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Relatively free bands of groups

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The subvarieties of the variety **CS** of all completely simple semigroups, along with their free objects, have been studied by V. V. RASIN [15], P. R. JONES [9] and by M. PETRICH and N. R. REILLY [14]. The lattice of subvarieties of the variety **B** of all bands has been constructed by A. P. BIRJUKOV [1], J. A. GERHARD [6] and C. F. FENNEMORE [5]; the defining laws of these varieties are known.

In this paper we observe that any regular semigroup is a subdirect product of any idempotent separating homomorphic image by any idempotent pure homomorphic image. This enables the construction of free objects of subvarieties of the variety **POBG** of all pseudo orthodox bands of groups in terms of relatively free bands and relatively free completely simple semigroups. It is shown that in any subvariety **V** of the variety **BG** of all bands of groups where $\mathbf{CS}\subseteqq\mathbf{V}\nsubseteqq\mathbf{POBG}$, the $\mathscr{H}$-classes of elements on 3 or more generators of the free objects are not free in any group variety. It is also shown that the free completely simple semigroup on a finite set is a retract of the free object on a countable set in any variety of completely regular semigroups that contains **CS**.

The first section includes a subdirect product decomposition of a regular semigroup and some preliminary results on varieties; it is shown that $\mathbf{RBG}\cap\mathbf{POBG}$ is a significant lower bound of the set of varieties **V**, $\mathbf{CS}\subseteqq\mathbf{V}\subseteqq\mathbf{BG}\setminus\mathbf{POBG}$, where **RBG** is the variety of all regular bands of groups. In the next section models of free objects in subvarieties of **POBG** are described, with an emphasis on those contained in $\mathbf{RBG}\cap\mathbf{POBG}$. The retract and $\mathscr{H}$-class results mentioned above are in the final section.

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1. Definitions and preliminary results

Suppose ϱ is a congruence on a regular semigroup S. Denote by $E(S)$ the set of idempotents of S. Define

$$\textit{trace of } \varrho = \operatorname{tr} \varrho = \varrho | E(S)$$

and

$$\textit{kernel of } \varrho = \ker \varrho = \{u \in S;\ (u, e) \in \varrho \text{ for some } e \in E(S)\}.$$

By FEIGENBAUM [4; Theorem 4.1], ϱ is completely determined by its trace and kernel. Note that if τ is also a congruence on S then $\operatorname{tr} \varrho \cap \operatorname{tr} \tau = \operatorname{tr}(\varrho \cap \tau)$. Also, by [8; proof of Lemma II.4.6], $\ker \varrho \cap \ker \tau = \ker(\varrho \cap \tau)$. By [10; Theorem 3.2], there exist least and greatest congruences on S with the same trace as ϱ (denoted respectively $\varrho_{\min}$ and $\varrho_{\max}$), or with the same kernel as ϱ (denoted respectively $\varrho^{\min}$ and $\varrho^{\max}$).

Lemma 1.1. *Let ϱ, τ and λ be congruences on a regular semigroup S such that $\varrho \subseteq \tau \subseteq \varrho_{\max}$ and $\varrho \subseteq \lambda \subseteq \varrho^{\max}$. Then S/ϱ is isomorphic to the subdirect product $\{(a\tau, a\lambda);\ a \in S\}$ of S/τ by S/λ.*

Proof. Since $\ker \lambda = \ker \varrho \subseteq \ker \tau$ and $\operatorname{tr} \tau = \operatorname{tr} \varrho \subseteq \operatorname{tr} \lambda$ then $\ker(\tau \cap \lambda) = \ker \varrho$ and $\operatorname{tr}(\tau \cap \lambda) = \operatorname{tr} \varrho$. So $\varrho = \tau \cap \lambda$ and the result follows (see [12; Proposition II.1.4]).

Throughout the paper $\mathbf{U}$ will denote the variety of all semigroups that have a unary operation, and X will denote a countably infinite set. The free object on X in $\mathbf{U}$ is denoted by $F_X^{\mathbf{U}}$. $F_X^{\mathbf{U}}$ is the smallest subsemigroup of the free semigroup on $X \cup \{(\ ,\ )^{-1}\}$ such that $X \subseteq F_X^{\mathbf{U}}$ and $(w)^{-1} \in F_X^{\mathbf{U}}$ for all $w \in F_X^{\mathbf{U}}$. We will write $w^{-1} = (w)^{-1}$ and $w^0 = ww^{-1}$.

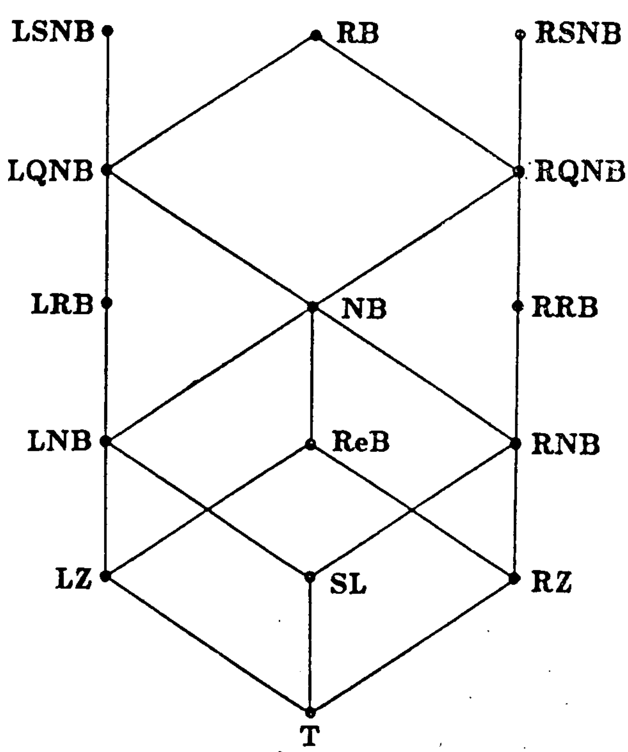
If $\mathbf{V}$ is a subvariety of $\mathbf{U}$ let $F_X^{\mathbf{V}}$ denote the free object in $\mathbf{V}$ on X, and let $\varrho_{\mathbf{V}}$ be the fully invariant congruence on $F_X^{\mathbf{U}}$ such that $F_X^{\mathbf{V}} \cong F_X^{\mathbf{U}}/\varrho_{\mathbf{V}}$. Denote by $L(\mathbf{V})$ the lattice of subvarieties of $\mathbf{V}$ and by $C(\mathbf{V})$ the lattice of fully invariant congruences on $F_X^{\mathbf{V}}$ (both ordered by inclusion). There is a lattice anti-isomorphism between $L(\mathbf{V})$ and $C(\mathbf{V})$ given by $\mathbf{W} \to \varrho_{\mathbf{W}}/\varrho_{\mathbf{V}}$. For $\mathbf{V} \subseteq \mathbf{W}$ in $L(\mathbf{U})$ let $[\mathbf{V}, \mathbf{W}] = \{\mathbf{Z} \in L(\mathbf{U});\ \mathbf{V} \subseteq \mathbf{Z} \subseteq \mathbf{W}\}$. For $Y \subseteq X$, let $F_Y^{\mathbf{V}}$ denote the subsemigroup of $F_X^{\mathbf{V}}$ generated in $\mathbf{V}$ by Y; $F_Y^{\mathbf{V}}$ is free on Y. We may regard $F_X^{\mathbf{V}}$ as being the set $F_X^{\mathbf{U}}$, subject to the laws of $\mathbf{V}$.

A semigroup is *completely regular* if and only if it is a union of its subgroups. It is well known that the class $\mathbf{CR}$ of all completely regular semigroups is a subvariety of $\mathbf{U}$ defined by the laws $xx^{-1}x = x$, $xx^{-1} = x^{-1}x$ and $(x^{-1})^{-1} = x$. So $\varrho_{\mathbf{CR}}$ is generated by $\{(uu^{-1}u, u), (uu^{-1}, u^{-1}u), ((u^{-1})^{-1}, u);\ u \in F_X^{\mathbf{U}}\}$.

By [10; Theorems 3.6, 4.2 and 4.3], for any $\mathbf{V} \in L(\mathbf{CR})$ then $(\varrho_{\mathbf{V}}/\varrho_{\mathbf{CR}})_{\min}$, $(\varrho_{\mathbf{V}}/\varrho_{\mathbf{CR}})^{\min}$, $(\varrho_{\mathbf{V}}/\varrho_{\mathbf{CR}})_{\max}$ and $(\varrho_{\mathbf{V}}/\varrho_{\mathbf{CR}})^{\max}$ are in $C(\mathbf{CR})$. Let $\mathbf{V}_{\max}$, $\mathbf{V}^{\max}$, $\mathbf{V}_{\min}$ and $\mathbf{V}^{\min}$ denote the varieties in $L(\mathbf{CR})$ that are respectively defined by these congruences.

It is usual when $\mathbf{V}\in L(\mathbf{B})$, the lattice of varieties of bands, to write **VG** for $\mathbf{V}_{max}$. **VG** is the variety of all semigroups $S\in\mathbf{CR}$ such that $\mathscr{H}$ is a congruence on S and $S/\mathscr{H}\in\mathbf{V}$.

Let **G** denote the variety of all groups, **CS** is the variety of all completely simple semigroups, and let **OBG** be the variety of all bands of groups that are orthodox. Let **POBG** denote the variety (see [7; Proposition 4.1]) of all $S\in\mathbf{BG}$ such that for each $e\in E(S)$, eSe is orthodox; S is called a *pseudo orthodox band of groups.* The following list, from [11], is of the bottom 15 varieties in $L(\mathbf{B})$ along with their defining laws as subvarieties of **B**: **T**=trivial variety $(x=y)$; **LZ**=left zero semigroups $(xy=x)$; **ReB**=rectangular bands $(xyx=x)$; **SL**=semilattices $(xy=yx)$; **LNB**=left normal bands $(xyz=xzy)$; **NB**=normal bands $(xyzx=xzyx)$; **LRB**=left regular bands $(xy=xyx)$; **LQNB**=left quasinormal bands $(xyz=xyxz)$; **RB**=regular bands $(xyzx=xyxzx)$; **LSNB**=left seminormal bands $(xyz=xyzxz)$; and the left-right duals **RZ, RNB, RRB, RQNB** and **RSNB** of **LZ, LNB, LRB, LQNB** and **LSNB** respectively. If $\mathbf{V}\in L(\mathbf{B})$ is not in the list then $\mathbf{V}\supseteq\mathbf{LSNB}\vee\mathbf{RB}$ or $\mathbf{V}\supseteq\mathbf{RSNB}\vee\mathbf{RB}$.



The following results are to be used later in the text. Define the *content* of $v \in F_X^{\mathbf{U}}$ to be

$$c(v) = \{\text{letters of } X \text{ appearing in } v\},$$

and for $\mathbf{V} \in L(\mathbf{CR})$ define

$$\mathscr{D}_{\mathbf{V}} = \{(u, v);\ u, v \in F_X^{\mathbf{U}} \text{ and } u\varrho_{\mathbf{V}}\, \mathscr{D}\, v\varrho_{\mathbf{V}}\}.$$

Theorem 1.2. (i) [2; Theorem 4.2]. *For* $u, v \in F_X^{\mathbf{U}}$, $(u, v) \in \mathscr{D}_{\mathbf{CR}}$ *if and only if* $c(u) = c(v)$.

(ii) $\mathscr{D}_{\mathbf{CR}}$ *is a congruence on* $F_X^{\mathbf{U}}$. *For* $\mathbf{V} \in L(\mathbf{CR})$ *either* $\varrho_{\mathbf{V}} \subseteq \mathscr{D}_{\mathbf{CR}}$ *and* $\mathbf{V} \supseteq \mathbf{SL}$ *or* $\varrho_{\mathbf{V}} \nsubseteq \mathscr{D}_{\mathbf{CR}}$ *and* $\mathbf{V} \subseteq \mathbf{CS}$.

Proof. Since $\mathscr{D}$ is the finest semilattice congruence on any completely regular semigroup then $\mathscr{D}_{\mathbf{CR}}$ is a congruence of $F_X^{\mathbf{U}}$ and $\varrho_{\mathbf{V}} \subseteq \mathscr{D}_{\mathbf{CR}}$ if and only if $\mathbf{V} \supseteq \mathbf{SL}$. If $\mathbf{V} \subseteq \mathbf{CS}$ then $\mathbf{V} \nsupseteq \mathbf{SL}$ and hence $\varrho_{\mathbf{V}} \nsubseteq \mathscr{D}_{\mathbf{CR}}$. Suppose $\varrho_{\mathbf{V}} \nsubseteq \mathscr{D}_{\mathbf{CR}}$. Then by (i) there exists $u, v \in F_X^{\mathbf{U}}$ such that $(u, v) \in \varrho_{\mathbf{V}}$ and $c(u) \neq c(v)$. We may assume that there exists $x \in c(u) \setminus c(v)$. Select finite subsets Y, Z of X and endomorphisms φ, ψ of $F_X^{\mathbf{U}}$ such that $c(x\varphi) = Y = c(z\psi)$ and $c(x\psi) = Z = c(z\varphi)$ for all $z \in X \setminus \{x\}$. Since $\varrho_{\mathbf{V}}$ is fully invariant and $(u, v) \in \varrho_{\mathbf{V}}$ then $(v\varphi, (u^0 v)\varphi), (v\psi, (u^0 v)\psi) \in \varrho_{\mathbf{V}}$ while $c(v\varphi) = Z$, $c(v\psi) = Y$ and $c((u^0 v)\varphi) = Y \cup Z = c((u^0 v)\psi)$. Hence by (i) $F_X^{\mathbf{U}}/\varrho_{\mathbf{V}}$ has just one $\mathscr{D}$-class and is therefore completely simple.

Theorem 1.3. *Suppose* $\mathbf{V} \in L(\mathbf{BG})$. *Then*

(i) $\mathbf{V}_{\max} \in L(\mathbf{OBG})$ *if and only if* $\mathbf{V} \cap \mathbf{B} \nsupseteq \mathbf{ReB}$,

(ii) $\mathbf{V}_{\max} \in L(\mathbf{POBG})$ *if and only if* $\mathbf{V} \cap \mathbf{B} \nsupseteq \mathbf{RB}$, *and*

(iii) $\mathbf{RBG} \cap \mathbf{POBG}$ *is the greatest lower bound in* $L(\mathbf{POBG})$ *of*

$$[\mathbf{CS}, \mathbf{BG}] \setminus L(\mathbf{POBG}).$$

Proof. Note that since $\mathscr{H}$ is the greatest idempotent separating congruence on $F_X^{\mathbf{V}}$, and $\mathscr{H}$ is a band congruence then $\mathbf{V}_{\min} = \mathbf{V} \cap \mathbf{B}$. Also observe that if $\mathbf{Z} \supseteq \mathbf{W}$ in $L(\mathbf{CR})$ then $\mathbf{Z}_{\max} \supseteq \mathbf{W}_{\max}$.

(i) Since $\mathbf{ReB}_{\max} = \mathbf{CS} \nsubseteq \mathbf{OBG}$ then $\mathbf{V}_{\max} \notin L(\mathbf{OBG})$ if $\mathbf{V} \cap \mathbf{B} \supseteq \mathbf{ReB}$. Conversely suppose $\mathbf{V} \cap \mathbf{B} \nsupseteq \mathbf{ReB}$; then $\mathbf{LRB} \supseteq \mathbf{V} \cap \mathbf{B}$ or $\mathbf{RRB} \supseteq \mathbf{V} \cap \mathbf{B}$. By duality, it suffices to assume $\mathbf{V} = \mathbf{V}_{\max} = \mathbf{LRBG}$, and to prove $\mathbf{V} \subseteq \mathbf{OBG}$. In this case $\mathbf{V}$ is defined as a subvariety of $\mathbf{BG}$ by $(xy)^0 = (xyx)^0$. So for any $e, f \in F_X^{\mathbf{U}}$ where $e\varrho_{\mathbf{V}}$ and $f\varrho_{\mathbf{V}}$ are idempotents,

$$ef\,\varrho_{\mathbf{V}}\, ef(ef)^0 f\,\varrho_{\mathbf{V}}\, ef(efe)^0 f\,\varrho_{\mathbf{V}}\, ef(efe)^0 ef\,\varrho_{\mathbf{V}}\, ef(ef)^0 ef\,\varrho_{\mathbf{V}}\, efef.$$

Thus $F_X^{\mathbf{V}}$ is orthodox.

(ii) The free completely simple semigroup with adjoined identity, $(F_X^{\mathbf{CS}})^1$, is not a pseudo-orthodox band of groups but it is a regular band of groups since it satisfies the law $(xyzx)^0=(xyxzx)^0$. Conversely, suppose $\mathbf{V}\cap\mathbf{B}\not\supseteq\mathbf{RB}$; so $\mathbf{V}\cap\mathbf{B}\subseteq\subseteq\mathbf{LSNB}$ or $\mathbf{V}\cap\mathbf{B}\subseteq\mathbf{RSNB}$. By duality we may assume $\mathbf{V}=\mathbf{V}_{\max}=\mathbf{LSNBG}$. Suppose $e, f, g\in F_X^{\mathbf{U}}$ such that $e\varrho_{\mathbf{V}}, f\varrho_{\mathbf{V}}$ and $g\varrho_{\mathbf{V}}$ are idempotents and $(efe, f), (ege, g)\in\varrho_{\mathbf{V}}$. Since $\mathbf{V}$ is defined in $L(\mathbf{BG})$ by $(xyz)^0=(xyzxz)^0$ then

$$(fg)^0\,\varrho_{\mathbf{V}}\,(fge)^0\,\varrho_{\mathbf{V}}\,(fgefe)^0\,\varrho_{\mathbf{V}}\,(fgf)^0\,\varrho_{\mathbf{V}}\,(fgf)^0f\,\varrho_{\mathbf{V}}\,(fg)^0f,$$

so $fg\,\varrho_{\mathbf{V}}\,fg(fg)^0g\,\varrho_{\mathbf{V}}\,fg(fg)^0fg\,\varrho_{\mathbf{V}}\,fgfg$. Hence $F_X^{\mathbf{V}}\in\mathbf{POBG}$ and the result follows.

(iii) By [7; Theorem 3.1 and Corollary 5.4], $L(\mathbf{BG})$ is modular and $\mathbf{POBG}==\mathbf{CS}\vee\mathbf{B}$. Therefore, since $\mathbf{RBG}\supseteq\mathbf{CS}$,

$$\mathbf{POBG}\cap\mathbf{RBG}=(\mathbf{CS}\vee\mathbf{B})\cap\mathbf{RBG}=\mathbf{CS}\vee(\mathbf{B}\cap\mathbf{RBG})=\mathbf{CS}\vee\mathbf{RB}.$$

By (ii) $\mathbf{CS}\vee\mathbf{RB}$ is a lower bound for $[\mathbf{CS},\mathbf{BG}]\setminus L(\mathbf{POBG})$. Furthermore if $\mathbf{V}\in L(\mathbf{POBG})$ is a lower bound for $[\mathbf{CS},\mathbf{BG}]\setminus L(\mathbf{POBG})$ then $\mathbf{V}\subseteq\mathbf{POBG}\cap\mathbf{RBG}$.

Lemma 1.4. *Suppose* $\mathbf{V}\in L(\mathbf{CR})$ *and* $\mathbf{W}\in[\mathbf{V},\mathbf{V}_{\max}\vee\mathbf{V}^{\max}]$. *Then* $\mathbf{W}==(\mathbf{W}\cap\mathbf{V}_{\max})\vee(\mathbf{W}\cap\mathbf{V}^{\max})$. *Furthermore* $\ker(\varrho_{\mathbf{W}}/\varrho_{\mathbf{CR}})=\ker(\varrho_{\mathbf{W}\cap\mathbf{V}_{\max}}/\varrho_{\mathbf{CR}})$.

Proof. The first statement is by [10; Theorem 5.4]. The second statement is proved in the initial part of the proof of [10; Theorem 5.1].

2. Free pseudo orthodox bands of groups

The lattice $L(\mathbf{CS})$ of completely simple semigroup varieties has been studied by several authors. In particular $F_X^{\mathbf{V}}$ has been characterized for $\mathbf{V}\in L(\mathbf{CS})$ in [9], [14] and [15].

Write $\leqq$ to mean "is embedded in", and omit the embedding details where they are obvious.

Theorem 2.1. (i) *If* $\mathbf{V}\in L(\mathbf{OBG})$ *then*

$$F_X^{\mathbf{V}}\cong\{(u\varrho_{\mathbf{V}\cap\mathbf{B}}, u\varrho_{\mathbf{V}\cap\mathbf{G}});\ u\in F_X^{\mathbf{U}}\}\leqq F_X^{\mathbf{V}\cap\mathbf{B}}\times F_X^{\mathbf{V}\cap\mathbf{G}}.$$

(ii) *If* $\mathbf{V}\in[\mathbf{ReB},\mathbf{POBG}]$ *then*

$$F_X^{\mathbf{V}}\cong\{(u\varrho_{\mathbf{V}\cap\mathbf{B}}, u\varrho_{\mathbf{V}\cap\mathbf{CS}});\ u\in F_X^{\mathbf{U}}\}\leqq F_X^{\mathbf{V}\cap\mathbf{B}}\times F_X^{\mathbf{V}\cap\mathbf{CS}}.$$

Proof. We have $\mathbf{T}_{\max}=\mathbf{G}$, $\mathbf{T}^{\max}=\mathbf{B}=\mathbf{ReB}^{\max}$ and $\mathbf{ReB}_{\max}=\mathbf{CS}$. By [13; Lemma 1] and [7; Corollary 5.4], $\mathbf{OBG}=\mathbf{B}\vee\mathbf{G}$ and $\mathbf{POBG}=\mathbf{B}\vee\mathbf{CS}$ respectively. By Lemma 1.4 then $\mathbf{V}\supseteq\mathbf{V}\cap\mathbf{G}\supseteq\mathbf{V}^{\min}$ in case (i) and $\mathbf{V}\supseteq\mathbf{V}\cap\mathbf{CS}\supseteq\mathbf{V}^{\min}$ in case (ii). Since $\mathbf{V}_{\min}=\mathbf{V}\cap\mathbf{B}$, the result is by Lemma 1.1.

This result can be refined, given more information on $F_X^{\mathbf{V}\cap\mathbf{B}}$ and $F_X^{\mathbf{V}\cap\mathbf{CS}}$.

The *head* $h(v)$ *of* $v\in F_X^{\mathbf{U}}$ is the first letter of X to appear in v. Dually the *tail*

$t(v)$ is the last letter of X to appear in v. The *initial part* $i(v)$ of v is the word obtained from v by retaining only the first occurrence of each letter from X. Dually define the *final part* $f(v)$ of v. Define $I=\{i(v);\ v\in F_X^{\mathbf{U}}\}$; so $I\subseteqq F_X^{\mathbf{U}}$ consists of finite strings of distinct letters from X. Then

$$(1) \qquad \varrho_{\mathbf{LNB}} = \{(u, v);\ u, v\in F_X^{\mathbf{U}} \text{ where } c(u) = c(v) \text{ and } h(u) = h(v)\}.$$

To see this note that the set is a fully invariant left normal band congruence on $F_X^{\mathbf{U}}$ that is contained in $\varrho_{\mathbf{SL}}\cap\varrho_{\mathbf{LZ}}$. Since the sublattice described in the diagram is convex, the congruence is $\varrho_{\mathbf{LNB}}$.

Likewise

$$(2) \qquad \varrho_{\mathbf{NB}} = \{(u, v)\in\varrho_{\mathbf{LNB}};\ t(u) = t(v)\},$$

$$(3) \qquad \varrho_{\mathbf{LRB}} = \{(u, v);\ u, v\in F_X^{\mathbf{U}} \text{ where } i(u) = i(v)\},$$

$$(4) \qquad \varrho_{\mathbf{LQNB}} = \{(u, v)\in\varrho_{\mathbf{LRB}};\ t(u) = t(v)\},$$

and

$$(5) \qquad \varrho_{\mathbf{RB}} = \{(u, v)\in\varrho_{\mathbf{LRB}};\ f(u) = f(v)\}.$$

Along with the well known results we readily get the following.

Theorem 2.2. *$F_X^{\mathbf{T}}=\{0\}$; $F_X^{\mathbf{LZ}}\cong X$ with multiplication $x\cdot y=x$;*
$F_X^{\mathbf{ReB}}\cong F_X^{\mathbf{LZ}}\times F_X^{\mathbf{RZ}}$; $F_X^{\mathbf{SL}}\cong\{Y\subseteqq X;\ |Y|<\infty\}$ under set union;
$F_X^{\mathbf{LNB}}\cong\{(x, Y);\ x\in Y\subseteqq X,\ |Y|<\infty\}\leqq F_X^{\mathbf{LZ}}\times F_X^{\mathbf{SL}}$;
$F_X^{\mathbf{NB}}\cong\{(x, y, Y);\ x, y\in Y\subseteqq X,\ |Y|<\infty\}\leqq F_X^{\mathbf{ReB}}\times F_X^{\mathbf{SL}}$,
$F_X^{\mathbf{LRB}}\cong I$ with multiplication $a\cdot b = i(ab)$;
$F_X^{\mathbf{LQNB}}\cong\{(a, x);\ a\in I,\ x\in c(a)\}\leqq F_X^{\mathbf{LRB}}\times F_X^{\mathbf{RZ}}$; and
$F_X^{\mathbf{RB}}\cong\{(a, b)\in I\times I;\ c(a) = c(b)\}\leqq F_X^{\mathbf{LRB}}\times F_X^{\mathbf{RRB}}$.

The free objects in other varieties of bands are not so easy to model.

Corollary 2.3. *Suppose* $\mathbf{V}\in L(\mathbf{LRBG})$ *and* $\mathbf{W}=\mathbf{V}\cap\mathbf{G}$. *If* $\mathbf{V}\in[\mathbf{SL}, \mathbf{SLG}]$ *then*

$$F_X^{\mathbf{V}}\cong\{(Y, g);\ g\in F_X^{\mathbf{W}},\ c(g)\subseteqq Y\subseteqq X,\ |Y|<\infty\}\leqq F_X^{\mathbf{SL}}\times F_X^{\mathbf{W}}.$$

If $\mathbf{V}\in[\mathbf{LNB}, \mathbf{LNBG}]$ *then*

$$F_X^{\mathbf{V}}\cong\{(x, Y, g);\ g\in F_X^{\mathbf{W}},\ \{x\},\ c(g)\subseteqq Y\subseteqq X,\ |Y|<\infty\}\leqq F_X^{\mathbf{LNB}}\times F_X^{\mathbf{W}}.$$

If $\mathbf{V}\in[\mathbf{LRB}, \mathbf{LRBG}]$ *then*

$$F_X^{\mathbf{V}}\cong\{(a, g)\in I\times F_X^{\mathbf{W}},\ c(g)\subseteqq c(a)\}\leqq F_X^{\mathbf{LRB}}\times F_X^{\mathbf{W}}.$$

Proof. With F_X^W replaced in these descriptions by F_X^U/ϱ_W it can be easily seen by Theorems 2.1 and 2.2 that the respective isomorphisms are given by $u\varrho_V\to(c(u), u\varrho_W)$, $u\varrho_V\to(h(u), c(u), u\varrho_W)$ and $u\varrho_V\to(i(u), u\varrho_W)$.

Select $h\in X$ and let $\{p_{yz};\, y, z\in X\setminus\{h\}\}$ be a set in one to one correspondence with $X\setminus\{h\}\times X\setminus\{h\}$. Put $p_{yz}=e$ if $y=h$ or $z=h$. By [9], [14] or [15], $F_X^{CS}\cong\mathcal{M}(H, X, X, P)$, a Rees matrix semigroup, where H is the free group with identity e freely generated by $\{exe, p_{yz};\, x, y, z\in X,\, y\neq h\neq z\}$, and P is the matrix with p_{yz} in row y and column z. $\mathcal{M}(H, X, X, P)$ is freely generated in **CS** by $\{(exe, x, x);\, x\in X\}$.

Also by [9], [14] and [15], if $\mathbf{V}\in[\mathbf{ReB}, \mathbf{CS}]$ then there is a unique normal subgroup N_V of H such that $F_X^V\cong\mathcal{M}(H/N_V, X, X, P/N_V)$.

Let ψ: $F_X^U\to\mathcal{M}(H, X, X, P)$ be the surjective homomorphism given by $x\psi=(exe, x, x)$ for all $x\in X$. Define φ: $F_X^U\to H$ by $u\psi=(u\varphi, h(u), t(u))$ for all $u\in F_X^U$. Then $x\varphi=exe$, $(xy)\varphi=x\varphi p_{xy}(y\varphi)$ and $u^{-1}\varphi=(p_{t(u)h(u)}(u\varphi)p_{t(u)h(u)})^{-1}$ for any $x, y\in X$ and $u\in F_X^U$. It follows that for $\mathbf{V}\in[\mathbf{ReB}, \mathbf{CS}]$ and $u, v\in F_X^U$ then $(u, v)\in\varrho_V$ if and only if $h(u)=h(v)$, $t(u)=t(v)$ and $u\varphi N_V=v\varphi N_V$.

Corollary 2.4. *Let* $\mathbf{V}\in[\mathbf{NB}, \mathbf{RBG}\cap\mathbf{POBG}]$ *and* $\mathbf{W}=\mathbf{V}\cap\mathbf{CS}$. *If* $\mathbf{V}\in[\mathbf{NB}, \mathbf{NBG}]$ *then*

$$F_X^V\cong\{((x, y, Y), (g, x, y));\ g\in H/N_W,\ \{x, y\}, c(g)\subseteq Y\subseteq X,\ |Y|<\infty\}\leqq F_X^{NB}\times F_X^W.$$

If $\mathbf{V}\in[\mathbf{LQNB}, \mathbf{LQNBG}]$ *then*

$$F_X^V\cong\{((a, x), (g, h(a), x));\ g\in H/N_W,\ a\in I,\ \{x\}, c(g)\subseteq c(a)\}\leqq F_X^{LQNB}\times F_X^W.$$

If $\mathbf{V}\in[\mathbf{RB}, \mathbf{RBG}\cap\mathbf{POBG}]$ *then*

$$F_X^V\cong\{((a, b), (g, h(a), t(b)));\ g\in H/N_W,\ a, b\in I,\ c(g)\subseteq c(a)=c(b)\}\leqq F_X^{RB}\times F_X^W.$$

Proof. By Theorems 2.1 and 2.2 it can be readily checked that the respective isomorphisms are given by $u\varrho_V\to((h(u), t(u), c(u)), (u\varphi N_V, h(u), t(u)))$, $u\varrho_V\to((i(u), t(u)), (u\varphi N_V, h(u), t(u)))$ and $u\varrho_V\to((i(u), f(u)), (u\varphi N_V, h(u), t(u)))$.

Note that there are repetitive symbols in the models; $h(a)$ and $t(b)$ are derivable from a and b. The repetitions are included so as to give a simple description of the multiplication.

Since the relatively free objects of **LZG** are known modulo **G** then by the corollaries the relatively free objects of **RBG**$\cap$**POBG** are known modulo **CS** and **G**.

By [12; Theorem IV.4.3], S is a normal band of groups if and only if S is a strong semilattice of completely simple semigroups. We can use Corollary 2.4 to characterize free objects of varieties in [**NB**, **NBG**] in these terms.

Suppose E is a semilattice and $\{S_\alpha;\, \alpha\in E\}$ is a disjoint set of semigroups. Suppose there exists a set of injective homomorphisms $\psi_{\alpha,\beta}$: $S_\alpha\to S_\beta$ for all $\alpha, \beta\in E$ where $\alpha\geqq\beta$, such that $\psi_{\alpha,\alpha}$ is the identity map and $\psi_{\alpha,\beta}\psi_{\beta,\gamma}=\psi_{\alpha,\gamma}$ for all $\alpha, \beta, \gamma\in E$

where $\alpha \geqq \beta \geqq \gamma$. Then $S = \bigcup_{\alpha \in E} S_\alpha$ with multiplication $a \cdot b = a\psi_{\alpha,\alpha\beta} b\psi_{\beta,\alpha\beta}$ for $a \in S_\alpha$ and $b \in S_\beta$ is called a *sturdy semilattice E of semigroups S_α; $\alpha \in E$ with transitive system* $\{\psi_{\alpha,\beta};\ \alpha, \beta \in E\}$ (see [12]).

Corollary 2.5. *If* $\mathbf{V} \in [\mathbf{NB}, \mathbf{NBG}]$ *and* $\mathbf{W} = \mathbf{V} \cap \mathbf{CS}$ *then*

$$F_X^{\mathbf{V}} \cong \{(Y, (g, x, y));\ g \in H/N_{\mathbf{W}},\ \{x, y\},\ c(g) \subseteqq Y \subseteqq X,\ |Y| < \infty\} \leqq F_X^{\mathbf{SL}} \times F_X^{\mathbf{W}}.$$

Hence $F_X^{\mathbf{V}}$ is a sturdy semilattice $F_X^{\mathbf{SL}}$ of semigroups $F_Y^{\mathbf{W}}$; $Y \in F_X^{\mathbf{SL}}$ with transitive system $\{\psi_{Y,Z};\ Y, Z \in F_X^{\mathbf{SL}}\}$ such that $\{x\psi_{\{x\},Y};\ x \in Y\}$ generates $F_Y^{\mathbf{W}}$. Conversely any such sturdy semilattices of semigroups is isomorphic to $F_X^{\mathbf{W}}$.

Proof. The subdirect decomposition is immediate by Corollary 2.4. So $D_Y = \{(Y, g, x, y);\ g \in H/N_{\mathbf{W}},\ \{x, y\},\ c(g) \subseteqq Y\}$ is a $\mathcal{D}$-class of the model and $D_Y \cong F_Y^{\mathbf{W}}$. With $\psi_{Y,Z}\colon D_Y \to D_Z$ given by $(Y, g, x, y) \to (Z, g, x, y)$ for $Z \supseteqq Y$ we see that $F_X^{\mathbf{V}}$ is a sturdy semilattice of the required form. Now suppose S is a sturdy semilattice $F_X^{\mathbf{SL}}$ of $F_Y^{\mathbf{W}}$; $Y \in F_X^{\mathbf{SL}}$ with transitive system $\{\psi'_{Y,Z};\ Y, Z \in F_X^{\mathbf{SL}}\}$ such that $\{x\psi'_{\{x\},Y};\ x \in Y\}$ generates $F_Y^{\mathbf{W}}$ for all Y. Define an automorphism η_Y of $F_Y^{\mathbf{W}}$ by $x\psi_{\{x\},Y}\eta_Y = x\psi'_{\{x\},Y}$ for all $x \in Y$. We have for $Z \supseteqq Y$, $\psi_{\{x\},Y}\eta_Y\psi'_{Y,Z} = \psi'_{\{x\},Y}\psi'_{Y,Z} = \psi'_{\{x\},Z} = \psi_{\{x\},Z}\eta_Z$. By [12; Exercise III.7.12.11] then $S \cong F_X^{\mathbf{V}}$.

3. Free non-pseudo orthodox bands of groups

This section begins with a description of $\mathcal{D}$-classes of relatively free completely regular semigroups that allows easy comparison of some properties of the relatively free objects.

Throughout, Y will denote a finite subset of X and $D_Y = \{u \in F_X^{\mathbf{U}};\ c(u) = Y\}$. D_Y is a unary subsemigroup of $F_X^{\mathbf{U}}$. Let ϱ be a congruence on D_Y such that D_Y/ϱ is completely simple. Select $e_Y = w^0$ for some $w \in D_Y$; so $e_Y\varrho \in E(D_Y/\varrho)$. For $u, v \in F_Y^{\mathbf{U}}$ define

$$e_{Yu,v} = ue_Y(e_Y vue_Y)^{-1}e_Y v. \tag{6}$$

We have $E(D_Y/\varrho) = \{e_{Yu,v}\varrho;\ u, v \in F_Y^{\mathbf{U}}\}$ since $e_{Yu,v}\varrho$ is an idempotent and for $r \in D_Y$, $r^0\ \varrho\ e_{Yr,r}$. For notational convenience write $e_{Y-u} = e_{Ye_Y,u}$ and $e_{Yu-} = e_{Yu,e_Y}$. Define

$$p_{Yu,v} = e_{Y-u}e_{Yv-};\quad u, v \in F_Y^{\mathbf{U}}. \tag{7}$$

By [3; Theorem 3.4], for any $u \in D_Y$ there is a unique $a\varrho$ such that $a\varrho\ \mathcal{H}\ e_Y\varrho$ and $u\ \varrho\ e_{Yu-}ae_{Y-u}$. In fact since $e_{Yu-}\varrho\ \mathcal{L} e_Y\varrho\ \mathcal{R}\ e_{Y-u}\varrho$ then $a\varrho = (e_Y ue_Y)\varrho$; so $u\ \varrho\ e_{Yu-}e_Y ue_Y e_{Y-u}$. Let H_Y be the unary subsemigroup of $F_X^{\mathbf{U}}$ generated by

$$\{e_Y ue_Y,\ p_{Yu,v};\ u, v \in F_Y^{\mathbf{U}}\}. \tag{8}$$

Then by [3; Theorem 3.4], H_Y/ϱ is the $\mathscr{H}$-class of $e_Y\varrho$ in D_Y/ϱ and

$$\{(e_{Yu-}he_{Y-v})\varrho;\ h\in H_Y\}$$

is the $\mathscr{H}$-class of $(e_{Yu-}e_{Y-v})\varrho$, $u, v\in D_Y$.

Suppose $\mathbf{V}\in[\mathbf{CS}, \mathbf{CR}]$. Let $S_{\mathbf{V}Y}$ be the completely simple subsemigroup of $D_Y/\varrho_\mathbf{V}$ that is generated by

$$\{(e_{Yx-}e_Yxe_Ye_{Y-x})\varrho_\mathbf{V};\ x\in Y\}.$$

Let T be a subsemigroup of a semigroup S and $\psi\colon S\to T'$ be a homomorphism. Then T is a *retract subsemigroup of S under ψ* if and only if there is an isomorphism $\varphi\colon T'\to T$, and $\varphi\psi$ is the identity map.

Theorem 3.1. *Let Y be a finite subset of X and $\mathbf{V}\in[\mathbf{CS}, \mathbf{CR}]$. Then $S_{\mathbf{V}Y}$ is a retract subsemigroup of $F_X^\mathbf{U}/\varrho_\mathbf{V}$ under $(\varrho_\mathbf{CS}/\varrho_\mathbf{V})^\#$. In particular $S_{\mathbf{V}Y}\cong F_Y^\mathbf{CS}$.*

Proof. Let $\psi\colon F_Y^\mathbf{U}/\varrho_\mathbf{V}\to F_Y^\mathbf{U}/\varrho_\mathbf{CS}$ be the surjective homomorphism determined by the action $(x\varrho_\mathbf{V})\psi=x\varrho_\mathbf{CS}$ for each $x\in Y$. So $\psi\circ\psi^{-1}$ is the restriction of $(\varrho_\mathbf{CS}/\varrho_\mathbf{V})$ to the subsemigroup $F_Y^\mathbf{U}/\varrho_\mathbf{V}$ of $F_X^\mathbf{U}/\varrho_\mathbf{V}$. We have $e_{Yx-}e_Yxe_Ye_{Y-x}\varrho_\mathbf{V}xe_Y(e_Yxe_Y)^{-1}e_Yx$, and $(xe_Y(e_Yxe_Y)^{-1}e_Y)\varrho_\mathbf{CS}$ is an idempotent that is $\mathscr{R}$-related to $x\varrho_\mathbf{CS}$. Hence $((e_{Yx-}e_Yxe_Ye_{Y-x})\varrho_\mathbf{V})\psi=x\varrho_\mathbf{V}\psi$ and ψ maps $S_{\mathbf{V}Y}$ onto $F_Y^\mathbf{U}/\varrho_\mathbf{CS}$. But $S_{\mathbf{V}Y}\in\mathbf{CS}$ so there is a surjective homomorphism $\varphi\colon F_Y^\mathbf{U}/\varrho_\mathbf{CS}\to S_{\mathbf{V}Y}$ given by $x\varrho_\mathbf{CS}\varphi=(e_{Yx-}e_Yxe_Ye_{Y-x})\varrho_\mathbf{V}$. The result follows.

The Theorem can be strengthened in the two variable case.

Theorem 3.2. *If $\mathbf{V}\in L(\mathbf{BG})$ and $\mathbf{W}=\mathbf{V}\cap\mathbf{NBG}$ then $F_{\{x,y\}}^\mathbf{V}\cong F_{\{x,y\}}^\mathbf{W}$.*

Proof. By [8; Lemma IV.4.6] it can be easily seen that $auva\ \varrho_\mathbf{B}\ avua$ for any $a, u, v\in F_{\{x,y\}}^\mathbf{U}$. So $(auva)^0\ \varrho_\mathbf{V}\ (avua)^0$ and hence $F_{\{x,y\}}^\mathbf{V}\in\mathbf{W}$. But $F_{\{x,y\}}^\mathbf{W}\in\mathbf{V}$, so the homomorphism $F_{\{x,y\}}^\mathbf{W}\to F_{\{x,y\}}^\mathbf{V}$ such that $x\to x$, $y\to y$ is an isomorphism.

Now suppose $\mathbf{V}\in L(\mathbf{BG})$ and Y is a finite subset of X. Let $a, b, c, d\in F_Y^\mathbf{U}$. If $(a, b), (c, d)\in\varrho_\mathbf{B}$ then since $\mathrm{tr}\ \varrho_\mathbf{BG}=\mathrm{tr}\ \varrho_\mathbf{B}$ we have by (6) and (7), $(e_{Ya,c}, e_{Yb,d})\in\varrho_\mathbf{BG}$, whence $(p_{Ya,c}, p_{Yb,d})\in\varrho_\mathbf{BG}$. So

(9) $$(p_{Ya,c}, p_{Yb,d})\in\varrho_\mathbf{V} \quad \text{if} \quad (a, b), (c, d)\in\varrho_\mathbf{B}.$$

Also by (7) $p_{Ya,c}\varrho_\mathbf{V}(e_Yae_Y)^{-1}e_Yace_Y(e_Yce_Y)^{-1}$ and since $H_Y/\varrho_\mathbf{V}$ is a group,

(10) $$e_Yace_Y\ \varrho_\mathbf{V}\ e_Yae_Yp_{Ya,c}e_Yce_Y.$$

By (9) and (10) $e_Yae_Y\ \varrho_\mathbf{V}\ e_Yaa^0e_Y\ \varrho_\mathbf{V}\ e_Yae_Yp_{Ya,a}e_Ya^0e_Y$ so

(11) $$e_Ya^0e_Y\ \varrho_\mathbf{V}\ p_{Ya,a}^{-1}.$$

Also $e_Y a e_Y \varrho_V e_Y a^2 a^{-1} e_Y \varrho_V e_Y a e_Y p_{Ya,a} e_Y a e_Y p_{Ya,a} e_Y a^{-1} e_Y$ so

(12) $$e_Y a^{-1} e_Y \varrho_V (p_{Ya,a} e_Y a e_Y p_{Ya,a})^{-1}.$$

Also note that since $(a^0, (h(a)t(a))^0) \in \varrho_{CS}$ while $e_{Ya-}\varrho_{CS}$ and $e_{Y-b}\varrho_{CS}$ are idempotents then by (6), $(e_{Ya-}, e_{Yh(a)-}), (e_{Y-b}, e_{Y-t(b)}) \in \varrho_{CS}$, so by (7)

(13) $$p_{Ya,b}\, \varrho_{CS}\, p_{Yt(a),h(b)}.$$

Lemma 3.3. *Suppose* $V \in [CS, BG]$. *Then* $V \in L(POBG)$ *if and only if* $(p_{Ya,b}, p_{Yt(a),h(b)}) \in \varrho_V$ *for some finite subset Y of X such that* $|Y| \geqq 3$ *and for all* $a, b \in F_Y^U$.

Proof. As noted in the proof of Theorem 2.1, $\mathbf{POBG} = \mathbf{ReB}_{\max} \vee \mathbf{ReB}^{\max}$. Then by [10; Theorem 3.4], $\mathbf{POBG} = (\mathbf{ReB}_{\max})^{\max} \cap (\mathbf{ReB}^{\max})_{\max}$. Since $\mathbf{ReB}_{\max} = \mathbf{CS}$ then $\mathbf{POBG} \in [\mathbf{CS}, \mathbf{CS}^{\max}]$ so $\ker \varrho_{POBG}/\varrho_{CR} = \ker \varrho_{CS}/\varrho_{CR}$. Thus if $\mathbf{CS} \subseteqq \mathbf{V} \subseteqq \mathbf{POBG}$ then $\ker \varrho_{CS}/\varrho_{CR} = \ker \varrho_V/\varrho_{CR}$. Then by (13), since $p_{Ya,b}\varrho_{CR}$ and $p_{Yt(a),h(b)}\varrho_{CR}$ are $\mathscr{H}$-related, $(p_{Yb,a}^{-1} p_{Yt(a),h(b)})\varrho_{CR} \in \ker \varrho_V/\varrho_{CR}$ so $(p_{Ya,b}, p_{Yt(a),h(b)}) \in \varrho_V$ for all $a, b \in F_Y^U$.

Conversely suppose $|Y| \geqq 3$ and $(p_{Ya,b}, p_{Yt(a),h(b)}) \in \varrho_V$ for all $a, b \in F_Y^U$. Then by (8), (10), (11) and (12), H_Y/ϱ_V is the group generated by

$$\{(e_Y x e_Y)\varrho_V,\ p_{Yx,y}\varrho_V;\ x, y \in Y\}.$$

We begin by showing that $\ker((\varrho_{CS}/\varrho_V)|(F_Y^U/\varrho_V)) = E(F_Y^U/\varrho_V)$. Recall that S_{VY} is a completely simple subsemigroup of D_Y/ϱ_V generated by $\{(e_{Yx-} e_Y x e_Y e_{Y-x})\varrho_V;\ x \in Y\}$. So there is a subgroup K_{VY} of H_Y/ϱ_V such that for each $x, y \in Y$, $\{(e_{Yx-} k e_{Y-y})\varrho_V;\ k \in K_{VY}\}$ is an $\mathscr{H}$-class in S_{VY}. We have $(e_Y x e_Y)\varrho_V \in K_{VY}$. Also, by (7), $(e_{Yy-} p_{Yx,y}^{-1} e_{Y-x})\varrho_V$ is an idempotent; it is $\mathscr{R}$-related to $(e_{Yy-} e_Y y e_Y e_{-yY})\varrho_V$ and $\mathscr{L}$-related to $(e_{Yx-} e_Y x e_Y e_{Y-x})\varrho_V$ so it is in S_{VY}. Hence $p_{Yx,y}\varrho_V \in K_{VY}$. It follows that $H_Y/\varrho_V = K_{VY}$, so the $\mathscr{H}$-classes of S_{VY} are $\mathscr{H}$-classes of D_Y/ϱ_V. Hence, since $D_Y/\varrho_V \in \mathbf{CS}$ and $\ker((\varrho_{CS}/\varrho_V)|S_{VY}) = E(S_{VY})$ by Theorem 3.1 then $\ker((\varrho_{CS}/\varrho_V)|(D_Y/\varrho_V)) = E(D_Y/\varrho_V)$.

Suppose $Z \subseteqq Y$. There is an endomorphism ψ of F_Y^U such that $x\psi = x$ if $x \in Z$ and $x\psi \in Z$ if $x \in Y \setminus Z$. Since ϱ_V is fully invariant then ψ induces an endomorphism φ of F_Y^U/ϱ_V given by $a\varrho_V\varphi = a\psi\varrho_V$. Define $e_Z = e_Y\psi$, so $e_Z\varrho_V = e_Y\varrho_V\varphi$ is an idempotent in D_Z/ϱ_V. Construct $p_{Zu,v}$ by (7) for $u, v \in F_Z^U$. Then

$$p_{Yu,v}\varrho_V\varphi = ((e_Y u e_Y)^{-1} e_Y u v e_Y (e_Y v e_Y)^{-1})\varrho_V\varphi = p_{Zu,v}\varrho_V.$$

Hence $(p_{Zu,v}, p_{Zt(u),h(v)}) \in \varrho_V$ for all $u, v \in F_Z^U$, and as above we get

$$\ker((\varrho_{CS}/\varrho_V)|(D_Z/\varrho_V)) = E(D_Z/\varrho_V).$$

Hence $\ker((\varrho_{CS})/\varrho_V)|(F_Y^U/\varrho_V)) = E(F_Y^U/\varrho_V)$.

Since $(x^0 F_Y^U x^0)/\varrho_{CS} \in \mathbf{OBG}$ it now follows that $(x^0 F_Y^U x^0)/\varrho_V \in \mathbf{OBG}$. But then $(x^0 y x^0 x^0 z x^0)^0 \, \varrho_V \, (x^0 y x^0)^0 (x^0 z x^0)^0$ for any $x, y, z \in Y$. So $(x^0 y x^0 x^0 z x^0)^0 = (x^0 y x^0)^0 (x^0 z x^0)^0$ is a law in $\mathbf{V}$ and $\mathbf{V} \in L(\mathbf{POBG})$.

The major result of this section can now be proved.

Theorem 3.4. *Suppose* $\mathbf{V} \in [\mathbf{POBG} \cap \mathbf{RBG}, \mathbf{BG}] \setminus L(\mathbf{POBG})$, *and* Y *is a finite subset of* X *such that* $|Y| \geqq 3$. *Then any* $\mathcal{H}$*-class of* F_X^U/ϱ_V *in the* $\mathcal{D}$*-class* D_Y/ϱ_V *is not a free group.*

Proof. Suppose $v \in Y$ and $u, w \in F_Y^U$. By (9), (10) and (11) we have

(14) $$e_Y uvwe_Y \, \varrho_V \, e_Y ue_Y p_{Yu, vw} e_Y ve_Y p_{Yv, w} e_Y we_Y,$$

(15a) $$e_Y uvwe_Y \, \varrho_V \, e_Y uv^0 e_Y p_{Yuv, vw} e_Y vwe_Y,$$

(15b) $$e_Y uvwe_Y \, \varrho_V \, e_Y uve_Y p_{Yuv, vw} e_Y v^0 we_Y,$$

(16a) $$e_Y uv^0 e_Y \, \varrho_V \, e_Y ue_Y p_{Yu, v} e_Y v^0 e_Y \, \varrho_V \, e_Y ue_Y p_{Yu, v} p_{Yv, v}^{-1},$$

(16b) $$e_Y v^0 we_Y \, \varrho_V \, p_{Yv, v}^{-1} p_{Yv, w} e_Y we_Y.$$

Then by (15a), (16a) and (14)

(17a) $$\begin{aligned} p_{Yuv, vw} e_Y vwe_Y \, &\varrho_V \, (e_Y uv^0 e_Y)^{-1} e_Y uvwe_Y \\ &\varrho_V \, p_{Yv, v} p_{Yu, v}^{-1} p_{Yu, vw} e_Y ve_Y p_{Yv, w} e_Y we_Y. \end{aligned}$$

Likewise by (15b), (16b) and (14)

(17b) $$e_Y uve_Y p_{Yuv, vw} \, \varrho_V \, e_Y ue_Y p_{Yu, vw} e_Y ve_Y p_{Yv, v}.$$

So by (10) and (17a), and (10) and (17b) respectively

(18a) $$\begin{aligned} e_Y uv^2 we_Y \, &\varrho_V \, e_Y uve_Y (p_{Yuv, vw} e_Y vwe_Y) \\ &\varrho_V \, e_Y ue_Y p_{Yu, v} e_Y ve_Y p_{Yv, v} p_{Yv, v}^{-1} p_{Yu, vw} e_Y ve_Y p_{Yv, w} e_Y we_Y, \end{aligned}$$

(18b) $$\begin{aligned} e_Y uv^2 we_Y \, &\varrho_V \, (e_Y uve_Y p_{Yuv, vw}) e_Y vwe_Y \\ &\varrho_V \, e_Y ue_Y p_{Yu, vw} e_Y ve_Y p_{Yv, v} e_Y ve_Y p_{Yv, w} e_Y we_Y. \end{aligned}$$

Comparing (18a) and (18b) then

$$p_{Yu, v} e_Y ve_Y p_{Yv, v} p_{Yu, v}^{-1} p_{Yu, vw} \, \varrho_V \, p_{Yu, vw} e_Y ve_Y p_{Yv, v}$$

whence

(19) $$(e_Y ve_Y p_{Yv, v})(p_{Yu, v}^{-1} p_{Yu, vw}) \, \varrho_V \, (p_{Yu, v}^{-1} p_{Yu, vw})(e_Y ve_Y p_{Yv, v}).$$

Alternatively we may repeat the above calculation with (14) replaced by $e_Y uvwe_Y \, \varrho_V \, e_Y ue_Y p_{Yu,v} e_Y ve_Y p_{Yuv,w} e_Y we_Y$ to get

$$(20) \qquad (p_{Yv,v} e_Y ve_Y)(p_{Yuv,w} p_{Yv,w}^{-1}) \, \varrho_V \, (p_{Yuv,w} p_{Yv,w}^{-1})(p_{Yv,v} e_Y ve_Y).$$

Let

$$\alpha = e_Y ve_Y p_{Yv,v}, \quad \beta = p_{Yu,v}^{-1} p_{Yu,vw}, \quad \gamma = p_{Yv,v} e_Y ve_Y, \quad \delta = p_{Yuv,w} p_{Yv,w}^{-1}.$$

Note that $\alpha\varrho_V$ is not an idempotent. To see this observe that for $v \in Y$ then as in the proof of Theorem 3.1, $(e_{Yv} - e_Y ve_Y e_{Y-v})\varrho_{CS} = v\varrho_{CS}$ which is not an idempotent in F_Y^U/ϱ_{CS}. But $(e_{Yv} - p_{Yv,v}^{-1} e_{Y-v})\varrho_{CS}$ is the idempotent $\mathscr{H}$-related to $v\varrho_{CS}$. Hence $(e_Y ve_Y, p_{Yv,v}^{-1}) \notin \varrho_{CS}$, so $\alpha\varrho_{CS} \neq e_Y \varrho_{CS} = \alpha^0 \varrho_{CS}$. Likewise $\gamma\varrho_{CS} \neq \gamma^0 \varrho_{CS}$.

Let A and B denote the subgroups of the $\mathscr{H}$-class H_Y/ϱ_V of $e_Y \varrho_V$ that are respectively generated by $\{\alpha\varrho_V, \beta\varrho_V\}$ and $\{\gamma\varrho_V, \delta\varrho_V\}$. Assume H_Y/ϱ_V is a free group. By (19) and (20), $\{\alpha\varrho_V, \beta\varrho_V\}$ and $\{\gamma\varrho_V, \delta\varrho_V\}$ are not sets of free generators of free groups, so A and B are free cyclic groups. Say $\lambda\varrho_V$ generates A for some $\lambda \in F_Y^U$ and $\alpha \, \varrho_V \, \lambda^m$, $\beta \, \varrho_V \, \lambda^n$. But $\alpha\varrho_{CS}$, and $\lambda\varrho_{CS}$, are not idempotents while by (13) $\beta\varrho_{CS} = \lambda^n \varrho_{CS}$ is idempotent, so $n=0$. Therefore $(p_{Yu,vw}, p_{Yu,v}) \in \varrho_V$, and likewise $(p_{Yuv,w}, p_{Yv,w}) \in \varrho_V$ for any $u, w \in F_Y^U$ and $v \in Y$. Of course $v = h(vw) = t(uv)$ so by (9) we now have $(p_{Ya,b}, p_{Yt(a),h(b)}) \in \varrho_V$ for all $a, b \in F_Y^U$; thus by Lemma 3.3 $\mathbf{V} \in L(\mathbf{POBG})$. This is a contradiction. Thus H_Y/ϱ_V is not a free group.

Remark. Since the subgroup S_{VY} of F_X^U/ϱ_V is isomorphic to F_Y^{CS} then for $|Y| \geqq 2$ and $\mathscr{H}$-class of S_{VY} is a free group on more than $|Y|$ free generators; that is, it generates the variety $\mathbf{G}$ of all groups. Hence any $\mathscr{H}$-class in D_Y/ϱ_V generates $\mathbf{G}$ and thus lies in no proper subvariety of $\mathbf{G}$.

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