers are all isomorphic — although to get a decomposition you have to specify the isomorphisms.

In the present schema this corresponds to introducing a "typing functor" (defined below), which allows a partial skeletonization of the fibers of the quotient action. To do this we will make the fibers into a category $\mathbf{Fib}(\lambda)$ where λ is the quotient map. A "coordinate system" will then be a category and an action (**Cat**-valued functor) which "includes" $\mathbf{Fib}(\lambda)$ in a certain sense. All this requires that the components of λ be split normal opfibrations, a condition which is vacuous in the discrete case. The main Theorem 4.1 then says that in the presence of a coordinate system the action can be decomposed into the wreath of the action on the (partially skeletonized) fibers and the quotient action.

Let **C** be a category, $F\colon \mathbf{C}\to\mathbf{Cat}$ and $G\colon \mathbf{C}\to\mathbf{Cat}$ functors, and $\lambda\colon F\to G$ a natural transformation. Then λ is *split* if for each object c of **C**, $\lambda c\colon Fc\to Gc$ is a split opfibration with splitting $Lc\colon Gc\to\mathbf{Cat}$, and for each $f\colon c\to d$ in **C**, the pair (Gf, Ff) is an $\mathbf{F}^{\circ}$-morphism. The latter requirement implies that for each object x of Gc, $Ff|Lc\,.\,x$ has values in $Ld(Gf\,.\,x)$, and for each $u\colon x\to y$ in Gc,

$$(4.1)\qquad \begin{array}{ccc} Lc.x & \xrightarrow{Ff|Lc.x} & Ld(Gf.x) \\ \big\downarrow{\scriptstyle Lc.u} & & \big\downarrow{\scriptstyle Ld(Gf.u)} \\ Lc.y & \xrightarrow[Ff|Lc.z]{} & Ld(Gf.y) \end{array}$$

commutes. If F and G are discrete, any natural transformation $\lambda\colon F\to G$ is split.

The fibers of λ, in other words the categories $Lc\,.\,x$ for c an object of **C** and x an object of Gc, are objects of a category $\mathbf{Fib}(\lambda)$. The arrows are the functors from $Lc\,.\,x$ to $Ld(Gf\,.\,y)$ given by (4.1) for each $f\colon c\to d$ in **C** and each $u\colon x\to y$ in Gc. Thus $\mathbf{Fib}(\lambda)$ is a subcategory of **Cat**.

A functor T: $\mathbf{Fib}(\lambda)\to\mathbf{Cat}$ is a *typing functor* if there is a natural isomorphism τ: $I_\lambda\to T$, where I_λ: $\mathbf{Fib}(\lambda)\to\mathbf{Cat}$ is inclusion. Extreme cases of typing functors are I_λ and a skeletonizing functor. An intermediate case is actually used in an application in Section 6.

(M, K, T) is a *coordinate system* for a split λ: $F\to G$ with splitting L if T is a typing functor for $\mathbf{Fib}(\lambda)$, $\mathbf{M}$ is a category and K: $M\to\mathbf{Cat}$ a functor for which

CS—1. For each object c of $\mathbf{C}$ there is a set Φ_c of functors P: $Gc\to\mathbf{M}$ for each of which $T\circ Lc$ is a subfunctor of $K\circ P$, and

CS—2. If f: $c\to d$ in $\mathbf{C}$ and P: $Gc\to\mathbf{M}$ in Φ_c, then there is Q: $Gd\to\mathbf{M}$ in Φ_d for which for each object x of Gc there is an arrow m: $Px\to Q(Gf\,.\,x)$ for which $Km|T(Lc\,.\,x)=T(Ff|Lc\,.\,x)$.

A transitive group action with a quotient always has a coordinate system. Let $\mathbf{C}$ be the group, F the action, G the quotient action, λ the quotient map, so the fibers form a system of imprimitivity. T is then a way of identifying all the fibers with one of them, $\mathbf{M}$ is the isotopy subgroup of that fiber with action K. P is then a constant map. Even a nontransitive group action with quotient has a coordinate system, but then $\mathbf{M}$ will be a disjoint union of isotopy subgroups regarded as categories.

If F, G: $\mathbf{C}\to\mathbf{Set}$, λ: $F\to G$ any natural transformation, then λ always has a coordinate system based on $\mathbf{Fib}(\lambda)$. This is discussed further in Section 6.

A functor H: $\mathbf{A}\to\mathbf{B}$ *lifts triangles* if for all arrows f of $\mathbf{A}$ and h, k of $\mathbf{B}$ for which $Hf\circ h$ and $k\circ Hf$ are defined, there are arrows u, v of $\mathbf{A}$ for which $f\circ u$ and $v\circ f$ are defined, and $Hu=h$, $Hv=k$. A decomposition ought to lift triangles, as I explain later. Too bad, because the decomposition is trivial to construct if it needn't lift triangles.

In the following theorem, F: $\mathbf{C}\to\mathbf{Cat}$, G: $\mathbf{C}\to\mathbf{Cat}$ are functors and λ: $F\to G$ a natural transformation. $\overline{\mathbf{G}}$ is the image of G in $\mathbf{Cat}$, and I_G: $\overline{\mathbf{G}}\to\mathbf{Cat}$ is inclusion.

Theorem 4.1. *If F is faithful and λ is split with coordinate system (M, K, T), then there is a subcategory $\mathbf{S}\subset\mathbf{M}\,\mathrm{wr}^{IG}\,\overline{G}$ and a triangle-lifting functor H: $\mathbf{S}\to\mathbf{C}$ for which $F\circ H$ is isomorphic to a subfunctor of the restriction of $K\,\mathrm{wr}\,I_G$ to* $\mathbf{S}$.

The proof is given in Section 5, and applications are discussed in Section 6.

If you think of this theorem as giving sufficient conditions for simulating a state-transition system triangularly (in the sense of Krohn, Langer and Rhodes [11]) by a wreath product or cascade of systems, then the simulation has the property that for any state and any transition from that state in the simulat*ed* system, there is at least one state and transition from it in the simulat*ing* system which mimics (functorially) the operation of the simulat*ed* system. Moreover *you can always simulate the next transition from the simulating state you find yourself in.* That is the meaning of triangle-lifting. Clearly it is a necessary property of typed-state simulations.

Note that the system $F\colon \mathbf{C} \to \mathbf{Cat}$ might very well allow a sequence of transitions which begin and end at the same state, but for which the simulation begins and ends at different states, behavior reminiscent of a path in a Riemann surface lying over a loop.

Theorem 4.1 is similar to, but apparently not exactly a generalization of, both Theorem 11.1 of WELLS [13] and the main theorem of WELLS [15].

5. Proof of Theorem 4.1. S is the subcategory of $\mathbf{M}\,\mathrm{wr}^{I_G}\,\bar{G}$ defined this way: an object of S is any pair (Gc, P) where c is an object of $\mathbf{C}$ and $P\colon Gc \to \mathbf{M}$ is a functor in Φ_c. An arrow $(Gf, \gamma)\colon (Gc, P) \to (Gd, Q)$ has $f\colon c \to d$ in $\mathbf{C}$ and γ any function from the objects of Gc to the arrows of $\mathbf{M}$ with the properties that for each object x of Gc,

$$\gamma x\colon Px \to Q(Gf.x), \tag{5.1}$$

$$T(Lc.x) \subset KPx, \tag{5.2}$$

$$T(Ff(Lc.x)) \subset KQ(Gf.x), \quad \text{and} \tag{5.3}$$

$$K(\gamma x)|T(Lc.x) = T(Ff|Lc.x). \tag{5.4}$$

There may not be such a γ for a given f, P, and Q as above, but for a given f and P there *is* a Q in Φ_d for which there is at least one such γ. That follows from CS—1 and CS—2.

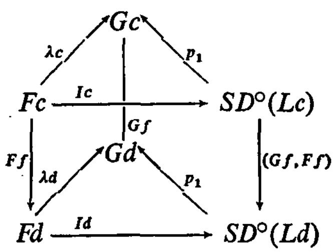
The functor $H\colon \mathbf{S} \to \mathbf{C}$ is defined by

$$H(Gf, \gamma) = f. \tag{5.5}$$

It is necessary to see that H is well-defined. Because $T(Lc\,.\,x)$ is naturally isomorphic to $Lc\,.\,x$, (5.4) says that the arrows which make up γ determine the effect of Ff on the categories $Lc\,.\,x$. Because (Gf, Ff) is a morphism in $\mathbf{F}^\circ$, Lemma 2.1 says that γ and Gf determine Ff. That determines f because F is faithful. It is clear that H is triangle lifting.

To show that $F \circ H$ is a subfunctor of the restriction of $K\,\mathrm{wr}\,I_G$ requires several steps. In the first place

(5.6)



commutes, where I_c is the natural isomorphism defined by $\eta_{\lambda c} = (\mathrm{id}_{Gc}, I_c)$ as in Section 2, and p_1 is first projection (representing the elements as ordered pairs as in

Section 2). This follows because (Gf, Ff) is an $\mathbf{F}^\circ$-morphism and $SN^\circ(\Lambda^\circ(Fc)) = SD^\circ(Lc)$ and $SN^\circ(\Lambda^\circ(Gf, Gf)) = (Gf, Gf)$.

Because T is a typing functor, there are natural isomorphisms τc, τd making this diagram of functors and natural transformations commute. The component of τc at x is $\tau(Lc\,.\,x)$, τ as in the definition of typing functor.

(5.7)
$$\begin{array}{ccc} Lc & \xrightarrow{\tau c} & T\circ Lc \\ {\scriptstyle \Lambda^\circ(Gf,Ff)}\downarrow & & \downarrow{\scriptstyle \Lambda^\circ(Gf,TFf)} \\ Ld & \xrightarrow[\tau d]{} & T\circ Ld. \end{array}$$

By (2.13)° and (2.14)°, the left vertical arrow is $\alpha\,.\,Ff$ and the right one is $T(\alpha\,.\,Ff)$. Applying these functors at an object x of Gc and using (2.14)° yields

(5.8)
$$\begin{array}{ccc} Lc.x & \xrightarrow{\tau(Lc.x)} & T(Lc.x) \\ \downarrow{\scriptstyle Ff|Lc.x} & & \downarrow{\scriptstyle T(Ff|Lc.x)} \\ Ld.Gf.x & \xrightarrow[\tau(Ld.Gf.x)]{} & T(Ld.Gf.x) \end{array}$$

(the right arrow is also $TFf|T(Lc\,.\,x)$). The point is not to prove that (5.8) commutes, which is easy, but to see for later use that (5.8) is (5.7) evaluated at x.

By definition of $\mathbf{S}$ there is an arrow $(Gf, \gamma)\colon (Gc, P) \to (Gd, Q)$ of $\mathbf{S}$ for which by (5.4) the following diagram commutes. The horizontal arrows are the inclusions of (5.2).

(5.9)
$$\begin{array}{ccc} T(Lc.x) & \rightarrowtail & KP.x \\ \downarrow{\scriptstyle T(Ff|Lc.x)} & & \downarrow{\scriptstyle K(\gamma x)} \\ T(Ld.Gf.x) & \rightarrowtail & KQ.Gf.x \end{array}$$

By (2.14)°, $\Lambda^\circ(Gf, Ff) = (Gf, \alpha_{Ff})$ (a **Scat**-morphism from Lc to Ld), where $\alpha_{Ff}\colon Lc \to Ld\circ Gf$ is a natural transformation whose component at an object x of Lc is $\alpha_{Ff}\,.\,x = Ff|Lc\,.\,x$. Then putting (5.8) and (5.9) together yields a commutative diagram

(5.10)
$$\begin{array}{ccc} Lc & \overset{i_c}{\rightarrowtail} & K\circ P \\ \downarrow{\scriptstyle \alpha_{Ff}} & & \downarrow{\scriptstyle K\gamma} \\ Ld\circ Gf & \underset{i_d Gf}{\rightarrowtail} & K\circ Q\circ Gf \end{array}$$

of functors and natural transformations with i_c, i_d monic. This yields a **Scat**-diagram

(5.11)
$$\begin{array}{ccc} Lc & \overset{(id_{Gc},\, i_c)}{\rightarrowtail} & K\circ P \\ \downarrow{\scriptstyle \Lambda^\circ(Gf,Ff)} & & \downarrow{\scriptstyle (Gf,\, K\gamma)} \\ Ld & \underset{(id_{Gd},\, i_d)}{\rightarrowtail} & K\circ Q. \end{array}$$

Applying the functor SD° then yields a diagram of categories and functors whose left vertical arrow is $SD^\circ(\Lambda^\circ(Gf, Ff))=(Gf, Ff)$: $SD^\circ(Lc)\to SD^\circ(Ld)$, the same as the right vertical arrow in (5.6). Pasting the front face of (5.6) and (5.11) together yields

(5.12)
$$\begin{array}{ccc} Fc & \rightarrowtail & SD^\circ(K\circ P) \\ {\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle SD^\circ(Gf,\,K\gamma)} \\ Fd & \rightarrowtail & SD^\circ(K\circ Q). \end{array}$$

Now to complete the proof of Theorem 4.1. By (5.5), the left vertical arrow in (5.12) is $(F\circ H)(Gf, v)$. By the definition of wreathing functors in Section 3 (warning — the G there is I_G here, the f there is Gf), the right vertical arrow is $SD^\circ(Gf, K\gamma)= =SD^\circ(\overline{K}(Gf, \gamma))=K \text{ wr } I_G(Gf, \gamma)$. Thus $F\circ H$ is isomorphic to a subfunctor of the restriction of $K \text{ wr } I_G$ to **S**, as required.

6. Applications of coordinate systems. If the actions in Theorem 4.1 are discrete (F and G are set-valued), there is no requirement on λ except that it be a natural transformation. Then the category **Fib**(λ) has only arrows corresponding to the horizontal arrows in (4.1). In *any* case, if λ is split, **Fib**(λ) itself, with $K=T$ the inclusion of **Fib**(λ) into **Cat**, is a coordinate system; in CS—1, $\Phi_c=\{Lc\}$ where Lc is the splitting, and in CS—2, $m=Ff$. Thus we have the following corollary, in which I_F is the inclusion of **Fib**(λ) in **Cat** and I_G the inclusion of Im G in **Cat**.

Corollary 6.1. *If* F: **C**$\to$**Cat** *is faithful,* G: **C**$\to$**Cat**, *and* λ: $F\to G$ *a split natural transformation, then there is a subcategory* **S** *of* **Fib**(λ) wr$^{I_G}\overline{G}$ *for which* F *is isomorphic in* **Scat** *to the restriction of* I_F wr I_G *to* **S**.

Corollary 6.2. *If* F: **C**$\to$**Set**, G: **C**$\to$**Set** *and* λ: $F\to G$ *is any natural transformation, then the conclusion to Corollary* 6.1 *holds.*

The preceding corollary, when **C** is a group, could be called the *natural* Kaloujnine—Krasner theorem. It embeds **C** into a groupoid. The Kaloujnine—Krasner embedding into a group is obtained by constructing an *unnatural* typing functor which identifies all the fibers with one by noncanonical isomorphisms.

If F, G are set-valued one can always construct a coordinate system which is minimal (in states) but excessively large in transitions this way: let γ be any set whose cardinality is the supremum of the cardinalities of all the sets $Lc\,.\,x$, and the typing functor T a collection of injections of $Lc\,.\,x$ into Y. Let **M** be Trans Y, the monoid of all transformations of Y, with K its natural action. This yields

Corollary 6.3. *If* F, G, λ *are as in Corollary* 6.2, *then there is a subcategory* **S** *of* Trans Y wr^{I_G} $\overline{G}$ *and a triangle-lifting functor* H: **S**$\to$**C** *for which* $F\circ H$ *is isomorphic to a subfunctor of* K wr I_G, *where* K *is the action of* Trans Y *on* Y.

A more complicated construction leads to a decomposition via a subfunctor instead of a quotient functor; nevertheless it is an application of Theorem 4.1.

Some concepts are necessary. A functor $F\colon \mathbf{C}\to\mathbf{Cat}$ is *separated* if for distinct objects c, c' of $\mathbf{C}$, c is not an object of Fc and $Fc\cap Fc'$ is empty. Every functor $F\colon \mathbf{C}\to\mathbf{Cat}$ is isomorphic in $\mathbf{Func}(\mathbf{C}, \mathbf{Cat})$ to a separated one. (In mathematical practice people commonly assume implicitly that set-valued functors are separated.) A *transversal* of a separated functor $F\colon \mathbf{C}\to\mathbf{Cat}$ is a function Y with domain the objects of $\mathbf{C}$ such that Yc is an object of Fc. Any separated functor has a transversal by the axiom of choice.

If $\mathbf{D}$ is a subcategory of $\mathbf{Cat}$, the *constant completion* of $\mathbf{D}$, denoted $\mathbf{D}^c$, is the category whose objects are the objects of $\mathbf{D}$ and whose arrows are the arrows of $\mathbf{D}$ plus all constant functors $K_y^A\colon A\to B$, where A, B are objects of $\mathbf{D}$ and y is an object of B.

Let $F, H\colon \mathbf{C}\to\mathbf{Cat}$ be functors with H a subfunctor of F. H is *isolated* in F if for each object c of $\mathbf{C}$, Hc is the union of one or more connected components of Fc. Thus if $u\colon x\to y$ in Fc and either x or y is an object of Hc then u is an arrow of Hc. Note that if F, H are set valued then H is automatically isolated.

If H is isolated in F and F is separated then $F/H\colon \mathbf{C}\to\mathbf{Cat}$ is the functor defined by

(6.1) $$(F/H)c = (Fc-Hc)\{c\} \quad \text{for } c \text{ an object of } \mathbf{C}$$

(remember $\{c\}$ is the trivial category with object c), and for $f\colon c\to c'$ in $\mathbf{C}$,

(6.2) $$(F/H)f.x=\begin{cases} c' & \text{if } x=c \text{ or } Ff.x \text{ is in } Hc' \\ F.fx & \text{otherwise.}\end{cases}$$

There is a natural transformation $\lambda_H\colon F\to F/H$, easily seen to be split, defined by

(6.3) $$\lambda_H c.y=\begin{cases} c & \text{if } y \text{ is in } Hc \\ y & \text{otherwise.}\end{cases}$$

Proposition 6.4. *Let* $F\colon \mathbf{C}\to\mathbf{Cat}$ *be a separated functor with isolated subfunctor* H. *Then there is a subcategory* $\mathbf{S}$ *of* $(\operatorname{Im} H)^c \operatorname{wr}^I(F/H)$ *and a triangle-lifting functor* $H\colon \mathbf{S}\to\mathbf{C}$ *for which* $H\circ F$ *is isomorphic to a subfunctor of* $J \operatorname{wr} I$, *where* J *is the inclusion of* $(\operatorname{Im} H)^c$ *in* $\mathbf{Cat}$ *and* I *the inclusion of* $\operatorname{Im}(F/H)$ *in* $\mathbf{Cat}$.

Proof. The objects of $\mathbf{Fib}(\lambda_H)$ are (a) the categories Hc for object c of $\mathbf{C}$, and (b) the categories $\{x\}$ where x is an object of Fc not in Hc. Arrows are of the form (a) $Hf\colon Hc\to Hd$ for arrows $f\colon c\to d$ in $\mathbf{C}$, and (b) $\{x\}\to\{y\}\to\{Ff.y\}$ where $u\colon x\to y$ is an arrow of Fc not in Hc and $f\colon c\to d$ in $\mathbf{C}$. Arrows of type (a) do not compose with arrows of type (b) in either order. Thus $Lc.c=Hc$, $Lc.x=\{x\}$ for

x an object of $Fc-Hc$, and for $f\colon c\to d$, $(Lc\,.f)c=Hf$, $(Lc\,.f)u=\{Ff\,.\,x\}\to\{Ff\,.\,y\}$ for $u\colon x\to y$ in $Fc-Hc$.

Define a typing functor T as follows. For objects Hc of $\mathbf{Fib}(\lambda_H)$, $T(Hc)=Hc$. For objects $\{x\}$ where x is an object of $Fc-Hc$, $T\{x\}=\{Yc\}$. For arrows $Hf\colon Hc\to Hd$, $T(Hf)=Hf$. For arrows $g\colon \{x\}\to\{y\}\to\{Ff\,.\,y\}$ where $u\colon x\to y$ in $Fc-Hc$ and $f\colon c\to d$ in $\mathbf{C}$, $Hg=\{Yc\}\to\{Yd\}$.

Then $((\operatorname{Im} H)^c, J, T)$ is a coordinate system. For CS—1, let $\Phi_c=\{K_{Hc}^{(F/H).c}\}$. For CS—2, let $f\colon c\to d$ in $\mathbf{C}$ and x be an object of $(F/H)c$. If $x=c$ set $m=Hf\colon Hc\to Hd$. If $x\in Fc-Hc$ and $Ff\,.\,x$ is in $Fd-Hd$, set $m=K_{Ff.x}^{Hc}$. If $Ff\,.\,x$ is in Hd, set $m=K_{Yd}^{Hc}$. It is straightforward to verify that CS—2 holds for this definition. The proposition now follows from Theorem 4.1.

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