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On the representation of distributive algebraic lattices. II

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1. Introduction

Around 1980, H. Bauer found a result which implies that countable distributive semilattices with 0 can be represented as semilattices of compact congruences of a lattice, whence it also follows that every lower bounded distributive algebraic lattice with countably many compact elements is the congruence lattice of a lattice. This proof, however, was not published. In [2], we proved that if D_1 and D_2 are finite distributive semilattices with 0 such that D_1 is a 0-subsemilattice of D_2, then D_1 and D_2 have a simultaneous representation (in a sense precisely defined in [3]) as semilattices of compact congruences of lattices L_1 and L_2, respectively. There we promised to show that this idea can be developed to a proof of the countable representation problem. Here we present this proof. We note that independently and by different methods H. Dobbertin [1] found another proof of the theorem.

It is easy to show that any finite subset of a distributive semilattice with 0 is contained in a finite distributive 0-subsemilattice. Hence it follows that for any countable distributive semilattice D with 0, there exist finite distributive semilattices $D_1, D_2, D_3, \dots$ with 0 and embeddings $\varepsilon_i\colon D_i \to D_{i+1}$, $i=1, 2, \dots$, such that D is the direct limit of the family $(\{D_i\}_{i\in N}, \{\varepsilon_i\}_{i\in N})$. Now let D and D_i, $i=1, 2, \dots$, be as above and fixed once and for all. We prove the following

Theorem. *There exist lattices* L_i, $i=1, 2, \dots$, *such that*

(α) $D_i \cong \mathrm{Con}(L_i)$ *under an isomorphism to be denoted by* φ_i, $i=1, 2, \dots$,

(β) L_i *has an embedding* λ_i *to* L_{i+1}, $i=1, 2, \dots$,

(γ) *if we denote by* $\mathrm{Con}(\lambda_i)$ *the mapping of* $\mathrm{Con}(L_i)$ *to* $\mathrm{Con}(L_{i+1})$ *induced by* λ_i *(that is the one that maps* $\Theta \in \mathrm{Con}(L_i)$ *to the congruence generated by*

*) This paper was left behind by András Huhn in the form of a first draft of a manuscript. Hans Dobbertin was kind to prepare it for publication.

$\{(a\lambda_i, b\lambda_i)\in L_{i+1}^2 | (a, b)\in \Theta\})$, *then the following diagram is commutative*

$$\begin{array}{ccc} D_i & \xrightarrow{\varepsilon_i} & D_{i+1} \\ \varphi_i \downarrow & & \downarrow \varphi_{i+1} \\ \mathrm{Con}\,(L_i) & \xrightarrow[\mathrm{Con}\,(\lambda_i)]{} & \mathrm{Con}\,(L_{i+1}) \end{array}$$

where ε_i denotes the identical embedding of D_i to D_{i+1}. In other words Con (λ_i) *represents* id_i.

Corollary. *Every countable distributive semilattice with* 0 *is isomorphic to the semilattice of all compact congruences of a lattice.*

To prove the Corollary from the Theorem, observe that the Con (L_i)'s form the same directed system (up to commuting isomorphisms) that the D_i's, whence their direct limit is also isomorphic with D. On the other hand, the L_i's also form a directed system and the congruence lattice of their direct limit is the direct limit of their congruence lattices (see PUDLÁK [3]). This proves the corollary.

2. The construction of L_j. Proof of (a)

First we define the following lattices. Let $i\leqq j$ be natural numbers. Let $D(i\to j)$ be the distributive lattice whose join-irreducibles are $(a_i, \dots, a_j), (a_{i+1}, \dots, a_j), \dots, (a_j)$, where $a_i, \dots, a_j$ are join-irreducibles of $D_i, \dots, D_j$, respectively, and $a_i\varepsilon_i\geqq a_{i+1}, a_{i+1}\varepsilon_{i+1}\geqq a_{i+2}, \dots$. Let these join-irreducibles be ordered componentwise, that is, let $(a_k, \dots, a_j)\leqq(a'_l, \dots, a'_j)$ iff $k\leqq l$ and $a_l\leqq a'_l, \dots, a_j\leqq a'_j$. Clearly, the set of join-irreducibles and their ordering determines $D(i\to j)$. Let $B(1\to j)$ be the Boolean lattice whose set of atoms is $\{[a] \mid a$ join-irreducible in $D(1\to j)\}$. Of course, instead of $[(a_1, \dots, a_j)]$ etc. we shall write $[a_1, \dots, a_j]$. Now there are some natural 0-1-embeddings. Each element of $D(i+1\to j)$ can be identified with an element of $D(i\to j)$ as follows: $x\in D(i+1\to j)$ is a join of join-irreducibles. These join-irreducibles are, however, join-irreducibles of $D(i\to j)$, too. Thus x can be identified with their join in $D(i\to j)$. This is a lattice 0-1-embedding and from now on we shall consider $D(i+1\to j)$ as a sublattice of $D(i\to j)$. Note that $D(j\to j)\cong D_j$ and will be identified with it. Furthermore, $D(1\to j)$ can be considered as a 0—1-sublattice of $B(1\to j)$, namely $x\in D(1\to j)$ can be identified with the join of all $[a]$, $a\leqq x$, a join-irreducible.

Now we define lattices $L(1\to j)$ as follows. Let $M(1\to j)$ consist of all triples $(x, y, z)\in(B(1\to j))^3$ satisfying $x\wedge y=x\wedge z=y\wedge z$. Let $L(1\to j)$ be the set of all those triples in $M(1\to j)$ also satisfying $z\in D(1\to j)$. Let $M(i\to j)$ $(i>1)$ consist of all those triples $(x, y, z)\in(D(i-1\to j))^3$ satisfying $x\wedge y=x\wedge z=y\wedge z$, and let

$L(i\to j)$ be the set of all those triples satisfying also $z\in D(i\to j)$. Now we describe the operations of $L(1\to l)$ and $L(i\to j)$, $i=2, \ldots, j$. The meet operations are the same as in $(B(1\to j))^3$ and in $(D(i-1\to j))^3$, respectively. We shall denote the joins in $(B(1\to j))^3$, $M(1\to j)$, $L(1\to j)$ by $\vee$, $\vee_M$, $\vee_L$, respectively and the join in $(D(i-1\to j))^3$, $M(i\to j)$, $L(i\to j)$ by $\vee$, $\vee_M$, $\vee_L$, respectively. This will cause no confusion. As $D(1\to j)$ is a sublattice of $B(1\to j)$, with every $z\in B(1\to j)$ we can associate an element $\bar{z}\in D(1\to j)$ which is the smallest element of $B(1\to j)$ such that $z\leqq \bar{z}$. Also, with any $z\in D(i-1\to j)$ $(i>1)$ we can associate a $\bar{z}\in D(i\to j)$, which is the smallest element of $D(i\to j)$ such that $z\leqq \bar{z}$. Now it is proven in SCHMIDT [4] that

$$(x, y, z)\vee_M(x', y', z') = (x\vee x', y\vee y', z\vee z')^{\sim},$$

where

$$(x, y, z)^{\sim} = \big(x\vee(y\wedge z), y\vee(x\wedge z), z\vee(x\wedge y)\big) \quad \text{for} \quad (x, y, z)\in\big(B(1\to j)\big)^3,$$

and

$$(x, y, z)\vee_L(x', y', z') = (x\vee x', y\vee y', z\vee z')^{\sim\wedge},$$

where

$$(x, y, z)^{\wedge} = \big(x\vee(y\wedge\bar{z}), y\vee(x\wedge\bar{z}), \bar{z}\big) \quad \text{for} \quad (x, y, z)\in M(1\to j).$$

The same proof as in [4], pp. 82—86 yields that this description remains valid for $(x, y, z)\in D(i-1\to j)$ as well as for $(x, y, z)\in M(i\to j)$. Now $L(1\to j)$ has an ideal isomorphic to $D(1\to j)$, namely the ideal $[(0, 0, 0), (0, 0, 1)]$, where 0 and 1 denote the bounds of $B(1\to j)$. The ideals $[(0, 0, 0), (1, 0, 0)]$ and $[(0, 0, 0), (0, 1, 0)]$ are isomorphic to $B(1\to j)$. Furthermore, the dual ideals $[(0, 1, 0), (1, 1, 1)]$ and $[(1, 0, 0), (1, 1, 1)]$ are isomorphic to $B(1\to j)$. All these proofs can be carried out by using the description of the operation of $L(1\to j)$. In fact, as an example, we prove that $[(1, 0, 0), (1, 1, 1)]$ is isomorphic to $D(1\to j)$. The elements of this interval are the elements $(1, y, z)$ with $z\in D(1\to j)$ and by $y\wedge 1=z\wedge 1=1\wedge 1$ we have $y=z$, that is, the elements of the interval are $(1, z, z)$, $z\in D(1\to j)$. Their meet is always formed componentwise and, using the previous description of the operation, is obvious, that the componentwise join is already invariant under $^{\sim}$ and $^{\wedge}$. Now we are ready to define L_j. Namely, similarly as the $L(1\to j)$, all the $L(i\to j)$, $i=2, \ldots, j$, have ideals isomorphic to $D(i-1\to j)$ and to $D(i\to j)$ (the proof is the same), so we can "glue them together" as shown in Figure 1. More exactly we form the direct product of the $L(i\to j)$'s. It has an ideal isomorphic to $L(i\to j)$ for all $i=1, \ldots, j$. We glue the bottom of this direct product to the top of $\prod_{i=2}^{j} M(i\to j)$. The latter has dual ideals isomorphic to $M(i\to j)$ for all $i=2, \ldots, j$. Now we identify, for all $i=1, 2, \ldots, j-1$, the ideal $[(0, 0, 0), (0, 0, 1)]$ of $L(i\to j)$ $(\subseteq\prod_i L(i\to j))$ with the dual ideal $[(0, 0, 1), (1, 1, 1)]$ of a copy of $M(i+1\to j)$.

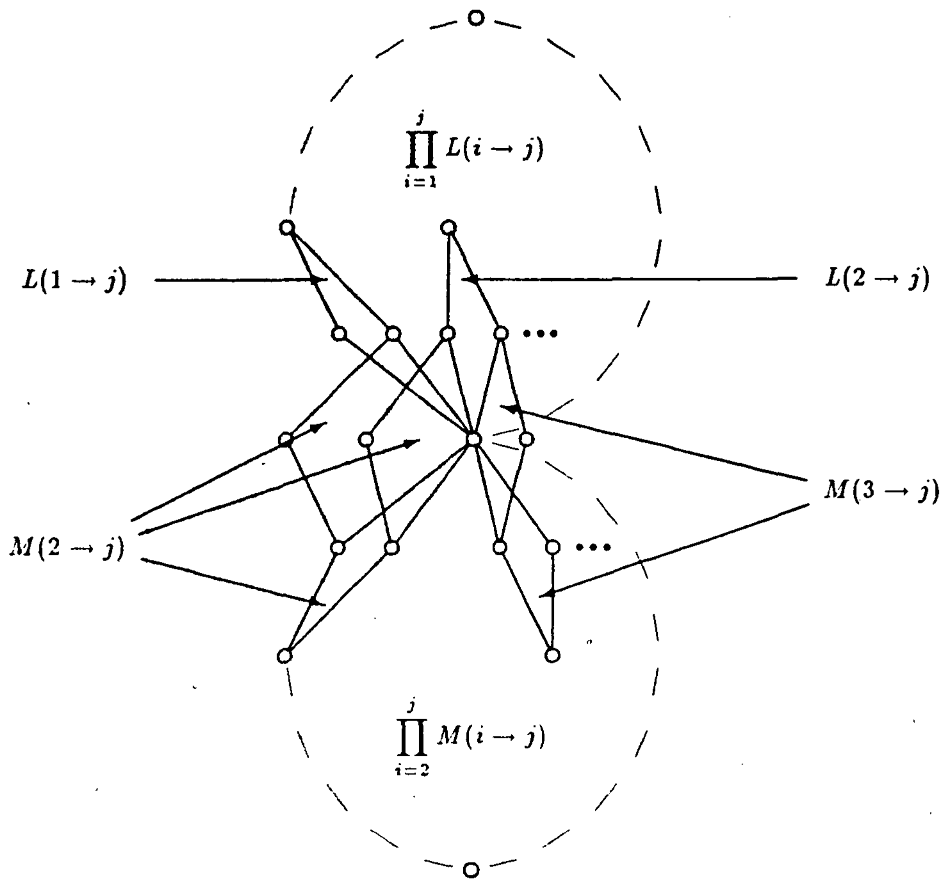

Figure 1

We identify the ideal $[(0,0,0),(0,0,1)]$ of this copy with the dual ideal $[(1,0,0),(1,1,1)]$ of the copy of $M_{\varepsilon}(i+1 \to j)$ which is a dual ideal in $\prod_{k=2}^{j} M(k \to j)$, and we identify the dual ideal $[(0,0,1),(1,1,1)]$ of this copy with the ideal $[(0,0,0),(0,0,1)]$ of a third copy of $M(i+1 \to j)$. Finally, we identify the dual ideal $[(0,0,1),(1,1,1)]$ of this third copy with the ideal $[(0,0,0),(1,0,0)]$ of $L(i+1 \to j)$ $(\subseteq \prod_{k=1}^{j} L(k \to j))$. The lattice we so obtain is L_j.

Now we have to prove (α). Consider any congruence α of L_j. First of all it splits into a join of congruences of the two direct products and of the joining $M(i \to j)$'s. By perspectivity, the generating pairs of these congruences can be transformed to the upper part $\prod_{i=1}^{j} L(i \to j)$, and there they factorize according to the direct representation, thus α is generated by pairs contained in the $L(i \to j)$'s (considered as ideals of $\prod L(i \to j)$). We shall prove that α is generated by an ideal of the interval $[(0, 0, 0), (0, 0, 1)] \cong D_j$ of $L(i \to j)$. As we mentioned, α is a join of principal congruences generated from the $L(i \to j)$'s. We may assume that α itself is such a principal congruence (because the join of ideals of $[(0, 0, 0), (0, 0, 1)] \subseteq J(j \to j)$ itself is an ideal).

Let α be generated by the pair $((x, y, z), (x', y', z'))$, where $(x, y, z), (x', y', z') \in L(k \to j)$, that is

$$x, y, x', y' \in D(k-1 \to j), \quad z, z' \in D(k \to j).$$

Then, forming the meets with $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, we obtain

$$(x, 0, 0)\, \alpha\, (x', 0, 0), \quad (0, y, 0)\, \alpha\, (0, y', 0), \quad (0, 0, z)\, \alpha\, (0, 0, z').$$

Hence $(x, 0, 0) \vee_L (0, 1, 0) = (x, 1, 0)^{\frown\frown} = (x, 1, x)^{\frown} = (x, 1, x)$, thus we have $(x, 1, x)\, \alpha\, (x', 1, x')$. Forming the meet of both sides with $(0, 0, 1)$, we get $(0, 0, x)\, \alpha\, (0, 0, x')$. Similarly $(0, 0, y)\, \alpha\, (0, 0, y')$. Thus the congruence generated by $((x, y, z), (x', y', z'))$ contains the pairs $((0, 0, x), (0, 0, x'))$, $((0, y, 0), (0, y', 0))$, $((0, 0, z), (0, 0, z'))$. It is also generated by them. We refer to p. 241 of [2] with which our notation coincides. Now $(0, 0, x)$, $(0, 0, x')$, etc. are contained in the copy $D(k-1 \to j)$, *which was used for the glueing in Figure 1.* Hence α is generated from $L(k-1 \to j)$ already (the generators can be transported by perspectivity), that is, by induction, it is generated from $L(1 \to j)$, and, finally, with the same computation as above, from $B(1 \to j)$. $B(1 \to j)$ is Boolean, hence α is generated by an ideal, say, by the pair $((0, 0, 0), (t, 0, 0))$, $(0, 0, 0), (t, 0, 0) \in L(1 \to j)$. Then it is also generated by

$$((0, 0, 0), (t, 0, 0)) \vee_L ((0, 1, 0), (0, 1, 0)) = ((0, 1, 0), (\bar{t}, 1, \bar{t})),$$

that is, by

$$((0, 1, 0), (\bar{t}, 1, \bar{t})) \wedge_L ((0, 0, 1), (0, 0, 1)) = ((0, 0, 0), (0, 0, \bar{t})),$$

which is an ideal of $D(1 \to j)$. By induction, it is generated by an ideal of D_j, as claimed.

3. The construction of the embeddings λ_j. Proof of (β)

First of all we define embeddings

$$\beta_{1j}\colon B(1 \to j) \to B(1 \to j+1) \quad \text{and} \quad \delta_{ij}\colon D(i \to j) \to D(i \to j+1),$$

whenever $i \leqq j$, as follows: The atoms of $B(1 \to j)$ are of the form $[a_1, \ldots, a_j]$, $a_1 \varepsilon_1 \geqq a_2$, $a_2 \varepsilon_2 \geqq a_3, \ldots, a_{j-1}\varepsilon_{j-1} \geqq a_j$ or of the form $[a_2, \ldots, a_j]$, $a_2 \varepsilon_2 \geqq a_3, \ldots, a_{j-1}\varepsilon_{j-1} \geqq a_j$, and so on, or of the form $[a_j]$, where $a_1, \ldots, a_j$ are join-irreducibles of $D_1, \ldots, D_j$, respectively. (These atoms are unordered.) We associate with $[a_i, \ldots, a_j]$

the join of all $[a_i, \ldots, a_j, a_{j+1}]$ in $B(1\to j+1)$, where $a_j\varepsilon_j \geqq a_{j+1}$, and a_{j+1} is a join-irreducible element in D_{j+1}. With the join of a set of atoms we associate the join of their images. This mapping is then denoted by β_1. β_{1j} clearly preserves 0 and the lattice operations, thus we only have to prove that it is one-to-one. In other words we have to prove that the dual mapping under Stone's duality is onto. This dual mapping associates with the atom $[a_1, \ldots, a_j, a_{j+1}]$ the atom $[a_1, \ldots, a_j]$, that is, we have to show that, for every atom $[a_1, \ldots, a_j]$ of $B(1\to j)$, there is an atom $[a_1, \ldots, a_j, a_{j+1}]$ of $B(1\to j+1)$ with $a_j\varepsilon_j \geqq a_{j+1}$, and this is evident as $a_j\varepsilon_j \neq 0$. Now we define δ_{ij}. The join-irreducibles of $D(i\to j)$ are of the form $(a_i, \ldots, a_j)$, $a_i\varepsilon_i \geqq a_{i+1}, \ldots, a_{j-1}\varepsilon_{j-1} \geqq a_j$, or $(a_{i+1}, \ldots, a_j)$, $a_{i+1}\varepsilon_{i+1} \geqq a_{i+2}, \ldots, a_{j-1}\varepsilon_{j-1} \geqq a_j$, and so on, or (a_j), and they are ordered componentwise. For $x\in D(i\to j)$, let $x\delta_{ij}$ be the join of all $(a_k, \ldots, a_j)$, where $(a_k, \ldots, a_j)$ is join-irreducible in $D(i\to j)$, $(a_k, \ldots, a_j) \leqq x$, and $a_j\varepsilon_j \geqq a_j+1$. δ_{ij} is a 0-preserving lattice embedding. The proof is the same as for β_{ij}, but we have to prove Priestley's duality, rather than Stone's duality. We need the following lemmas.

Lemma 1. *Let* $x\in B(1\to j)$. *Then* $\bar{x}\delta_{1j} = \overline{x\delta_{1j}}$.

Lemma 2. *Let* $x\in D(i-1\to j)$, $i-1<j$. *Then* $\bar{x}\delta_{ij} = \overline{x\delta_{i-1,j}}$.

Proof of Lemma 1. Let $(a_1, \ldots, a_j, a_{j+1}) \in D(1\to j+1)$ such that

$$(a_1, \ldots, a_j, a_{j+1}) \leqq \bar{x}\delta_{1j} \quad \text{and} \quad (a_1, \ldots, a_j, a_{j+1})$$

is join-irreducible. Then $(a_1, \ldots, a_j) \in \bar{x}$. Hence there is a join-irreducible element $(b_1, \ldots, b_j)$ in $D(1\to j)$ such that $(b_1, \ldots, b_j) \geqq (a_1, \ldots, a_j)$ and $(b_1, \ldots, b_j)$ occurs in the join-representation of $\bar{x}$, that is, $[b_1, \ldots, b_j]$ occurs in the join-representation of x. Then $[b_1, \ldots, b_j] \leqq x$. Hence $[b_1, \ldots, b_j, a_j+1] \leqq x\beta_{1j}$, that is, $(a_1, \ldots, a_j, a_j+1) \leqq$ $\leqq (b_1, \ldots, b_j, a_j+1) \leqq \overline{x\beta_{1j}}$. Conversely, if $(a_1, \ldots, a_j, a_j+1) \leqq \overline{x\beta_{1j}}$, then

$$(a_1, \ldots, a_j, a_j+1) \leqq (b_1, \ldots, b_j, b_j+1),$$

where $(b_1, \ldots, b_j, b_j+1)$ occurs in the join-representation of $\overline{x\beta_{1j}}$, that is $[b_1, \ldots, b_j, b_j+1]$ occurs in the join-representation of $x\beta_{1j}$. Hence $[b_1, \ldots, b_j, b_j+1] \leqq$ $\leqq x\beta_{1j}$. Then $[b_1, \ldots, b_j] \leqq x$ (see the definition of β_{1j}), $(b_1, \ldots, b_j) \leqq \bar{x}$, thus $(a_1, \ldots, a_j) \leqq \bar{x}$ and $(a_1, \ldots, a_j, a_{j+1}) \leqq \bar{x}\delta_{1j}$.

Proof of Lemma 2. Let $(a_i, \ldots, a_j, a_{j+1}) \leqq \bar{x}\delta_{ij}$, join-irreducible in $D(i\to j+1)$. Then $(a_i, \ldots, a_j) \leqq \bar{x}$, that is, $(a_i, \ldots, a_j) \leqq (b_i, \ldots, b_j)$, where $(b_i, \ldots, b_j)$ occurs in the join-representation of $\bar{x}$, that is, for a suitable join-irreducible $b_{i-1}\in D_{i-1}$ with $b_{i-1}\varepsilon_{i-1} \geqq b_i$, $(b_{i-1}, b_i, \ldots, b_j)$ occurs in the join-representation of x. Hence $(b_{i-1}, b_i, \ldots, b_j, a_{j+1}) \leqq x\delta_{i-1,j}$, that is, $(a_i, \ldots, a_j, a_{j+1}) \leqq (b_i, \ldots, b_j, a_{j+1}) \leqq \overline{x\delta_{i-1,j}}$. Conversely, $(a_i, \ldots, a_j, a_{j+1}) \leqq \overline{x\delta_{i-1,j}}$. Then $(a_i, \ldots, a_j, a_{j+1}) \leqq (b_i, \ldots, b_j, b_{j+1})$, where $(b_i, \ldots, b_j, b_{j+1})$ occurs in the join-representation of $\overline{x\delta_{i-1,j}}$, that is, for suitable b_{i-1} with $b_{i-1}\varepsilon_{i-1}\geqq b_i$, $(b_{i-1}, b_i, \ldots, b_{j+1})$ occurs in the join-representation of $x\delta_{i-1,j}$. This means, that $(b_{i-1}, b_i, \ldots, b_j)\leqq x$. Hence $(b_i, \ldots, b_j)\leqq \bar{x}$, that is $(a_i, \ldots, a_j, a_{j+1})\leqq(b_i, \ldots, b_j, a_{j+1})\leqq \bar{x}\delta_{ij}$.

Now we are ready to prove (β). First we prove that $L(1\to j)$ can be embedded to $L(1\to j+1)$. Consider the elements $(x\beta_{1j}, y\beta_{1j}, z\delta_{1j})\in L(1\to j+1)$ with $x, y\in B(1\to j)$, $z\in D(1\to j)$. These triples form a $\wedge$-subsemilattice of $L(1\to j+1)$. Now consider two such triples (x, y, z), $(x', y', z')\in L(1\to j)$, and let λ_{1j} denote the mapping $(\beta_{1j}, \beta_{1j}, \delta_{1j})$ described above. Then

$$[(x, y, z)\vee_{L(1\to j)}(x', y', z')]\lambda_{1j} = (x\vee x', y\vee y', z\vee z')^{\sim\wedge}\lambda_{1j} =$$
$$= [(x\vee x', y\vee y', z\vee z')^{\sim\wedge}](\beta_{1j}, \beta_{1j}, \delta_{1j}),$$
$$(x, y, z)\lambda_{1j}\vee_{L(1\to j)}(x', y', z')\lambda_{1j} = (x\beta_{1j}, y\beta_{1j}, z\delta_{1j})\vee_{L(1\to j)}(x'\beta_{1j}, y'\beta_{1j}, z'\delta_{1j}) =$$
$$= [(x\vee x', y\vee y', z\vee z')(\beta_{1j}, \beta_{1j}, \delta_{1j})]^{\sim\wedge}.$$

Now it is evident, that the operator $\sim$ and $(\beta_{1j}, \beta_{1j}, \delta_{1j})$ are permutable, and Lemma 1 shows that the same is true for $\wedge$ and $(\beta_{1j}, \beta_{1j}, \delta_{1j})$.

Finally we remark that the embedding λ_{1j} coincides with β_{1j} on $B(1\to j)$ considered as the ideal $[(0, 0, 0), (1, 0, 0)]$ of $L(1\to j)$ and coincides with δ_{1j} on $D(1\to j)$ considered as the ideal $[(0, 0, 0), (0, 0, 1)]$ of $L(1\to j)$.

Now $L(1\to j)$ can also be embedded to $L(i\to j+1)$ $(i\leqq j)$ by the embedding $\lambda_{ij}=(\delta_{i-1,j}, \delta_{i-1,j}, \delta_{i-1,j})$. The proof is the same as above, but we have to use Lemma 2 instead of Lemma 1. Furthermore, λ_{ij} coincides with $\delta_{i-1,j}$ on the copy of $D(i-1\to j)$ used in the glueing of Figure 1 and it coincides with δ_{ij} on the copy of $D(i\to j)$ used in the glueing. Thus we can glue together the λ_{ij}'s to get an embedding λ_j of L_j to L_{j+1}.

4. Proof of (γ)

We need a last lemma.

Lemma 3. *Let* $x\in D_{j-1}$. *Then* $x\delta_{j-1}=x\varepsilon_{j-1}$, *where* δ_{j-1} *stands for* $\delta_{j-1,j-1}$ *and* ε_{j-1} *maps* D_{j-1} *to* $D_j\subseteqq D(j-1\to j)$.

Proof. Let a_j be a join-irreducible element in D_j such that $a_j\leqq\overline{x\delta_{j-1}}$. Then $a_j\leqq b_j$ for some b_j in the join-representation of $\overline{x\delta_{j-1}}$. Thus, for some join-irreducible $b_{j-1}\in D_{j-1}$ with $b_{j-1}\varepsilon_{j-1}\geqq b_j$, (b_{j-1}, b_j) is in the join-representation of $x\delta_{j-1}$. Hence $(b_{j-1}, b_j)\leqq x\delta_{j-1}$, thus $b_{j-1}\leqq x$. Now $x\varepsilon_{j-1}$ is the join of all a'_j with $b'_{j-1}\varepsilon_{j-1}\geqq a'_j$ and b'_{j-1} $(\leqq x)$ join-irreducible. Thus $b_j\leqq x\varepsilon_{j-1}$, whence $a_j\leqq x\varepsilon_{j-1}$.

Conversely, let $a_j \leqq x\varepsilon_{j-1}$. Then $a_j \leqq a_{j-1}\varepsilon_{j-1}$ for some a_{j-1} ($\leqq x$) join-irreducible of D_{j-1}, which can be proved as follows. x is a join of join-irreducibles a_γ, $\gamma \in P$, of D_{j-1}. $a_j \leqq (\bigvee_\gamma a_\gamma)\varepsilon_{j-1} = (\bigvee_\gamma a_\gamma \varepsilon_{j-1})$. As a_j is join-irreducible (hence join-prime), it is less than or equal to one of the components in this join. (Notice, that this is the point of the proof which cannot be generalized to arbitrary directed systems.) Hence $(a_{j-1}, a_j) \leqq x\delta_{j-1}$, that is, $a_j \leqq \overline{x\delta}_{j-1}$.

Now the proof of (γ) is to prove that, for $d \in D_{j-1}$, $d\varepsilon_{j-1}\varphi_j = d\varphi_{j-1}\gamma_{j-1}$, where $\gamma_{j-1} = \mathrm{Con}\,(\lambda_{j-1})$. Now $d\varphi_{j-1}$ is the congruence generated by $[(0, 0, 0), (0, 0, d)]$ of the copy of $L(j-1 \to j-1)$ used in Figure 1 (constructed with $j-1$ instead of j, that is representing L_{j-1}). λ_{j-1} takes this interval to the interval $[(0, 0, 0), (0, 0, d\delta_{j-1})]$ of the copy of $L(j-1 \to j)$ used in the construction of L_j. Thus $d\varphi_{j-1}\gamma_{j-1}$ is generated by this interval. It is also generated (by perspectivity) by the interval $[(0, 0, 0), (d\delta_{j-1}, 0, 0)]$ of $L(j \to j)$. But then further generating pairs are

$$((0,0,0),\ (0,0,d\delta_{j-1})) \vee ((0,1,0),\ (0,1,0)) = ((0,1,0),\ (\overline{d\delta}_{j-1}, 0, \overline{d\delta}_{j-1}))$$

and

$$((0,1,0),\ (\overline{d\delta}_{j-1}, 0, \overline{d\delta}_{j-1})) \wedge ((0,0,1),\ (0,0,1)) = ((0,0,0),\ (0,0,\overline{d\delta}_{j-1})).$$

Using Lemma 3, we have that $d\varphi_{j-1}\gamma_{j-1}$ is generated by $((0, 0, 0), (0, 0, \overline{d\varepsilon}_{j-1}))$. On the other hand, $d\varepsilon_{j-1}\varphi_{j-1}$ is evidently generated by the pair $((0, 0, 0), (0, 0, \overline{d\varepsilon}_{j-1}))$ of the copy of $L(j \to j)$ used to construct L_j. This completes the proof.

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