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Wreath product decomposition of categories. II*)

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1. Introduction. In this paper, I prove a theorem which shows how to decompose a functor F: $\mathbf{C} \to \mathbf{Cat}$ into the wreath product of two functors, given a right ideal and a "wide" subcategory of $\mathbf{C}$ which together generate $\mathbf{C}$ (this is made precise in Section 2).

The decomposition is in the sense of Krohn—Rhodes theory: the functor F is not *embedded* in a wreath product, but rather a subfunctor of the wreath product maps *onto* F, like a covering space. This is in contrast to the decomposition theorem of WELLS [4], although of course any embedding is an example of decomposition in the present sense. The theorem in this paper actually generalizes one of the decomposition techniques used in proving the Krohn—Rhodes Theorem (KROHN—RHODES [2], EILENBERG [1], WELLS [3]), although it works just as well for infinite categories. Note that one of the corollaries of the decomposition theorem in WELLS [4] generalizes another of the techniques used in proving the Krohn—Rhodes Theorem.

My hope is that the decomposition techniques described here and in WELLS [4] will be useful in developing a theory of "state-transition systems with structured, typed states". This is discussed in WELLS [4] so I will say no more about it here.

The present paper is self-contained except for the terminology developed in Section 2.3 of WELLS [4].

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2. Statement of the theorem. If $\mathbf{C}$, $\mathbf{D}$ are categories and x an object of $\mathbf{D}$, the *constant functor* $K_x^{\mathbf{C}}$ takes all objects of $\mathbf{C}$ to x and all arrows to 1_x. The *constant completion* of a subcategory $\mathbf{A}$ of $\mathbf{Cat}$ consists of the subcategory of $\mathbf{Cat}$ consisting of everything in $\mathbf{A}$ and all constant functors K_x^a where a is an object of $\mathbf{A}$ and x is an object of some object of $\mathbf{A}$.

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If $\mathbf{C}$ is a small category, the *global* hom *functor* $\mathbf{C}_*: \mathbf{C} \to \mathbf{Cat}$ takes an object c to the set of all arrows into c, and $f: c \to d$ to the function from $\mathbf{C}_* c$ to $\mathbf{C}_* d$ which takes $x: a \to c$ to $f \circ x$. C_* is set valued, regarded as a discrete-category-valued functor.

The *constant completion* of a small category $\mathbf{C}$, denoted $\mathbf{C}^c$, is the constant completion in the sense defined earlier of the image of $\mathbf{C}_*$. $\mathbf{C}_*$ is injective, and I shall identify $\mathbf{C}$ with its image, so that $\mathbf{C}_* f: \mathbf{C}_* c \to \mathbf{C}_* d$ *is* $f: c \to d$. I shall write K_x^c for $K_x^{\mathbf{C}_* c}$. This has the following notational consequences:

a) $K_x^c: c \to d$ where x is an arrow with codomain d. (The notation does not determine dom x.)

b) If $K_x^c: c \to d$ and $g: d \to e$ then $g \circ K_x^c = K_{g \circ x}^c$.

c) If $K_x^c: c \to d$ and $h: b \to c$ then $K_x^c \circ h = K_x^b$.

d) If it is defined, $K_y^d \circ K_x^c = K_y^c$.

The inclusion $\mathbf{C}_*: \mathbf{C}^c \to \mathbf{Cat}$ is denoted $d_{\mathbf{C}}$.

A subclass I of arrows of a category $\mathbf{C}$ is a *right ideal* if for any arrow f of $\mathbf{C}$ and g of I, if $g \circ f$ is defined then it is in I. An example of a right ideal is any Grothendieck topology on $\mathbf{C}$. If I is a right ideal (which need not be a subcategory of $\mathbf{C}$), I^1 denotes the subcategory consisting of all objects and identity arrows of $\mathbf{C}$ and all arrows of I.

A subcategory $\mathbf{D}$ of $\mathbf{C}$ is *wide* if it has the same objects as $\mathbf{C}$. If $\mathbf{C} = \mathbf{D} \circ I$ for some subcategory $\mathbf{D}$ and right ideal I then $\mathbf{C}$ is *generated* by $\mathbf{D}$ and I. A functor $H: \mathbf{A} \to \mathbf{B}$ *lifts triangles* if for all arrows f of $\mathbf{A}$ and h, k of $\mathbf{B}$ for which $Hf \circ h$ and $k \circ Hf$ are defined, there are arrows u, v of $\mathbf{A}$ for which $f \circ u$ and $v \circ f$ are defined and $Hu = h$, $Hv = k$. The motivation for requiring this property in wreath product decompositions is discussed in WELLS [4, § 4].

Theorem. *Let* $\mathbf{C}$ *be a small category and* $G: \mathbf{C} \to \mathbf{Cat}$ *a functor. Let* $\mathbf{D}$ *be a wide subcategory and* I *a right ideal which generate* $\mathbf{C}$. *Then there is a subcategory* $\mathbf{S}$ *of* I^1 wr $\mathbf{D}^c$ *(action by* $J_{\mathbf{D}}$*), a triangle-lifting functor* $H: \mathbf{S} \to \mathbf{C}$ *and a surjective natural transformation*

$$\theta: W \to G \circ H \quad \textit{where} \quad W = [(G|I^1) \text{ wr } J_{\mathbf{D}}]|\mathbf{S}.$$

Note. This theorem cannot be strengthened to make $G \circ H$ a subfunctor of W, even when G is set valued and the categories are all monoids.

3. Proof of the Theorem. For an object c of $\mathbf{C}$, let $\delta^c: \mathbf{D}_* c \to I^1$ be the function taking an arrow to its domain, and $i^c: \mathbf{D}_* c \to I^1$ the function taking an arrow to the identity arrow of its domain.

Define $\mathbf{S}$ as follows. An object of $\mathbf{S}$ is any pair (c, δ^c) for any object c of $\mathbf{C}$. Arrows are of the following two forms.

$$(f, i^b): (b, \delta^b) \to (c, \delta^c) \tag{3.1}$$

for all arrows $f\colon b\to c$ in $\mathbf{D}$, and

(3.2) $$(K_g^c, \mathbf{C}_* h|\mathbf{D}_* c)\colon(c, \delta^c) \to (e, \delta^e)$$

for all $h\colon c\to d$ in I^1 and $g\colon d\to e$ in $\mathbf{D}$.

Let's check that (3.2) makes sense ((3.1) is easier). An arrow of I^1 wr $\mathbf{D}^c$ must by definition be of the form $(f, \lambda)\colon (c, P)\to(d, Q)$ where $f\colon c\to d$, $P\colon \mathbf{D}_* c\to I^1$, $Q\colon \mathbf{D}_* d\to I^1$, and $\lambda\colon P\to Q\circ J_{\mathbf{D}} f$ is a natural transformation (note that $\mathbf{D}_*$ is discrete so there are no commutativity conditions for natural transformations here). Here, $K_g^c\colon \mathbf{D}_* c\to \{g\}\subset\mathbf{D}_* e$. For an object $f\colon b\to c$ of $\mathbf{D}_* c$ the component of the natural transformation must be an arrow from $\delta^c f=b$ to $(\delta^e\circ K_g^c)f=\delta^e g=d$. This works because $\mathbf{C}_* h\,.\,f=h\circ f\colon b\to d$.

Define the functor $H\colon \mathbf{S}\to\mathbf{C}$ by $H(f, i^b)=f$ and $H(K_g^c, \mathbf{C}_* h)=g\circ h$.

We have the following formulas for composition of arrows in $\mathbf{S}$, which prove that H is a functor. H is bijective on objects, so lifts triangles.

(3.3) $$(g, \delta^c)\circ(f, \delta^b) = (g\circ f, \delta^b)$$

for $f\colon b\to c$, $g\colon c\to d$ in $\mathbf{D}$

(3.4) $$(K_g^c, \mathbf{C}_* h)\circ (f, \delta^b) = (K_g^b, \mathbf{C}_*(h\circ f))$$

for $f\colon b\to c$, $h\colon c\to d$, $g\colon d\to e$ in $\mathbf{D}$.

(3.5) $$(g, \delta^d)\circ(K_m^b, \mathbf{C}_* k) = (K_{g\circ m}^b, \mathbf{C}_* k)$$

for $k\colon b\to c$ in I, $m\colon c\to d$, $g\colon d\to e$ in $\mathbf{D}$.

(3.6) $$(K_n^e, \mathbf{C}_* m)\circ(K_g^c, \mathbf{C}_* h) = (K_n^c, \mathbf{C}_*(m\circ g\circ h))$$

for $h\colon c\to d$, $m\colon e\to p$ in I, $g\colon d\to e$, $n\colon p\to q$ in $\mathbf{D}$.

To simplify notation in the definition of θ, the component of θ at an object (b, δ^b) of $\mathbf{S}$ will be denoted θb. First note that for each object b, $W(b, \delta^b)$ is the disjoint union of categories Ga indexed by all arrows $f\colon a\to b$ of $\mathbf{D}$. This follows from the definition of the wreath product of functors in WELLS [4, §3]: An object of $W(b, \delta^b)$ is a pair (f, x) with $f\colon a\to b$ (some a) and x an object of Ga. An arrow has to look like $(f, r)\colon (f, x)\to(f, y)$ where $r\colon x\to y$ in Ga, $f\colon a\to b$ in $\mathbf{D}$, since $\mathbf{D}_* b$ is a set (discrete category).

Now, to define the component $\theta b\colon W(b, \delta^b)\to G\circ H(b, \delta^b)=Gb$, set

(3.7) $$\theta b.(f, r) = Gf.r,$$

for $f\colon a\to b$ in $\mathbf{D}$, r an arrow of Ga.

To prove that θ is a natural transformation requires (after applying the definition of H) proving the following diagrams commute.

(3.8) $$\begin{tikzcd} W(b, \delta^b) \arrow[r, "\theta b"] \arrow[d, "W(g, \delta^b)"'] & Gb \arrow[d, "Gg"] \\ W(c, \delta^c) \arrow[r, "\theta c"'] & Gc, \end{tikzcd}$$

for $g\colon b\to c$ in $\mathbf{D}$, and

$$\begin{tikzcd} W(b,\delta^b) \arrow[r, "\theta b"] \arrow[d, "W(K_g^b,\, C_* h)"'] & Gb \arrow[d, "G(g\circ h)"] \\ W(c,\delta^c) \arrow[r, "\theta c"'] & Gc \end{tikzcd} \tag{3.9}$$

for $h\colon b\to c$ in I and $g\colon c\to d$ in $\mathbf{D}$.

These facts follow from an easy application of the definitions. Given $f\colon a\to b$ in $\mathbf{D}$ and starting at the upper left corner of (3.8), the northeast route gives $(f,r)\mapsto (Gf).r\mapsto(Gg\circ Gf).r$ and the southwest route gives $(f,r)\mapsto(g\circ f,r)\mapsto G(g\circ f).r$. For (3.9) the corresponding chases are $(f,r)\mapsto Gf.r\mapsto\big(G(g\circ h)\circ Gf\big).r$ and $(f,r)\mapsto \big(g, G(h\circ f).r\big)\mapsto\big(Gg\circ G(h\circ f)\big).r$.

This proves the Theorem.

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