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Interval filling sequences and additive functions

ZOLTÁN DARÓCZY and IMRE KÁTAI

1. Introduction. Interval filling sequences have been defined in our paper [1]. Let Λ denote the set of all real sequences, for which the conditions $\lambda_n > \lambda_{n+1} > 0$ $(n \in \mathbf{N})$ and $L := \sum_{n=1}^{\infty} \lambda_n < \infty$ hold.

Definition 1.1. We call the sequence $\{\lambda_n\} \in \Lambda$ *interval filling*, if for any $x \in [0, L]$ there exists a sequence $\{\varepsilon_n\}$, $\varepsilon_n \in \{0, 1\}$ $(n \in \mathbf{N})$, such that

$$x = \sum_{n=1}^{\infty} \varepsilon_n \lambda_n. \tag{1.1}$$

We have the following result ([1]):

Theorem 1.2. *The sequence* $\{\lambda_n\} \in \Lambda$ *is interval filling if and only if*

$$\lambda_n \leqq \sum_{i=n+1}^{\infty} \lambda_i \tag{1.2}$$

for any $n \in \mathbf{N}$.

Let $\{\lambda_n\} \in \Lambda$ be an interval filling sequence. For $x \in [0, L\}$ we define by induction on n

$$\varepsilon_n(x) := \begin{cases} 1 & \text{if} \quad \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n \leqq x, \\ 0 & \text{if} \quad \sum_{i=1}^{n-1} \varepsilon_i(x)\lambda_i + \lambda_n > x. \end{cases} \tag{1.3}$$

It is known ([1]) that

$$x = \sum_{n=1}^{\infty} \varepsilon_n(x)\lambda_n. \tag{1.4}$$

We call the representation (1.4) of the number x the *regular* expansion of x.

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Definition 1.3. Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence and $a_n\in\mathbf{C}$ such that $\sum_{n=1}^{\infty}|a_n|<\infty$. Then we call the function

$$F(x):=\sum_{n=1}^{\infty}\varepsilon_n(x)\,a_n \quad (x\in[0,L]) \tag{1.5}$$

additive (with respect to the interval filling sequence $\{\lambda_n\}\in\Lambda$), where $\varepsilon_n(x)$ denotes the digits (0, 1) determined by algorithm (1.3).

In this paper we give an exact description of the set of those points in which an additive function is *continuous.* Following this, with the help of quasiregular expansions we give a criterium for the continuity in $[0, L]$ of additive functions. Thus we generalize our results obtained in [2] which referred to special interval filling sequences

$$\lambda_n:=\frac{1}{q^n} \quad (1<q\leqq 2).$$

As to further properties of continuous additive functions, we refer to our result in [2], according which there exist an interval filling sequence and a function F continuous and additive with respect to it, such that this function is nowhere differentiable in $[0, L]$.

In this paper $\{\lambda_n\}\in\Lambda$ will denote an arbitrary but fixed interval filling sequence, even if we do not emphasize it explicitely.

2. Finite numbers. Finite numbers will play a fundamental role in the sequel.

Definition 2.1. Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence. We call the number $x\in[0, L]$ *finite*, if there exists $N\in\mathbf{N}$ such that $\varepsilon_n(x)=0$ for $n>N$. If x is finite and $\varepsilon_m(x)=1$ moreover $\varepsilon_n(x)=0$ for $n>m$, then we say that x has *length* m, and write $h(x)=m$. We define $h(0)=0$, i.e. $x=0$ is also a finite number.

Let $N\in\mathbf{N}$ and

$$V_N:=\{t\mid t\in[0,L],\ h(t)\leqq N\} \tag{2.1}$$

the set of finite numbers having length not greater than N. For $0<x\leqq L$ we put

$$b_N(x):=\max\{t\mid t\in V_N,\ t<x\} \tag{2.2}$$

and call this number the *left neighbour* of x in V_n.

Lemma 2.1. *Let* $0<x\leqq L$ *be arbitrary. Then for any* $b_N(x)<y<x$ *we have*

$$\varepsilon_n(y)=\varepsilon_n[b_N(x)] \quad \text{if} \quad n\leqq N. \tag{2.3}$$

Proof. If $b_N(x)<y<x$ then let

$$y=\sum_{n=1}^{N}\varepsilon_n(y)\,\lambda_n+\sum_{n=N+1}^{\infty}\varepsilon_n(y)\,\lambda_n.$$

Clearly

$$S_N(y) := \sum_{n=1}^{N} \varepsilon_n(y)\lambda_n \in V_N.$$

The inequality $b_N(x) < S_N(y) \leqq y < x$ is impossible by the definition of $b_N(x)$. Thus $S_N(y) \leqq b_N(x)$. Now $S_N(y) < b_N(x)$ implies the existence of a first index $k \in \{1, 2, \dots, N\}$ such that $\varepsilon_k(y) = 0$ and $\varepsilon_k[b_N(x)] = 1$. From this, by algorithm (1.3),

$$b_N(x) \geqq \sum_{n=1}^{k-1} \varepsilon_n[b_N(x)] + \lambda_k = \sum_{n=1}^{k-1} \varepsilon_n(y)\lambda_n + \lambda_k > y$$

follows, a contradiction. Thus $S_N(y) = b_N(x)$, and this implies (2.3).

3. Additive functions.

Theorem 3.1. *Let F: $[0, L] \to \mathbf{C}$ be an additive function. Then F is continuous at every nonfinite point x.*

Proof. Let $0 < x < L$ be a nonfinite number. Let $\varepsilon > 0$. Then there exists $N_0 \in \mathbf{N}$ such that

$$2 \sum_{n=N_0+1}^{\infty} |a_n| < \varepsilon.$$

Let $N > N_0$ be such that $x < \sum_{n=1}^{N} \lambda_n \in V_N$ and put

$$j_N(x) := \min \{t \mid t \in V_N,\, x < t\}.$$

Then $x < j_N(x)$. We assert that

$$b_N[j_N(x)] < x < j_N(x). \tag{3.1}$$

As a matter of fact, $b_N[j_N(x)] \neq x$ because x is nonfinite, and $x < b_N[j_N(x)]$ would contradict the definition of $j_N(x)$.

If $b_N[j_N(x)] < y < j_N(x)$ (i.e. if y is in the neighbourhood (3.1) of x), then by Lemma 2.1

$$\varepsilon_n(y) = \varepsilon_n\{b_N[j_N(x)]\} = \varepsilon_n(x) \quad \text{for} \quad n \leqq N,$$

whence

$$|F(x) - F(y)| = \Big|\sum_{n=1}^{\infty} \varepsilon_n(x)\, a_n - \sum_{n=1}^{\infty} \varepsilon_n(y) a_n\Big| =$$

$$= \Big|\sum_{n=N+1}^{\infty} [\varepsilon_n(x) - \varepsilon_n(y)\, a_n\Big| \leqq 2 \sum_{n=N+1}^{\infty} |a_n| < \varepsilon,$$

i.e. F is continuous at x.

We still have to consider the case $x=L$ (L is a nonfinite number). Here we must prove continuity from the left. Now

$$b_N(L) = \max\{t \mid t\in V_N,\ t < L = x\} = \sum_{n=1}^{N} \lambda_n.$$

Hence, if $b_N(L)<y<L$ then by Lemma 2.1 $\varepsilon_n(y)=1$ for $n \leqq N$. This implies

$$|F(L)-F(y)| = \Big|\sum_{n=N+1}^{\infty} [1-\varepsilon_n(y)]\, a_n\Big| \leqq 2 \sum_{n=N+1}^{\infty} |a_n| < \varepsilon$$

for $N>N_0$, i.e. F is left continuous in $x=L$.

Theorem 3.2. *Let F: $[0, L]\to \mathbf{C}$ be an additive function. Then F is right continuous at every finite point $x\in[0, L]$.*

Proof. Let x be finite and $m=h(x)$. Then for any $\varepsilon>0$ there exists $N>m$ such that

$$\sum_{n=N+1}^{\infty} |a_n| < \varepsilon.$$

Now $x\in V_N$. We have by definitions $b_N[j_N(x)]=x$. Hence by Lemma 2.1 for any

$$x = b_N[j_N(x)] < y < j_N(x)$$

the relation

$$\varepsilon_n(y) = \varepsilon_n\{b_N[j_N(x)]\} = \varepsilon_n(x) \quad (n \leqq N)$$

holds. Hence

$$|F(x)-F(y)| = \Big|\sum_{n=1}^{N} \varepsilon_n(x)\, a_n - \sum_{n=1}^{\infty} \varepsilon_n(y)\, a_n\Big| =$$

$$= \Big|\sum_{n=N+1}^{\infty} \varepsilon_n(y)\, a_n\Big| \leqq \sum_{n=N+1}^{\infty} |a_n| < \varepsilon,$$

i.e. F is right continuous in x.

4. Examples.

Example 4.1. Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence. Let moreover $a_1=a_2=1$ and $a_n=0$ for $n>2$. The additive function determined by the sequence a_n is

$$F(x) = \begin{cases} 0 & \text{for} \quad 0 \leqq x < \lambda_2, \\ 1 & \text{for} \quad \lambda_2 \leqq x < \lambda_1+\lambda_2, \\ 2 & \text{for} \quad \lambda_1+\lambda_2 \leqq x \leqq L. \end{cases}$$

Clearly, this function is not continuous at the finite points λ_2, $\lambda_1+\lambda_2$. On the basis of this the question arises, how exact are Theorems 3.1 and 3.2. The answer is given by the following example.

Example 4.2. *There exists with respect to the interval filling sequence* $\left\{\lambda_n:=\frac{1}{2^n}\right\}\in\Lambda$ *an additive function* F *which is noncontinuous at every finite point* $x>0$.

Proof. We have $L:=\sum_{n=1}^{\infty}\frac{1}{2^n}=1$ and the algorithm (1.3) yields the unique dyadic respresentation of the numbers $x\in[0, 1]$. The numbers $\frac{l}{2^n}\,(0\leq l<2^n)$ and only these are finite, any other number is nonfinite. Let $a_n:=\frac{1}{n^2}$ for which $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$, and let

$$F(x):=\sum_{n=1}^{\infty}\frac{\varepsilon_n(x)}{n^2}$$

for any $x\in[0, 1]$. Let still $x\in]0, 1[$ be finite and $h(x)=m\geq 1$. Then

$$x=\sum_{n=1}^{m-1}\frac{\varepsilon_n(x)}{2^n}+\frac{1}{2^m}.$$

Let $N>m$ and

$$x_N:=\sum_{n=1}^{m-1}\frac{\varepsilon_n(x)}{2^n}+\frac{0}{2^m}+\frac{1}{2^{m+1}}+\dots+\frac{1}{2^N}. \tag{4.1}$$

Since the right hand side of (4.1) is a regular expansion of x_N, we get

$$F(x_N)=\sum_{n=1}^{m-1}\frac{\varepsilon_n(x)}{n^2}+\frac{1}{(m+1)^2}+\dots+\frac{1}{N^2} \tag{4.2}$$

If F were continuous in x, then $x_N\to x$ would imply $F(x_N)\to F(x)$ $(N\to\infty)$. However from (4.2) we get

$$\lim_{N\to\infty}F(x_N)=\sum_{n=1}^{m-1}\frac{\varepsilon_n(x)}{n^2}+\frac{\pi^2}{6}-\frac{1}{1^2}-\dots-\frac{1}{m^2},$$

and this would imply

$$F(x)=\sum_{n=1}^{m-1}\frac{\varepsilon_n(x)}{n^2}+\frac{1}{m^2}=\lim_{N\to\infty}F(x_N),$$

i.e.

$$\frac{1}{m^2}=\frac{\pi^2}{6}-\frac{1}{1^2}-\dots-\frac{1}{m^2},$$

which is a contradiction, because π^2 is not rational.

5. Quasiregular expansions. Let $\{\lambda_n\}\in\Lambda$ be an interval filling sequence. For $x\in[0, L]$, by induction on n, let

$$\text{(5.1)}\qquad \varepsilon_n^*(x) := \begin{cases} 1 & \text{for} \quad \sum_{i=1}^{n-1} \varepsilon_i^*(x)\lambda_i + \lambda_n < x, \\ 0 & \text{for} \quad \sum_{i=1}^{n-1} \varepsilon_i^*(x)\lambda_i + \lambda_n \geqq x. \end{cases}$$

Theorem 5.1. *For any* $x\in[0, L]$ *we have*

$$\text{(5.2)}\qquad x = \sum_{n=1}^{\infty} \varepsilon_n^*(x)\lambda_n.$$

Proof. (i): For $x=0$ and $x=L$ (5.2) is trivially valid. (ii): If $0<x<L$ and $\varepsilon_n^*(x)=0$ for infinitely many values of n, then $N_0 := \{n | n\in N,\ \varepsilon_n^*(x)=0\}$ is an infinite set. If $n\in N_0$ then

$$0 \leqq x - \sum_{i=1}^{\infty} \varepsilon_i^*(x)\lambda_i \leqq x - \sum_{i=1}^{n-1} \varepsilon_i^*(x)\lambda_i \leqq \lambda_n$$

whence by $\lambda_n\to 0$ $(n\in N_0,\ n\to\infty)$ (5.2) follows. (iii): If $0<x<L$ and $\varepsilon_n^*(x)=0$ holds only for finitely many values of n, then let N be the greatest index, for which $\varepsilon_N^*(x)=0$ (i.e. $\varepsilon_n^*(x)=1$ if $n>N$). Then

$$x - \sum_{i=1}^{N-1} \varepsilon_i^*(x)\lambda_i \leqq \lambda_N \leqq \sum_{i=N+1}^{\infty} \lambda_i = \sum_{i=N+1}^{\infty} \varepsilon_i^*(x)\lambda_i$$

whence

$$x \leqq \sum_{i=1}^{\infty} \varepsilon_i^*(x)\lambda_i,$$

i.e. (5.2) holds.

Definition 5.2. We call the representation (5.2) the *quasiregular* expansion of x.

Lemma 5.3. *If* $0<x\leqq L$ *then* $\varepsilon_N^*(x)=1$ *for infinitely many values of* n.

Proof. Suppose the contrary, and let N be the largest index with $\varepsilon_N^*(x)=1$. Then

$$x = \sum_{i=1}^{N} \varepsilon_i^*(x)\lambda_i = \sum_{i=1}^{N-1} \varepsilon_i^*(x)\lambda_i + \lambda_N$$

and so by (5.1) $\varepsilon_N^*(x)=0$, a contradiction.

Lemma 5.4. *If* $0<x\leqq L$ *is a nonfinite number, then* $\varepsilon_n(x)=\varepsilon_n^*(x)$ *for every* $n\in N$, *i.e. the regular and quasiregular expansions coincide.*

Proof. Suppose the contrary, and let k be the first index for which $\varepsilon_k(x) \neq \varepsilon_k^*(x)$. By the definitions of $\varepsilon_k(x)$ and $\varepsilon_k^*(x)$ then we have $\varepsilon_k(x)=1$ and $\varepsilon_k^*(x)=0$. Hence

$$\sum_{i=1}^{k-1} \varepsilon_i(x)\lambda_i+\lambda_k \leqq x$$

and

$$\sum_{i=1}^{k-1} \varepsilon_i^*(x)\lambda_i+\lambda_k \geqq x.$$

Now $\varepsilon_i(x)=\varepsilon_i^*(x)$ for $i=1, 2, \ldots, k-1$; hence the previous inequalities yield

$$x = \sum_{i=1}^{k-1} \varepsilon_i(x)\lambda_i+\lambda_k,$$

i.e. x is finite, a contradiction.

Quasiregular expansions make it possible to determine for a number $0<x\leqq L$ its left neighbour $b_N(x)$ (see Definition 2.1), and to describe exactly the regular expansion of the latter. This we formulate in the following statement.

Theorem 5.5. *If* $0<x\leqq L$ *then*

$$b_N(x) = \sum_{n=1}^{N} \varepsilon_n^*(x)\lambda_n, \tag{5.3}$$

where the right hand side is the regular expansion of $b_N(x)$, *i.e.*

$$\varepsilon_n[b_N(x)] = \varepsilon_n^*(x) \quad \textit{for} \quad n = 1, 2, \ldots, N. \tag{5.4}$$

Proof. Suppose that, contradicting our assertion, there exists $z\in V_N$ such that $b_N(x)<z<x$.

(i) If x is *nonfinite*, then its regular and quasiregular expansions coincide. Let $x=\sum_{n=1}^{\infty} \varepsilon_n(x)\lambda_n$. Then $b_N(x)=\sum_{n=1}^{N} \varepsilon_n(x)\lambda_n$. Let $z=\sum_{n=1}^{N} \varepsilon_n(z)\lambda_n$. Since $b_N(x)<z$, there exists a first index $k\in\{1, 2, \ldots, N\}$ such that $\varepsilon_k(x)\neq\varepsilon_k(z)$. This is only possible if $\varepsilon_k(z)=1$ and $\varepsilon_k(x)=0$. Hence

$$z = \sum_{i=1}^{k-1} \varepsilon_i(z)\lambda_i+\lambda_k+\sum_{i=k+1}^{N} \varepsilon_i(z)\lambda_i \geqq$$

$$\geqq \sum_{i=1}^{k-1} \varepsilon_i(z)\lambda_i+\lambda_k = \sum_{i=1}^{k-1} \varepsilon_i(x)\lambda_i+\lambda_k > x,$$

a contradiction.

(ii) If x is *finite*, then let $h(x)=m\geqq 1$, i.e.

$$x = \sum_{n=1}^{m-1} \varepsilon_n(x)\lambda_n+\lambda_m.$$

Then

$$\lambda_m = \sum_{i=1}^{\infty} \varepsilon_i^*(\lambda_m)\lambda_i = \sum_{i=m+1}^{\infty} \varepsilon_i^*(\lambda_m)\lambda_i$$

because $\varepsilon_i^*(\lambda_m)=0$ for $i=1, 2, \dots, m$. Hence

$$x = \sum_{n=1}^{m-1} \varepsilon_n(x)\lambda_n + \sum_{i=m+1}^{\infty} \varepsilon_i^*(\lambda_m)\lambda_i. \tag{5.5}$$

Clearly, the right hand side of (5.5) is the *quasiregular* expansion of x, i.e.

$$\varepsilon_n^*(x) = \begin{cases} \varepsilon_n(x) & \text{for } n = 1, 2, \dots, m-1, \\ 0 & \text{for } n = m, \\ \varepsilon_n^*(\lambda_m) & \text{for } n = m+1, m+2, \dots. \end{cases} \tag{5.6}$$

If $m \geqq N$ then the proof is the same as in (i). If $m<N$, then let $z=\sum_{n=1}^{N} \varepsilon_n(z)\lambda_n$. Now by $b_N(x)<z$ there exists a first index $m<k\leqq N$ such that $\varepsilon_k(z)=1$ and $\varepsilon_k^*(x)=0$. Hence

$$z = \sum_{i=1}^{k-1} \varepsilon_i(z)\lambda_i + \lambda_k + \sum_{i=k+1}^{N} \varepsilon_i(z)\lambda_i \geqq$$

$$\geqq \sum_{i=1}^{k-1} \varepsilon_i(z)\lambda_i + \lambda_k = \sum_{i=1}^{k-1} \varepsilon_i^*(x)\lambda_i + \lambda_k \geqq x,$$

and this contradicts the condition $z<x$.

6. Quasiadditive functions. The notion of quasiadditive function will be defined in analogy to that of additive function.

Definition 6.1. Let $a_n \in \mathbf{C}$ and $\sum_{n=1}^{\infty} |a_n| < \infty$. The function F: $[0, L] \to \mathbf{C}$ is said to be *quasiadditive* if

$$F(x) = \sum_{n=1}^{\infty} \varepsilon_n^*(x) a_n \tag{6.1}$$

for any $x \in [0, L]$, where $\varepsilon_n^*(x)$ denotes the digits 0, 1 determined by algorithm (5.1).

Remark. If $a_n \in \mathbf{C}$ $\left(\sum_{n=1}^{\infty} |a_n| < \infty\right)$ then this sequence determines an *additive* function (say F_1), and a *quasiadditive* function (say F_2). By Lemma 5.4. $F_1(x)=F_2(x)$ holds for any *nonfinite* $x \in [0, L]$, and trivially also for $x=0$ and $x=L$. Hence, in general, the two functions differ only at the *finite* points $0<x<L$.

Definition 6.2. We call the function F: $[0, L] \to \mathbf{C}$ *biadditive*, if it is both additive and quasiadditive.

Lemma 6.3. *The additive function* F: $[0, L]\to\mathbf{C}$ *determined by the sequence* $a_n\in\mathbf{C}$ $\left(\sum_{n=1}^{\infty}|a_n|<\infty\right)$ *is biadditive if and only if*

(6.2)
$$a_n = \sum_{i=n+1}^{\infty} \varepsilon_i^*(\lambda_n)\, a_i$$

is satisfied for every $n\in\mathbf{N}$.

Proof. (i): If F is also quasiadditive, then

$$\lambda_n = \sum_{i=n+1}^{\infty} \varepsilon_i^*(\lambda_n)\,\lambda_i$$

implies

$$a_n = F(\lambda_n) = \sum_{i=n+1}^{\infty} \varepsilon_i^*(\lambda_n)\, a_i,$$

i.e. (6.2) holds. (ii): If (6.2) is valid, then by the foregoing it suffices to show that (6.1) holds for every *finite* number $0<x<L$. Let $h(x)=m\geqq 1$ and

$$x = \sum_{n=1}^{m-1} \varepsilon_n(x)\,\lambda_n+\lambda_m = \sum_{n=1}^{m-1} \varepsilon_n(x)\,\lambda_n+\sum_{i=m+1}^{\infty} \varepsilon_i^*(\lambda_m)\,\lambda_i.$$

Then by (5.6) we know the quasiregular representation of x, hence using (6.2) we get

$$F(x) = \sum_{n=1}^{m-1} \varepsilon_n(x)\, a_n+a_m =$$

$$= \sum_{n=1}^{m-1} \varepsilon_n(x)\, a_n+\sum_{i=m+1}^{\infty} \varepsilon_i^*(\lambda_m)\, a_i = \sum_{n=1}^{\infty} \varepsilon_n^*(x)\, a_n,$$

i.e. (6.1) holds.

Lemma 6.4. *If* F: $[0, L]\to\mathbf{C}$ *is additive and continuous in* $[0, L]$, *then* F *is quasiadditive (i.e.* F *is biadditive).*

Proof. The function F is left continuous at every λ_n, where

$$\lambda_n = \sum_{i=n+1}^{\infty} \varepsilon_i^*(\lambda_n)\,\lambda_i.$$

Let $N>n$ and

(6.3)
$$b_N(\lambda_n) = \sum_{i=n+1}^{N} \varepsilon_i^*(\lambda_n)\,\lambda_i.$$

Then by Theorem 5.5 the right hand side of (6.3) is a regular expansion and $b_N(\lambda_n)\to\lambda_n$ (for $N\to\infty$), hence by continuity

$$a_n = F(\lambda_n) = \lim_{\substack{N\to\infty\\ N>n}} F[b_N(\lambda_n)] =$$

$$= \lim_{\substack{N\to\infty\\ N>n}} \sum_{i=n+1}^{N} \varepsilon_i^*(\lambda_n)\, a_i = \sum_{i=n+1}^{\infty} \varepsilon_i^*(\lambda_n)\, a_i$$

for every $n\in\mathbf{N}$, i.e. (6.2) holds. From Lemma 6.3. it follows immediately that F is quasiadditive (i.e. biadditive).

Remark. By Lemma 6.4 quasiadditivity is a *necessary* condition for the continuity of an additive function F; also, by Lemma 6.3 it is necessary that for the sequence $a_n\in\mathbf{C}$ $\left(\sum_{n=1}^{\infty}|a_n|<\infty\right)$ the difference equations (6.2) $(n=1, 2, \ldots)$ should be valid.

7. Continuous additive functions.

Theorem 7.1. *An additive function* F: $[0, L]\to\mathbf{C}$ *is continuous in* $[0, L]$ *if and only if it is quasiadditive (i.e. biadditive).*

Proof. By Theorems 3.1—3.2 and Lemma 6.4. it will be sufficient to show that if F is also quasiadditive then it is left continuous at every *finite* point $0<x<L$.

For the sequence $a_n\in\mathbf{C}$ determining the additive function F it is clearly true that for any $\varepsilon>0$ there exists N_0 such that $N>N_0$ implies

$$2\sum_{n=N+1}^{\infty}|a_n| < \varepsilon.$$

Let x be finite and $h(x)=m\geqq 1$, i.e.

$$x = \sum_{i=1}^{m-1} \varepsilon_i(x)\,\lambda_i+\lambda_m.$$

If $N>m$ then

$$b_N(x) = \sum_{i=1}^{m-1} \varepsilon_i(x)\,\lambda_i+\sum_{i=m+1}^{N} \varepsilon_i^*(\lambda_m)\,\lambda_i$$

is a regular expansion (Theorem 5.5), and in case $b_N(x)<y<x$ we have by Lemma 2.1

$$\varepsilon_n(y) = \varepsilon_n[b_N(x)] \qquad (n = 1, 2, \ldots, N).$$

Hence by the quasiadditivity of F we get from (6.2)

$$|F(x)-F(y)| = \Big|\sum_{i=1}^{m-1} \varepsilon_i(x)\, a_i + a_m -$$

$$-\sum_{i=1}^{m-1} \varepsilon_i(x)\, a_i - \sum_{i=m+1}^{N} \varepsilon_i^*(\lambda_m)\, a_i - \sum_{i=N+1}^{\infty} \varepsilon_i(y)\, a_i\Big| =$$

$$= \Big|\sum_{i=N+1}^{\infty} [\varepsilon_i^*(\lambda_m)-\varepsilon_i(y)]\, a_i\Big| \leqq 2 \sum_{i=N+1}^{\infty} |a_i| < \varepsilon,$$

i.e. F is left continuous at x.

Corollary. *Let* $a_n \in \mathbf{C}$ $\left(\sum_{n=1}^{\infty} |a_n| < \infty\right)$ *and* F: $[0, L] \to \mathbf{C}$ *the additive function determined by the sequence* a_n. *Then for the continuity of* F *in* $[0, L]$ *it is necessary and sufficient that the difference equations* (6.2) *should be valid for every* $n \in \mathbf{N}$.

Remark. For $1<q<2$ and $\lambda_n := 1/q^n$ the previous statement has been proved in [2].

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(Z.D.)
DEPARTMENT OF MATHEMATICS
L. KOSSUTH UNIVERSITY
DEBRECEN 4010, HUNGARY

(I.K.)
DEPARTMENT OF MATHEMATICS
L. EÖTVÖS UNIVERSITY
BUDAPEST, MÚZEUM KRT. 6—8
1088 HUNGARY