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On the integral of fundamental polynomials of Lagrange interpolation

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1. Introduction. Let $X=\{x_{kn}\}$, $n=1, 2, \ldots;\ 1\leqq k\leqq n$, be a triangular interpolatory matrix in $[-1, 1]$, i.e.

$$-1 \leqq x_{nn} < x_{n-1,n} < \ldots < x_{1n} \leqq 1, \quad n = 1, 2, \ldots. \tag{1.1}$$

If, sometimes omitting the superfluous notation,

$$\omega(x) = \omega_n(X, x) = \prod_{k=1}^{n} (x-x_k), \quad n = 1, 2, \ldots,$$

then

$$l_k(x) = l_{kn}(X, x) = \frac{\omega(x)}{\omega'(x_k)(x-x_k)}, \quad k = 1, 2, \ldots, n, \tag{1.2}$$

are the corresponding fundamental polynomials of Lagrange interpolation. It is well known that the so called Lebesgue function and Lebesgue constant

$$\lambda_n(x) := \lambda_n(X, x) := \sum_{k=1}^{n} |l_k(x)|, \quad \Lambda_n := \Lambda_n(X) := \max_{-1\leqq x\leqq 1} \lambda_n(x) \tag{1.3}$$

are of fundamental importance considering the convergence and divergence properties of the Lagrange interpolation. Many important properties can be found in [1]—[7] and in their references.

One of them is as follows.

There exists a constant $c_1>0$ *such that we have for arbitrary X.*

$$\int_{-1}^{1} \sum_{k=1}^{n} |l_{kn}(X, x)|\, dx > c_1 \log n. \tag{1.4}$$

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This statement, proved by P. ERDŐS and J. SZABADOS [3][1)], was explicitly formulated, perhaps first, in P. Erdős [2, p. 242], where he also stated (without proof) that

To every $\varepsilon>0$ there exists a $\delta>0$ so that the number of indices k, $1\leqq k\leqq n$, for which

$$\int_{-1}^{1}|l_k(x)|\,dx<\frac{\delta\log n}{n} \tag{1.5}$$

is less than εn, and the number of k's for which

$$\int_{-1}^{1}|l_k(x)|\,dx<\frac{c_2}{n}\quad\text{is less than}\quad\frac{c_3\log n}{n}.$$

2. Results.

2.1. From (1.5) one could easily obtain (1.4). The first result in this paper gives another statement by which we can get again (1.4).

Let $x_{0n}=1$, $x_{nn}=-1$, $l_{0n}(x)=l_{n+1,n}(x)=0$,

$$J_{kn}=[x_{k+1,n},x_{kn}],\quad 0\leqq k\leqq n. \tag{2.1}$$

First a remark. *If for a fixed k, $0\leqq k\leqq n$, $|J_{kn}|>\delta_n:=\dfrac{75\log n}{n}$, then* $\int_{-1}^{1}\lambda_n(x)\,dx\geqq 4n$ $(n\geqq n_1)$ which is even stronger than (1.4) (see [3, case 1] and [3, (5)]; the last formula shows that $|J_{kn}|\leqq 25\log\Lambda_n/n$ if $k\neq 0, n$; but it can easily be proved for J_0 and J_n, too).

I.e. the real problem is to settle those so called "*short*" intervals J_{kn}, for which $|J_{kn}|\leqq\delta_n$.

The short interval J_{kn} is said to be *exceptional* iff for a given sequence $\varepsilon=\{\varepsilon_n\}_{n=1}^{\infty}$, $0<\varepsilon_n\leqq 2$,

$$\frac{1}{|J_{kn}|}\int_{-1}^{1}\big(|l_k(x)|+|l_{k+1}(x)|\big)\,dx<c\varepsilon_n\log n \tag{2.2}$$

(where c can be taken as 71680). Further, let $k\in K_n$ iff J_{kn} is exceptional. We prove

Theorem 2.1. *If $\varepsilon=\{\varepsilon_n\}$ is given then for any fixed n the total measure of intervals for which* (2.2) *is valid, could not exceed ε_n, or which is the same,*

$$\sum_{k\in K_n}|J_{kn}|\leqq\varepsilon,\quad n=1,2,\dots. \tag{2.3}$$

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1) (1.4) is an easy consequence of another statement in [1, Theorem 2] which was proved by P. Erdős and P. Vértesi (cf. [6] and [7]).

Now let us suppose that for a fixed n all the intervals J_{kn} are short. Then, using Theorem 2.1 with $\varepsilon_n=1$, we can write

$$\int_{-1}^{1} \lambda_n(x)\,dx \geqq \frac{1}{2} \int_{-1}^{1} \sum_{k=1}^{n} (|l_k(x)|+|l_{k+1}(x)|)\,dx \geqq$$

$$\geqq \frac{1}{2} \sum_{k\notin K_n} \int_{-1}^{1} (|l_k(x)|+|l_{k+1}(x)|)\,dx \geqq \frac{c}{2}\log n \sum_{k\notin K_n} |J_{kn}| \geqq \frac{c}{2}\log n,$$

i.e. we obtain (1.4).

2.2. The next theorem gives information on both short and long intervals. The interval J_{kn} is *bad* iff for a given $\varepsilon>0$

$$\int_{-1}^{1} (|l_k(x)|+|l_{k+1}(x)|)\,dx < \eta(\varepsilon)\,\frac{\log n}{n}, \quad n \geqq n_0(\varepsilon), \tag{2.4}$$

where $\eta(\varepsilon)$ can be choosen as $(10^2\cdot 14336)^{-1}\varepsilon^2$.[2] Further, let $k\in T_n$ iff J_{kn} is bad. Then we prove

Theorem 2.2. *By the previous notations*

$$\sum_{k\in T_n} |J_{kn}| \leqq \varepsilon \quad \textit{if} \quad n \geqq n_0(\varepsilon). \tag{2.5}$$

2.3. Finally we remark that analogous results can be proved for a fixed interval $[a,b]\subset[-1,1]$. We omit the details.

3. Proof.

3.1. Proof of Theorem 2.1. If for a fixed n, $0<\varepsilon_n<(c\log n)^{-1}$, then by

$$|l_k(x)|+|l_{k+1}(x)| \geqq 1 \quad \text{if} \quad x\in J_k, \quad k=0,1,\dots,n, \quad n\geqq 1, \tag{3.1}$$

(cf. [4, Lemma 4] for $k\neq 0, n$; if $k=0$ (or n), (3.1) comes from $l_1(x)\geqq 1$, $x\geqq x_1$ (or $l_n(x)\geqq 1$, $x\leqq x_n$)) we get

$$\int_{-1}^{1} (|l_k(x)|+|l_{k+1}(x)|)\,dx \geqq \int_{J_k} (\dots) \geqq |J_k| > |J_k|\,c\varepsilon_n\log n, \quad \text{i.e.}$$

there is no exceptional interval. That means from now on we can suppose

$$\varepsilon_n \geqq \frac{1}{c\log n}, \quad n=2,3,\dots. \tag{3.2}$$

[2]) Instead of ε, we can choose a sequence $\{\varepsilon_n\}$ which would give $\eta(\varepsilon_n)$ in (2.3). I hint with a finer argument the relation $\eta(\varepsilon_n)=c\varepsilon_n$ can be proved.

We introduce the following notations

$$J_k(q) = J_{kn}(q) = [x_{k+1}+q|J_k|, x_k-q|J_k|] \quad (0 \leqq k < n), \tag{3.3}$$

where $0 \leqq q \leqq 1/2$. Let $z_k = z_{kn}(q)$ be defined by

$$|\omega_n(z_k)| = \min_{x\in J_k(q)} |\omega_n(x)|, \quad k = 0, 1, \ldots, n, \tag{3.4}$$

finally let

$$|J_i, J_k| = \max(|x_{i+1}-x_k|, |x_{k+1}-x_i|) \quad (0 \leqq i, k \leqq n).$$

In [5, Lemma 4.2] we proved

Lemma 3.1. *If* $1 \leqq k, r < n$ *then for arbitrary* $0 < q \leqq 1/2$

$$|l_k(x)|+|l_{k+1}(x)| \geqq q^2 \frac{|\omega_n(z_r)|}{|\omega_n(z_k)|} \frac{|J_k|}{|J_r, J_k|} \quad \text{if} \quad x \in J_r(q). \tag{3.5}$$

Later we shall also [6, Lemma 3.2]:

Lemma 3.2. *Let* $I_k = [a_k, b_k]$, $1 \leqq k \leqq t$, $t \geqq 2$, *be any* t *intervals in* $[-1, 1]$ *with* $|I_k \cap I_j| = 0$, $(k \neq j)$, $|I_k| \leqq \varrho$ $(1 \leqq k \leqq t)$, $\sum_{k=1}^{t} |I_k| = \mu$. *Supposing that for certain integer* $R \geqq 2$ *we have* $\mu \geqq 2^R \varrho$, *there exists the index* s, $1 \leqq s \leqq t$, *such that*

$$S := \sum_{k=1}^{t} \frac{|I_k|}{|I_s, I_k|} \geqq \frac{R}{8} \mu. \tag{3.6}$$

I_s *will be called accumilation interval of* $\{I_k\}_{k=1}^{t}$.

(Here and later mutatis mutandis we apply the previous notations for arbitrary intervals.)

Note that we do not require $b_k \leqq a_{k+1}$.

Let $\sum_{k\in K_n'} |J_k| := \mu_n$, where $K_n' := K_n \setminus \{0, n\}$. If for a fixed $n \geqq n_0(\varepsilon_n)$, $\mu_n \leqq \varepsilon_n/2$, (2.3) holds true. So we investigate those $n \geqq n_0(\varepsilon)$ $\mu_n \geqq \varepsilon_n/10$, say.

We now apply Lemma 3.2 for the exceptional J_{kn}'s with $\mu = \mu_n$, $\varrho = \delta_n$ and $R = [\overset{2}{\log}\, n^{1/7}]+1$, $n \in N$, $n \geqq n_0(\varepsilon)$ (shortly $n \in N_1$).

Denote by $M_1 = M_{1n}$ the accumulation interval. Dropping M_1, we apply Lemma 3.2 again for the remaining exceptional intervals with $\mu = \mu_n - |M_1| > \mu_n/2$ and the above ϱ and R, supposing $\mu_n \geqq \varrho^{R+1}$ whenever $n \in N_1$. We denote the accumulation interval by M_2. At the i-th step $(2 \leqq i \leqq \psi_n)$ we drop $M_1, M_2, \ldots, M_{i-1}$ and apply Lemma 3.2 for the remaining exceptional intervals with $\mu = \mu_n - \sum_{j=1}^{i-1} |M_j|$ using the same ϱ and R.

Here ψ_n is the first index for which

$$\sum_{i=1}^{\psi_n-1} |M_i| \leqq \frac{\mu_n}{2} \quad \text{but} \quad \sum_{i=1}^{\psi_n} |M_i| > \frac{\mu_n}{2}, \quad n \in N_1. \tag{3.7}$$

If we denote by $M_{\psi_n+1}, M_{\psi_n+2}, \dots, M_{\varphi_n}$ the remaining (i.e. not accumulation) exceptional intervals, by (3.6) we can write

$$\sum_{k=r}^{\varphi_n} \frac{|M_k|}{|M_r, M_k|} \geqq \frac{\mu_n \log n}{112} \quad \text{if} \quad 1 \leqq r \leqq \psi_n \quad (n \in N_1). \tag{3.8}$$

Now we have

$$\sum_{k \in K_n} \int_{-1}^{1} (|l_k(x)| + |l_{k+1}(x)|)\, dx \geqq \sum_{k=1}^{\varphi_n} \sum_{r=1}^{\varphi_n} \int_{M_r} (|l_k(x)| + |l_{k+1}(x)|)\, dx \geqq ^{3)} \tag{3.9}$$

$$\geqq \sum_{k=1}^{\varphi_n} \sum_{r=1}^{\varphi_n} (1-2q)|M_r|\, q^2 \left|\frac{\omega(\bar z_r)}{\omega(\bar z_k)}\right| \frac{|M_k|}{|M_r, M_k|} \geqq$$

$$\geqq \frac{q^2(1-2q)}{2} \sum_{k=1}^{\varphi_n} \sum_{r=1}^{\varphi_n} \left(\left|\frac{\omega(\bar z_r)}{\omega(\bar z_k)}\right| + \left|\frac{\omega(\bar z_k)}{\omega(\bar z_r)}\right| \right) \frac{|M_r|\,|M_k|}{|M_r, M_k|} \geqq$$

$$\geqq q^2(1-2q) \sum_{k=1}^{\psi_n} |M_k| \sum_{r=k}^{\varphi_n} \frac{|M_r|}{|M_r, M_k|} > \frac{\mu_n^2 \log n}{16 \cdot 2 \cdot 2 \cdot 112} \quad \text{if} \quad q = \frac{1}{2}$$

(see (3.5), (3.7) and (3.8); we used that $x + x^{-1} \geqq 2$).

On the other hand, by (2.2)

$$\sum_{k \in K_n} \int_{-1}^{1} (|l_k(x)| + |l_{k+1}(x)|)\, dx < c\varepsilon_n \log n \sum_{k \in K_n} |J_k| = c\varepsilon_n \mu_n \log n$$

i.e. $\mu_n^2 \log n < 7168\, c\varepsilon_n \mu_n \log n$, from where by $\mu_n \geqq \varepsilon_n/10$ $1 < 71\,680c$, a contradiction if $c = (71\,680)^{-1}$ and $n \geqq n_0$.

If $n \leqq n_0$, by (3.1) we have for arbitrary k, $0 \leqq k \leqq n$,

$$\int_{-1}^{1} (|l_k(x)| + |l_{k+1}(x)|)\, dx \geqq \int_{J_k} (\dots) \geqq |J_k| \geqq |J_k| 2c \log n_0 \geqq |J_k| c\varepsilon_n \log n$$

whenever $2c \log n_0 \leqq 1$. Considering, that if $n_0 = 10^{420}$, $2c \log n_0 \leqq 1$, indeed. But then for $n \leqq n_0(\varepsilon)$, $K_n = \emptyset$, which gives the statement for arbitrary $n \geqq 2$.

3.2. Proof of Theorem 2.2. If $|J_{kn}| \geqq \delta_n$, then by (3.1)

$$\int_{-1}^{1} (|l_k(x)| + |l_{k+1}(x)|)\, dx \geqq \int_{J_k} (\dots)\, dx \geqq |J_k| \geqq \frac{75 \log n}{n},$$

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3) We denote the fundamental polynomials corresponding to M_k, by $l_k(x)$ and $l_{k+1}(x)$, the corresponding minimums are $|\omega(\bar z_k)|$.

i.e. a long interval could not be bad. Considering the short intervals, again we suppose that $\mu_n := \sum_{k\in T_n} |J_{kn}| \geqq \varepsilon/10$ to get a contradiction. Then, as above, we obtain that for $n \geqq n_0(\varepsilon)$

$$\frac{\mu_n^2 \log n}{7168} < \sum_{k\in T_n} \int_{-1}^{1} \left(|l_k(x)| + |l_{k+1}(x)|\right) dx := P.$$

By (2.3), $P < |T_n| \eta(\varepsilon) \dfrac{\log n}{n} \leqq 2\eta(\varepsilon) \log n$, i.e.

$$\frac{\varepsilon^2 \log n}{10^2 \cdot 7168} \leqq \frac{\mu_n^2 \log n}{7168} < P \leqq 2\eta(\varepsilon) \log n,$$

a contradiction, if $\eta(\varepsilon) = (10^2 \cdot 14\,336)^{-1} \varepsilon^2$, $n \geqq n_0(\varepsilon)$.

References

[1] P. Erdős, Problems and results on the theory of interpolation. I, *Acta Math. Hungar.*, **9** (1958), 381—388.

[2] P. Erdős, Problems and results on the theory of interpolation. II, ibid, **12** (1961), 235—244.

[3] P. Erdős, J. Szabados, On the integral of the Lebesgue function of interpolation, ibid, **32** (1978), 191—195.

[4] P. Erdős, P. Turán, On interpolation. II, *Ann. of Math.*, **41** (1940), 510—553.

[5] P. Erdős, P. Vértesi, On the almost everywhere divergence of Lagrange interpolatory polynomials for arbitrary system of nodes, *Acta Math. Hungar.*, **36** (1980), 71—89 and **38** (1981), 263.

[6] P. Erdős, P. Vértesi, On the Lebesgue function of interpolation, *Functional Analysis and Approximation ISNM 60*, Birkhäuser Verlag (Basel, 1981), 299—309.

[7] P. Vértesi, New estimation for the Lebesque function of Lagrange interpolation, *Acta Math. Hungar.*, **40** (1982), 21—27.

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