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The asymptotic log likelihood function for a class of stationary processes

SÁNDOR VERES

The study of the weak consistency of maximum likelihood (ML) estimators for stationary processes in the scalar case was initiated by WHITTLE [7]. The strong consistency of the ML estimates for parameters of ARMA processes has intensively been dealt with by some authors. HANNAN [5] and DUNSMUIR and HANNAN [2] have given the strong laws of large numbers and the central limit theorem for ML estimates of ARMA processes. RISSANEN and CAINES [6] constructed the likelihood function via the innovation process. They proved the uniform P a.s. convergence of the log likelihood function over a compact set of parameters with fixed McMillan degree or Kronecker indices. A similar problem has been investigated by ARATÓ [1] in the continuous time case. These results show that one of the possible methods for proving the strong consistency of ML estimators is to show the P a.s. uniform convergence of the log likelihood function to the asymptotic one.

Our main aim in this paper is to extend the earlier results on the P a.s. uniform convergence of the log likelihood function. In a natural parameter domain the corresponding set of spectral densities would contain a sequence of spectral densities of stationary processes approaching to nonstationary processes, which is not allowed here. However we have that (i) the convergence holds not only over a special compact set of parameters, but on an arbitrary compact set of spectral densities, (ii) the convergence is shown for a wider class of processes then the ARMA processes.

1. Introduction. We shall consider discrete time r-dimensional stationary processes having exponentially bounded covariances. For $0<K, 0<\alpha<1$, let $S(K,\alpha)$ denote the set of spectral densities $\Phi(\omega)$, $\omega\in[-\pi,\pi]$ such that the sequences of covariance matrices

$$C_t = \int_{-\pi}^{\pi} \Phi(\omega) e^{it\omega}\, d\omega, \quad C_t^+ = \int_{-\pi}^{\pi} \Phi(\omega)^{-1} e^{it\omega}\, d\omega, \quad t\in\mathbf{Z}$$

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are uniformly bounded by the powers of α:

$$\|C_t\| \leqq K\alpha^{|t|} \quad \|C_t^+\| \leqq K\alpha^{|t|} \quad t\in\mathbf{Z} \tag{1.1}$$

where $\|v\|^2=v_1^2+\ldots+v_n^2$ is the norm of a vector $v=(v_1, \ldots, v_n)'$ and $\|A\|^2=\sup_{\|u\|=1}\|Au\|^2$ for a matrix A. The transpose of A will be denoted by A'. Let $\mathcal{S}=\bigcup_{K>0,\alpha<1} S(K,\alpha)$.

The log likelihood function is defined as usual by

$$L_n(y_n, \Phi) = \log\det\Gamma_n(\Phi)+\frac{1}{2}y_n'\Gamma_n(\Phi)^{-1}y_n \tag{1.2}$$

where

$$\Gamma_n = \begin{bmatrix} C_0 & C_1 \dots C_{n-1} \\ C_{-1} & C_0 \dots C_{n-2} \\ \vdots & \ddots \vdots \\ C_{-n+1} & C_0 \end{bmatrix}$$

is the Toeplitz matrix (see Grenander and Szegő [3]) composed from the autocovariance matrix sequence C_t, $t\in\mathbf{Z}$. In the following $\xrightarrow{P}$ denotes convergence in probability. Convergence with probability 1 will be written as P a.s.

2. Convergence of the log likelihood function. Introduce on $\mathcal{S}$ the metric ϱ

$$\varrho(\Phi, \Psi) = \operatorname*{ess\,sup}_{\omega\in[-\pi,\pi]} \sup_{1\leqq j,k\leqq r} |\Phi_{jk}(\omega)-\Psi_{jk}(\omega)| \quad \Phi, \Psi\in\mathcal{S}.$$

Theorem 1. *Let $S\subseteqq S(K,\alpha)$ be compact. If y_t, $t\in\mathbf{Z}$ is a Gaussian stationary process with spectral density $\Phi_0\in\mathcal{S}$ then with probability* 1

$$L_n(y_n, \Phi) \to \frac{1}{2\pi}\int_{-\pi}^{\pi}[\log\det\Phi(\omega)+\operatorname{tr}\Phi^{-1}(\omega)\Phi_0(\omega)]\,d\omega \tag{2.1}$$

as $n\to\infty$, uniformly for $\Phi\in S$.

Proof. The proof is based on some lemmas and properties of Toeplitz matrices.

Lemma 1. *With the above notations*

$$\frac{1}{n}\log\det\Gamma_n(\Phi) \to \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\det\Phi(\omega)\,d\omega \tag{2.2}$$

as $n\to\infty$ uniformly for $\Phi\in S$.

Proof. This statement is an extension of Szegő's classical theorem and we refer to Gyires [4] for its proof.

Lemma 2. *For each $\Phi \in S$ holds the convergence*

$$\bar{y}_n = \frac{1}{n} y_n' \Gamma_n(\Phi)^{-1} y_n \xrightarrow{P} \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{tr} \Phi^{-1}(\omega) \Phi_0\{\omega) \, d\omega, \tag{2.3}$$

where Φ_0 is the spectral density of y_t, $t \in \mathbf{Z}$.

Proof. The proof of this lemma is a straightforward modification for the vectorial case of a result of GRENANDER and SZEGŐ [3] in Section 11.5.

It will be proved by Lemmas 3.1—3.5 that $\bar{y}_n$ converges P a.s. uniformly to some function, then by Lemma 2 the limit function is the right hand side of (2.3) P a.s.

The proof of Theorem 1 will be based on an approximation of the matrix Γ_n with another matrix L_n defined in the following way. Let U_n be an orthogonal matrix of order nr composed from the r-order matrices

$$[U_n]_{\mu\nu} = n^{-1/2} e^{2\pi i \nu\mu} I_r, \quad \mu, \nu = 1, 2, \dots, r,$$

where I_r is the r-order identity matrix. We define Φ_p by

$$\Phi_p(ix) = \sum_{-p}^{p} \left(1 - \frac{|\nu|}{p}\right) C_\nu e^{i\nu x}, \quad x \in [-\pi, \pi], \quad p = 1, 2, \dots.$$

Let D_n be an nr-order matrix with the r-order matrices $[D_n]_{\nu\nu} = \Phi_p(2\pi i\nu/n)$ in the diagonal and 0 everywhere else, i.e. $[D_n]_{\mu\nu} = 0$, if $\mu \neq \nu$.

Now we define L_n by $L_n = U_n^* D_n U_n$ and $\underline{C}_\nu$ is given by $\underline{C}_\nu = (1 - |\nu|/p) C_\nu$ if $|\nu| < p$ and $\underline{C}_\nu = 0$ if $|\nu| \geqq p$.

Lemma 3.1. *Let p be in the above definition of L_n $p = p(n) = [n^{1/2+\varepsilon}]$, where $1/4 < \varepsilon < 1/2$ is a once for all fixed, but arbitrary number. Then for all natural numbers k*

$$\frac{1}{n} y_n' (\Gamma_n^k - L_n^k) y_n \to 0$$

as $n \to 0$ uniformly over S P a.s.

Proof. For the proof we shall consider the matrices K_n defined by

$$[K_n]_{\mu\nu} = \underline{C}_{\nu-\mu} \quad \text{for} \quad \nu, \mu = 1, 2, \dots, n,$$

and the convergence of the sequences

$$\frac{1}{n} y_n' (L_n^k - K_n^k) y_n \tag{2.4}$$

and

$$\frac{1}{n} y_n' (K_n^k - \Gamma_n^k) y_n \tag{2.5}$$

will be examined.

First we show the convergence of (2.4). Using the notations

$$W_n = L_n^{k-1} + L_n^{k-2} K_n + \ldots + K_n^{k-1}$$

and

$$M = \sup_{\Phi \in S} \operatorname*{ess\,sup}_{x \in [-\pi, \pi]} \|\Phi(x)\|$$

we have

(2.6) $$\|W_n\| \leqq k \cdot M^{k-1}$$

because of the inequalities

$$\|L_n\| \leqq \sup_{x \in [-\pi, \pi]} \|\Phi_p(ix)\| \leqq \operatorname*{ess\,sup}_{x \in [-\pi, \pi]} \|\Phi(x)\| \leqq M$$

and

$$\|K_n\| \leqq \sup_{x \in [-\pi, \pi]} \|\Phi_p(ix)\| \leqq \operatorname*{ess\,sup}_{x \in [-\pi, \pi]} \|\Phi(x)\|.$$

Introduce the notation $\underline{z}_n = W_n \underline{y}_n$, then the expression in (2.4) takes the form

(2.7) $$\frac{1}{n} \underline{y}_n'(L_n - K_n) \underline{z}_n.$$

Since the process $\{y_n, n \in \mathbf{Z}\}$ is stationary and ergodic so is the scalar process $\{\|y_n\|^2 \colon n \in \mathbf{Z}\}$. This implies that the averages

$$n^{-1}(\|y_1\|^2 + \|y_2\|^2 + \ldots + \|y_n\|^2) = n^{-1}\|\underline{y}_n\|^2$$

converge P a.s., and therefore the sequence $\{n^{-1}\|\underline{y}_n\|^2 \colon n \in \mathbf{N}\}$ is bounded P a.s. by a number $K(\omega)$, which depends on the elementary event ω. This implies that the sequence $n^{-1}\|\underline{z}_n\|^2$, $n \in \mathbf{N}$ is bounded P a.s., indeed

$$n^{-1}\|\underline{z}_n\|^2 \leqq n^{-1}\|W_n\|^2\|y_n\|^2 < k \cdot M^{k-1} n^{-1}\|\underline{y}_n\|^2 \leqq k \cdot M^{k-1} K\{\omega).$$

The r-order matrix block of L_n at place (ν, μ) can easily be computed as

$$[L_n]_{\nu,\mu} = \frac{1}{n} \sum_{j=1}^{n} e^{-2\pi i \nu j} I_r \Phi_p(2\pi i j/n) e^{2\pi i j \mu} I_r = \sum_{m=-\infty}^{\infty} \underline{C}_{\nu-\mu+mn}.$$

Now we deduce the following sequence of inequalities

(2.8) $$\left|\frac{1}{n} \underline{y}_n'(L_n - K_n) \underline{z}_n\right| =$$

$$= \left|\frac{1}{n}(y_1' \underline{C}_{p-1} z_{n-p+1} + y_2' \underline{C}_{p-1} z_{n-p+2} + \ldots + y_p' \underline{C}_{p-1} z_n) + \right.$$

$$+ \frac{1}{n}(y_1' \underline{C}_{p-2} z_{n-p+2} + \ldots + y_{p-1}' \underline{C}_{p-2} z_n) + \ldots + \frac{1}{n} y_1' \underline{C}_0 z_n +$$

$$+ \frac{1}{n}(y_{n-p+1}' \underline{C}_{1-p} z_p + y_{n-p+2}' \underline{C}_{1-p} z_{p-1} + \ldots + y_n' \underline{C}_{n-p} z_1) +$$

$$
\begin{aligned}
&+\frac{1}{n}(y'_{n-p+2}\underline{C}_{2-p}z_{p-1}+\ldots+y'_n\underline{C}_{2-p}z_1)+\ldots+\frac{1}{n}y'_n\underline{C}_0z_1\Big| \leqq \\
&\leqq n^{-1}\|\underline{C}_{p-1}\|(\|y_1\|\,\|z_{n-p+1}\|+\|y_2\|\,\|z_{n-p+2}\|+\ldots+\|y_p\|\,\|z_n\|)+ \\
&+n^{-1}\|\underline{C}_{p-2}\|(\|y_1\|\,\|z_{n-p+2}\|+\ldots+\|y_{p-1}\|\,\|z_n\|)+\ldots+n^{-1}\|\underline{C}_0\|\,\|y_1\|\,\|z_n\|+ \\
&+n^{-1}\|\underline{C}_{1-p}\|(\|y_{n-p+1}\|\,\|z_p\|+\|y_{n-p+2}\|\,\|z_{p-1}\|+\ldots+\|y_n\|\,\|z_1\|+ \\
&+n^{-1}\|\underline{C}_{2-p}\|(\|y_{n-p+2}\|\,\|z_{p-1}\|+\ldots+\|y_n\|\,\|z_1\|)+\ldots+n^{-1}\|\underline{C}_0\|\,\|y_n\|\,\|z_1\| \leqq \\
&\leqq (p/n)^{1/2}\|\underline{C}_{p-1}\|\,p^{-1/2}\|y_p\|\,n^{-1/2}(\|z_{n-p+1}\|^2+\ldots+\|z_n\|^2)^{1/2}+ \\
&+(p/n)^{1/2}\|C_{p-2}\|\,p^{-1/2}\|y_{p-1}\|\,n^{-1/2}(\|z_{n-p+2}\|^2+\ldots+\|z_n\|^2)^{1/2}+\ldots \\
&\ldots+(p/n)^{1/2}\|C_0\|\,p^{-1/2}\|y_0\|\,n^{-1/2}\|z_n\|+ \\
&+(p/n)^{1/2}\|C_{1-p}\|\,n^{-1/2}(\|y_{n-p+1}\|^2+\ldots+\|y_n\|^2)^{1/2}p^{-1/2}\|z_p\|+\ldots \\
&\ldots+(p/n)^{1/2}\|C_0\|\,n^{-1/2}\|y_n\|\,p^{-1/2}\|z_1\| \leqq \\
&\leqq (p/n)^{1/2}\Big(\sum_{\nu=0}^{p-1}\|C_\nu\|\Big)K(\omega)^{1/2}K_0(\omega)^{1/2}+ \\
&+(p/n)^{1/2}\Big(\sum_{\nu=1-p}^{0}\|C_\nu\|\Big)K_0(\omega)^{1/2}K(\omega)^{1/2}
\end{aligned}
$$

P a.s., where the notation $K_0(\omega)=k\cdot M^{k-1}$ was used. The last inequality follows from the simple relations

$$n^{-1}(\|z_{n-p+i}\|^2+\ldots+\|z_n\|^2) \leqq K_0(\omega), \quad i=1,2,\ldots,p$$

and

$$n^{-1}(\|y_{n-p+i}\|^2+\ldots+\|y_n\|^2) \leqq K(\omega), \quad i=1,2,\ldots,p.$$

But $\sum_{\nu=-\infty}^{\infty}\|C_\nu\|<\infty$ and therefore both summands in (2.8) converge to 0 as $n\to\infty$ P a.s. which implies that the expression in (2.7) converges to 0 P a.s. which was to be proved.

For proving (2.5) the following lemma can be applied, which gives an approximation of the powers of Toeplitz matrices.

Lemma 3.2. *For each* $k\geqq 1$

$$\|K_n^k-\Gamma_n^k\| \to 0 \quad as \quad n\to\infty. \tag{2.9}$$

Proof. Define the norm $|\cdot|$ for n-order symmetric matrices Γ by $|\Gamma|^2=\frac{1}{n}\sum_{i}\Gamma_{i,j}^2$. Using the inequality $|\Gamma|\leqq\|\Gamma\|$ it follows that

$$
\begin{aligned}
|K_n-\Gamma_n|^2 &= 2n^{-1}\sum_{\nu=1}^{p}(\nu^2/p^2)(n-\nu)|C_\nu|^2+2n^{-1}\sum_{\nu=p}^{n}(n-\nu)|C_\nu|^2 \leqq \\
&\leqq 2\cdot p^{-2}\sum_{\nu=1}^{p}\nu^2\|C_\nu\|^2+2n^{-1}(n-p)\sum_{\nu=p}^{\infty}\|C_\nu\|^2.
\end{aligned}
$$

In the last sum

$$\sum_{\nu=1}^{\infty} \nu^2 \|C_\nu\|^2 < \infty \quad \text{and} \quad \sum_{\nu=p}^{\infty} \|C_\nu\|^2 \leqq O(\alpha^p).$$

Thus

$$|K_n - \Gamma_n|^2 \leqq O(n^{-2\varepsilon-1}) + O(\alpha^{[n^{1/2+\varepsilon}]}),$$

from which we can conclude that

$$|K_n - \Gamma_n| \leqq o(n^{-1/2}).$$

Denote the eigenvalues of the matrix $V_n = n^{-1}(K_n - \Gamma_n)$ by $\lambda_1^{(n)}, \ldots, \lambda_{nr}^{(n)}$. Then

(2.10) $$|V_n|^2 = \frac{1}{nr} \sum_{i=1}^{nr} |\lambda_i^{(n)}|^2 \geqq \frac{1}{nr} \|V_n\|^2$$

and by the preceding inequality

(2.11) $$|V_n| \leqq o(n^{-3/2}).$$

By (2.10) and (2.11)

(2.12) $$\|V_n\| \leqq o(n^{-1}).$$

Since $\|K_n\| \leqq M$ and $\|\Gamma_n\| \leqq M$

$$n^{-1}\|K_n^k - \Gamma_n^k\| \leqq n^{-1}\|K_n - \Gamma_n\| \cdot k \cdot M^{k-1} = \|V_n\| k \cdot M^{k-1}.$$

Finally it follows from (2.11) and (2.12) that

$$\|n^{-1}(K_n^k - \Gamma_n^k)\| \leqq o(n^{-1})$$

and therefore Lemma 3.2 can be concluded. Applying (2.9) we have

(2.13) $$|n^{-1} y_n'(K_n^k - \Gamma_n^k) y_n| \leqq \|y_n\|^2 o(n^{-1}) = n^{-1}\|y_n\|^2 o(1).$$

But $n^{-1}\|y_n\| < K(\omega)$ P a.s., and by the inequality (2.13) it yields

$$\frac{1}{n} y_n'(K_n^k - \Gamma_n^k) y_n \to 0,$$

as $n \to \infty$ P a.s., and this was to be proved.

This means that both expressions in (2.4) and (2.5) tend to 0 as $n \to \infty$ P a.s., and this completes the proof of Lemma 3.

For later proofs we introduce the term

$$C(n, j) = \frac{1}{n} \sum_{\nu=1}^{n} e^{-2\pi i j\nu/n} \Phi^k_{p(n)}(2\pi i\nu/n), \quad n = 1, 2, \ldots, \quad j = 0, \pm 1, \ldots,$$

which plays an important role in the theory of Toeplitz matrices and the following two statements are valid.

Lemma 3.3. *Using the previous notation for $p(n)$ there is an $L_0 > 0$ such that*

$$\|C(n, j)\| \leqq L_0 (2p(n))^{k-1} \alpha^j$$

holds.

Lemma 3.4. *There are numbers A and B, which do not depend on n or j such that*

$$\|C(j) - C(n, j)\| \leqq A \cdot p(n)^{-1} + B \cdot jn^{-1}$$

holds, where

$$C(j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi^k(x) \cdot e^{-ij\omega} \, d\omega, \quad j = 0, \pm 1, \pm 2, \ldots.$$

The proofs of Lemma 3.3 and Lemma 3.4 can easily be given using the definitions of $C(n, j)$ and $C(j)$.

Lemma 3.5. *The sequence*

$$\frac{1}{n} \underline{y}'_n L^k_n \underline{y}_n, \quad n = 1, 2, \ldots$$

converges as $n \to \infty$ uniformly over S P *a.s..*

Proof. Taking into consideration the previous definitions

$$\frac{1}{n} \underline{y}'_n L^k_n \underline{y}_n = \frac{1}{n} \sum_{m, l=1}^{n, n} \sum_{\nu=1}^{n} \frac{1}{n} e^{2\pi i (m-l)\nu/n} y'_l \Phi^k_p(2\pi i\nu/n) y_m = \tag{2.14}$$

$$= \frac{1}{n} \sum_{m, l=1}^{n, n} y'_l C(n, l-m) y_m = \frac{1}{n} \sum_{j=-n+1}^{n-1} \sum_{\nu=a}^{f} y'_\nu C(n, j) y_{\nu+j},$$

where the notations $a = \max(1, -j+1)$ and $f = \min(n-j, n)$ were used.

Using the stationary and ergodic property of $\{y_n \colon n \in \mathbf{Z}\}$ and Lemma 7 for fixed j, with the notation $\gamma_j = \operatorname{tr} \Gamma_j C(j)$

$$\frac{1}{n} \sum_{\nu=a}^{f} y'_\nu C(n, j) y_{\nu+j} \to E(y'_1 C(j) y_{1+j}) = \gamma_j \tag{2.15}$$

holds uniformly over S P a.s..

To prove this convergence we show that taking in (2.14) $C(j)$ in place of $C(n,j)$ we have uniformly the same limit P a.s.. This comes from

$$(2.16)\qquad \left|\frac{1}{n}\sum_{j=-n+1}^{n-1}\sum_{\nu=a}^{f} y'C(n,j)y_{\nu+j}-\frac{1}{n}\sum_{j=-n+1}^{n-1}\sum_{\nu=a}^{f} y'_\nu C(j)y_{\nu+j}\right| \leqq$$

$$\leqq \frac{1}{n}\sum_{j=-n+1}^{n-1}\sum_{\nu=a}^{f}\|y_\nu\|\,\|C(n,j)-C(j)\|\,\|y_{\nu+j}\| \leqq$$

$$\leqq \frac{1}{n}\sum_{l=-j(n)}^{j(n)}\sum_{\nu=a}^{f}\|y_\nu\|\,\|C(n,l)-C(l)\|\,\|y_{\nu+j}\|+$$

$$+\frac{1}{n}\sum_{n\geqq|l|>j(n)}\sum_{\nu=a}^{f}\|y_\nu\|\left(\|C(n,l)\|+\|C(l)\|\right)\|y_{\nu+j}\| \leqq$$

$$\leqq \sum_{l=-j(n)}^{j(n)} p(n,j)n^{-1}\sum_{\nu=1}^{n}\|y_\nu\|^2+\sum_{n\geqq l>j(n)} 2L_0\left(2p(n)\right)^{k-1}\alpha^{|l|}\frac{1}{n}\sum_{\nu=1}^{n}\|y_\nu\|^2 \leqq$$

$$\leqq \left(2j(n)+1\right)p(n,j)\cdot K(\omega)+2L_0\left(2p(n)\right)^{k-1}K(\omega)\frac{\alpha^{j(n)}}{1-\alpha}$$

P a.s. for all elementary events, where the notation $p(n,j)=Ap(n)^{-1}+Bjn^{-1}$ was used. Now we choose $j(n)$ so that $j(n)/p(n)\to 0$ as $n\to\infty$. Let e.g. $j(n)=[n^{1/2-\varepsilon/2}]$. The on the right hand side of (2.16) both expressions tend to 0. Indeed, the convergence of the first term is obvious, and the convergence of the second easily follows from

$$p(n)^{k-1}\alpha^{[n^{1/2-\varepsilon/2}]}\to 0, \quad \text{as} \quad n\to\infty.$$

Now choose an arbitrary fixed $j_0>1$ and take the limit of the first $2j_0+1$ terms in

$$(2.17)\qquad \frac{1}{n}\sum_{l=-n+1}^{n-1}\sum_{\nu=a}^{f} y'_\nu C(l)y_{\nu+l} =$$

$$= \sum_{l=-j_0}^{j_0}\frac{1}{n}\sum_{\nu=a}^{f} y'_\nu C(l)y_{\nu+l}+\sum_{n\geqq|l|>j_0}\frac{1}{n}\sum_{\nu=a}^{f} y'_\nu C(l)\,y_{\nu+l}.$$

The second term can be majorized with the aid of (1.3)

$$(2.18)\qquad \left|\sum_{n\geqq|l|>j_0}\frac{1}{n}\sum_{\nu=a}^{f} y'_\nu C(l)y_{\nu+l}\right| \leqq 2V_0\alpha^{j_0}(1-\alpha)^{-1}K(\omega).$$

Then it follows by (2.15), (2.17) and (2.18) that

$$\sum_{l=-j_0}^{j_0}\gamma_l-V(j_0) \leqq \underline{\lim}_{n\to\infty},\ \overline{\lim}\,\frac{1}{n}\sum_{l=-n+1}^{n-1}\sum_{\nu=a}^{f} y'_\nu C(l)\,y_{\nu+l} \leqq \sum_{l=-j_0}^{j_0}\gamma_l+V(j_0)$$

uniformly over S P a.s. for all $j_0 > 1$, where the notation $V(j_0) = 2V_0\alpha^{j_0}(1-\alpha)^{-1}K(\omega)$ was used. This implies by (2.16) that

$$\lim_{n\to\infty} \frac{1}{n} \underline{y}_n' L_n^k \underline{y}_n = \lim_{n\to\infty} \frac{1}{n} \sum_{l=-n+1}^{n-1} \sum_{\nu=a}^{f} y_\nu' C(l) y_{\nu+l} = \sum_{l=-\infty}^{\infty} \gamma_l$$

uniformly over S P a.s., which completes the proof of the lemma.

Lemma 3. $\frac{1}{n} \underline{y}_n' \Gamma_n^{-1} \underline{y}_n$ *converges uniformly over* S P *a.s. as* $n\to\infty$.

Proof. By Lemma 3.1 $n^{-1}\underline{y}_n' \Gamma_n^k \underline{y}_n$ has the same limit as $n^{-1}\underline{y}_n' L_n^k \underline{y}_n$ uniformly over S P a.s.. Therefore

$$\frac{1}{n} \underline{y}_n (I_n - c\Gamma_n)^k \underline{y}_n, \tag{2.19}$$

converges uniformly P a.s. as $n\to\infty$, too. Here I_n denotes the unit matrix of order nr. We shall choose $c>0$ so that

$$\|I_n - c\Gamma_n(\Phi)\| < \chi, \quad n = 1, 2, \dots \tag{2.20}$$

be valid with a fixed $0<\chi<1$ over S. The existence of such $c>0$ will be assured by $0 < \min_{\Phi\in S} \operatorname{ess\,sup}_{x\in[-\pi,\pi]} \|\Phi(x)\| \le \max_{\Phi\in S} \operatorname{ess\,sup}_{x\in[-\pi,\pi]} \|\Phi(x)\|$. Indeed,

$$\|I_n - c\Gamma_n\| = \max_{\|u\|=1} |u'(I_n - c\Gamma_n)u| =$$

$$= \max\left\{1 - c\cdot \min_{\|u\|=1} u'\Gamma_n u,\; c\cdot \max_{\|u\|=1} u'\Gamma_n u - 1\right\} = \chi_n^c(\Phi)$$

and therefore it is enough to choose $c = c_0$ so that

$$c_0 \cdot \max_{\Phi\in S,\, x\in[-\pi,\pi]} \|\Phi(x)\| - 1 < 1$$

be valid.

It follows that there is a fixed $\chi < 1$ such that

$$\|I_n - c_0\cdot\Gamma_n(\Phi)\| = \chi_n^c(\Phi) < \chi$$

uniformly over S for all $n = 1, 2, \dots$.

Now we can apply a natural expansion of Γ_n^{-1}:

$$\Gamma_n^{-1} = c_0\big(I_n + (I_n - c_0\Gamma_n) + (I_n - c_0\Gamma_n)^2 + \dots\big).$$

Thus we get the series

$$\frac{1}{n} \underline{y}_n' \Gamma_n^{-1} \underline{y}_n = c_0 \sum_{k=0}^{\infty} \frac{1}{n} \underline{y}_n' (I_n - c_0\Gamma_n)^k \underline{y}_n. \tag{2.21}$$

Using Lemmas 3—6 we conclude that the terms in the series (2.21) converge uniformly P a.s., an can be evaluated by

$$\left|\frac{1}{n}\underline{y}_n'(I_n-c_0\Gamma_n)^k\underline{y}_n\right| \leqq \frac{1}{n}\sum_{v=1}^{n}\|y_v\|^2\chi^k \leqq K(\omega)\chi^k. \tag{2.22}$$

By the previous convergence results we may use the notation

$$r(k) = \lim_{n\to\infty}\frac{1}{n}\underline{y}_n'(I_n-c_0\Gamma_n)^k\underline{y}_n \quad \text{P a.s.}. \quad k = 1, 2, \ldots. \tag{2.23}$$

Then by (2.21), (2.22) and (2.23) for all fixed $k_0\in\mathbf{N}$

$$\sum_{k=0}^{k_0} r(k)-K(\omega)\chi^{k_0}(1-\chi)^{-1} \leqq \underline{\lim},\ \overline{\lim}_{n\to\infty}\frac{1}{n}\underline{y}_n'\Gamma_n^{-1}\underline{y}_n \leqq$$

$$\leqq \sum_{k=0}^{k_0} r(k)+K(\omega)\chi^{k_0}(1-\chi)^{-1}$$

holds uniformly over S P a.s.. This implies

$$\lim_{n\to\infty}\frac{1}{n}\underline{y}_n'\Gamma_n^{-1}(\Phi)\underline{y}_n = \sum_{k=0}^{\infty} r(k) \tag{2.24}$$

where the convergence is uniform over S P a.s., completing the proof of Lemma 3 and thus the proof of Theorem 1 too.

3. Strong consistency. As a consequence of Theorem 1 the following consistency theorem can be concluded for processes with exponentially stable covariances.

Theorem 2. *Let $S\subseteq S(K,\alpha)$ be a compact set of spectral densities and let y_t, $t\in\mathbf{Z}$ be a Gaussian stationary process with spectral density $\Phi_0\in S$. Then for the estimates $\hat{\Phi_n}$ obtained by minimizing $L_n(\underline{y}_n, \Phi)$ over S*

$$\hat{\Phi_n} \to \Phi_0 \tag{3.1}$$

P *a.s. as $n\to\infty$, where the convergence is considered in metric ϱ of the uniform convergence on S.*

Proof. The proof follows by a standard argumentation from Theorem 1 and the following lemma.

Lemma 4. *$L(\Phi, \Phi_0)$ is continuous with Φ as a variable on S and attains its minimum value over S only at $\Phi=\Phi_0$.*

Proof. For all $x\in[-\pi, \pi]$ the matrices $\Phi(x)$ and $\Phi_0(x)$ are positive definite. Therefore the matrix $\Phi^{-1}\Phi_0(x)$ must have positive eigenvalues $\lambda_1(x), \lambda_2(x), \ldots$ $\ldots, \lambda_r(x)$, although $\Phi^{-1}\Phi_0(x)$ is not necessarily symmetric.

By the inequality $\log\lambda\leqq\lambda-1$, $\lambda>0$ we have

$$\sum_{i=1}^{r}\log\lambda_i(x)-\sum_{i=1}^{r}\lambda_i(x)+r\leqq 0 \tag{3.2}$$

and thus

$$\log\det\Phi^{-1}\Phi_0(x)-\operatorname{tr}[\Phi^{-1}\Phi_0(x)]+r\leqq 0$$

that can be written in the form

$$r+\log\det\Phi_0(x)-\big(\log\det\Phi(x)+\operatorname{tr}[\Phi^{-1}\Phi_0(x)]\big)\leqq 0.$$

Taking the integral of both sides over $[-\pi,\pi]$ we have

$$L(\Phi_0,\Phi_0)\leqq L(\Phi,\Phi_0)$$

and equality is here only if equality holds in (3.2) for all $x\in[-\pi,\pi]$, which implies $\lambda_1(x)=\lambda_2(x)=\ldots=\lambda_r(x)=1$, $x\in[-\pi,\pi]$ and this is equivalent to $\Phi(X)=\Phi_0(X)$, $x\in[-\pi,\pi]$.

Remark. Assume that the topological space $\underline{\Theta}$ is a parametrization for stationary processes with exponentially bounded covariances, i.e. there is an injective continuous map $\tau\colon\underline{\Theta}\to S(K,\alpha)$ such that the process with parameter $\Theta\in\underline{\Theta}$ has spectral density $\tau(\Theta)$. Let $\Theta^c\subseteqq\underline{\Theta}$ be a compact subset of parameters. If the observed process y_t, $t\in\mathbf{Z}$ has parameter $\Theta_0\in\Theta^c$ then by Theorem 2 the estimates computed by minimizing $L_n(y_n,\tau(\Phi))$ over Θ^c are strongly consistent.

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DEPARTMENT OF PROBABILITY THEORY
EÖTVÖS LORÁND UNIVERSITY
MÚZEUM KRT. 6—8
1088 BUDAPEST, HUNGARY