Acta Sci. Math., 53 (1989), 33—42

Congruences on semigroups of quotients

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Introduction. PETRICH and others [2, 5, 6, 7] have studied semigroups V which are ideal extensions of a semigroup S by the quotient semigroup $T=V/S$. These extensions are classified by their homomorphic image in the translational hull $\Omega(S)$ of S. Most often S is required to be weakly reductive so that S is embedded in $\Omega(S)$. On the other hand, given a right quotient filter Σ on S, the semigroup $Q(S)$ of right quotients of S can be defined and all right S-systems $M\supseteqq S$ for which M/S is torsion can be classified by their homomorphic image in $Q(S)$. Often S is required to be strongly torsion free so that S is embedded in $Q(S)$. When M is strongly torsion free, M is isomorphic to an S-subsystem of $Q(S)$ and so may receive a semigroup structure from $Q(S)$. The author [4] has shown that these two concepts are special instances of a common generalization.

In this paper we study semigroups V containing the strongly torsion free semigroup S with $T=V/S$ torsion, called semigroup extensions of S by torsion T. In this situation T is an (S, S)-system which may not be a semigroup. However, $T^*=T\setminus\{0\}$ has a partial multiplication for pairs $t, t'\in T^*$ with $tt'\notin S$ in V. This partial multiplication is associative. When considering ideal extensions, the (S, S)-system T has a trivial scalar multiplication. In our situation, the (S, S)-system structure on T is not trivial and plays an important role.

In Section 1, the necessary definitions concerning semigroups of quotients are given and the semigroup extensions V of S are characterized in terms of an (S, S)-homomorphism $\theta\colon T^*\to Q$ which preserves any partial multiplication in T^*. This characterization is reminiscent of the characterization of ideal extensions due to CLIFFORD [1]. In Section 2, semigroup congruences υ on V are characterized in terms of the restriction $\sigma=\upsilon|_S$ of υ to S, and the (S, S)-system congruence τ on T inherited from υ when S/σ is strongly torsion free. In Section 3, the semigroup V/υ is shown to be an extension of S/σ by a quotient S-system of T. In Section 4, the special case of extensions determined by partial homomorphisms is considered.

Received October 10, 1985, and in revised form October 23, 1986.

1. Extensions of semigroups. Let S be a semigroup with zero. (In this paper S will always have a 0 unless otherwise noted.)

Definition. A *right quotient filter* on S is a nonempty collection Σ of right ideals of S satisfying

(i) if $A \in \Sigma$, $A \subseteq B$, a right ideal of S, then $B \in \Sigma$;

(ii) if $A, B \in \Sigma$ then $A \cap B \in \Sigma$;

(iii) if $A \in \Sigma$, $s \in S$ then $s^{-1}A = \{x \in S \mid sx \in A\} \in \Sigma$; and

(iv) if I is a right ideal of S, $A \in \Sigma$, and $a^{-1}I \in \Sigma$ for all $a \in A$, then $I \in \Sigma$.

HINKEL [3] calls such right quotient filters "special".

For $A \in \Sigma$, let $\mathrm{Hom}(A, S) = \{f\colon A \to S \mid f(ax) = f(a)x$ for all $x \in S$, $a \in A\}$. Let $\mathbf{B} = \bigcup_{A \in \Sigma} \mathrm{Hom}(A, S)$, then $\mathbf{B}$ is a semigroup under composition with multiplication of $f\colon A \to S$, $g\colon B \to S$ defined by the composition $f \circ g\colon C \to S$ where

$$C = \{b \in B \mid g(b) \in A\}$$

which is in Σ. Define the relation γ on $\mathbf{B}$ by $f\gamma g$ if and only if there is some $A \in \Sigma$ with $f(a) = g(a)$ for all $a \in A$. γ is a semigroup congruence and $Q = \mathbf{B}/\gamma$ is the *semigroup of right quotients* of S with respect to Σ.

Let M be a right S-system and define a relation δ on M by $m\delta m'$ if and only if for some $A \in \Sigma$, $ma = m'a$ for all $a \in A$. δ is called the *torsion congruence* on M. M is *strongly torsion free* if δ is the identity relation, and M is *torsion* if $\delta = M \times M$. For each $s \in S$, the γ class of the mapping $\lambda_s\colon S \to S$ given by left multiplication by s is denoted by $[s]$ and the mapping $[\ ]\colon S \to Q$ is a semigroup homomorphism. If S is strongly torsion free, $[\ ]$ is a monomorphism and we identify S with its image $[S]$ in Q.

Definition. A *partial (S, S)-algebra* T is a partial groupoid which is, at the same time, an (S, S)-system, for which $(ts)t' = t(st')$ for all $s \in S$ whenever both products are defined.

Let V be a (S, S)-system which is also a semigroup. If V satisfies $(vs)v' = v(sv')$ for all $v, v' \in V$, $s \in S$, we call V an *S-algebra*. A semigroup V containing S as a subsemigroup is clearly an S-algebra. Let $T = V/S$, the Rees quotient (S, S)-system. T has a partial associative multiplication of nonzero elements $t, t' \in T$ if $tt' \notin S$ (as an element of V) inherited from V, and so is a partial (S, S)-algebra. We denote $T \setminus \{0\}$ by T^* and note that $V = T^* \cup S$ as sets.

In general, given a partial (S, S)-algebra T, we wish to define a semigroup multiplication on $V = T^* \cup S$ extending the partial multiplication in T^* and the multiplication in S. If such a multiplication can be defined, we call V a *semigroup extension of S by T*.

Definition. Let Q be a semigroup of right quotients of S with respect to a right quotient filter Σ, and let T be a partial (S, S)-algebra. A mapping $\theta\colon T^* \to Q$ is a *partial homomorphism* if

(i) whenever $t, t' \in T^*$, and tt' is defined, $\theta(tt') = (\theta t)(\theta t')$, and

(ii) if $t \in T^*$, $s \in S$ and $ts \neq 0$ $[st \neq 0]$, then $\theta(ts) = (\theta t)s$ $[\theta(st) = s\theta(t)]$.

When S is strongly torsion free and T is torsion, the desired multiplication on $V = T^* \cup S$ can be defined as shown by the following theorem.

Theorem 1. *Let Σ be a right quotient filter on S and S be strongly torsion free. Let T be a partial (S, S)-algebra. If $\theta\colon T^* \to Q$ is a partial homomorphism satisfying $(\theta a)(\theta b) \in S$ if $a, b \in T^*$, ab undefined and $s(\theta b) \in S$ $[(\theta b)s \in S]$ if $sb = 0$ $[bs = 0]$, then $V = T^* \cup S$ is a semigroup under the multiplication*

$$a*b = \begin{cases} (\theta a)(\theta b) & \text{if } a, b \in T^*, ab \text{ undefined} \\ a(\theta b) & \text{if } a \in S, b \in T^*, ab = 0 \\ (\theta a)b & \text{if } a \in T^*, b \in S, ab = 0 \\ ab & \text{otherwise.} \end{cases}$$

Conversely, every semigroup extension V of S by torsion $T = V/S$ can be constructed in this manner.

Proof. The proof of the direct part of the theorem consists of verifying the associative law. The proof is tedious but not difficult so only the verification that $(a*b)*c = a*(b*c)$ for $a, b, c \in T^*$ is given. If ab, bc, $a(bc)$ and $(ab)c$ are defined, then

$$(a*b)*c = ab*c = (ab)c = a(bc) = a*bc = a*(b*c).$$

If ab and bc are defined while $(ab)c$ is not, then

$$(a*b)*c = ab*c = (\theta ab)(\theta c) = (\theta a\theta b)\theta c = \theta a(\theta b\theta c) = \theta a\theta(bc) = a*bc =$$
$$= a*(b*c).$$

If bc is defined while ab is not, then

$$(a*b)*c = (\theta a\theta b)*c = (\theta a\theta b)\theta c = \theta a(\theta b\theta c) = \theta a\theta(bc) = a*bc = a*(b*c).$$

Since the case bc undefined, ab defined is similar to the previous case, we consider the case where ab, bc are both undefined. In this case

$$(a*b)*c = (\theta a\theta b)*c = (\theta a\theta b)\theta c = \theta a(\theta b\theta c) = a*(\theta b\theta c) = a*(b*c).$$

Conversely, let $V = T^* \cup S$ be a semigroup extension of S by T where T is torsion. We define $\theta\colon V \to Q$ to be the natural mapping given as follows: for $v \in V$, since T is torsion $v^{-1}S = \{s \in S \mid vs \in S\} \in \Sigma$ so we define θv to be the γ-class of $g\colon v^{-1}S \to S$ given by $g(a) = va$. Clearly θ is a semigroup homomorphism of V into Q whose restriction to S is the identity. By abuse of notation, denote the restriction of θ to T^* by θ. Clearly, θ is a partial homomorphism satisfying $\theta a\,\theta b\in S$ if ab is undefined, and $s(\theta b)\in S$ $[(\theta b)s\in S]$ if $sb=0$ $[bs=0]$.

Moreover, if juxtaposition denotes the multiplication in V, then if $a, b\in T^*$, ab undefined, then $a*b=\theta a\,\theta b=\theta(ab)=ab$; if $as=0$, $a\in T^*$, $s\in S$, then $a*s=$ $=(\theta a)s=\theta(as)=as$; if $sa=0$, $a\in T^*$, $s\in S$, then $s*a=s\theta a=\theta(sa)=sa$; and otherwise, ab is defined in T^*, or $ab\in S$ in V, and in both cases $a*b=ab$.

2. Congruences on V. Let Σ be a right quotient filter on S, let S be strongly torsion free, and V be a semigroup containing S with $T=V/S$ a torsion partial (S, S)-algebra. To describe this situation we say that *V is a semigroup extension of S by torsion T.*

Definition. Let σ be a semigroup congruence on S and P be a (S, S)-subsystem of T with the following property:

(1) For each $p\in P^*$ there is $s\in S$ and $A\in\Sigma$ with the property that $pa\sigma sa$ for all $a\in A$.

In this case we say the p is *σ-linked* to s. (Note that T/P inherits a partial multiplication from T.)

Let τ be a 0-restricted multiplication preserving (S, S)-congruence on T/P satisfying

(2) if $x\tau y$, $s\sigma t$ and $xs, yt\in S$ then $xs\sigma yt$.

The relation $(\sigma, P, \tau)=\upsilon$ on $V=T^*\cup S$ is defined as follows:

for $x, y\in T\setminus P$, $x\upsilon y$ if and only if $x\tau y$;

for $x, y\in P^*$, $x\upsilon y$ if and only if there are $s, t\in S$ σ-linked to x and y (respectively) with $s\sigma t$;

for $x\in P^*$, $s\in S$, $x\upsilon s$ if and only if $s\upsilon x$ if and only if there is $t\in S$ σ-linked to x and $t\sigma s$; and

$\upsilon|_S=\sigma$.

A congruence σ on S is *strongly torsion free* if S/σ is a strongly torsion free semigroup with respect to the right quotient filter Σ/σ with base $\{\sigma^\#(A)|A\in\Sigma\}$ where $\sigma^\#$ is the canonical semigroup homomorphism from S to S/σ.

Lemma. *Σ/σ is a right quotient filter on S/σ.*

Proof. Property (i) is clear, while (ii) follows from elementary properties of the function $\sigma^\#$.

(iii) Let $B\in\Sigma/\sigma$ and $t=\sigma^\#(s)\in S/\sigma$. Let $A\in\Sigma$ with $\sigma^\#(A)\subseteq B$. Then $s^{-1}A\in\Sigma$ and $\sigma^\# s\sigma^\#(s^{-1}A)\subseteq B$ and $(\sigma^\# s)^{-1}B\in\Sigma/\sigma$.

(iv) Let $\sigma^{\#}(I)$ be a right ideal of S and $B\in\Sigma/\sigma$. Let $A\in\Sigma$ with $\sigma^{\#}(A)\subseteqq B$. Let $(\sigma^{\#}a)^{-1}\sigma^{\#}(I)\in\Sigma/\sigma$ for all $a\in A$. Then without loss of generality, $a^{-1}I\in\Sigma$ for all $a\in A$ so $I\in\Sigma$ and $\sigma^{\#}(I)\in\Sigma/\sigma$.

Theorem 1. *If σ is a strongly torsion free semigroup congruence on S, then $\upsilon=(\sigma, P, \tau)$ is a semigroup congruence on V whose restriction to S is strongly torsion free. Moreover, every semigroup congruence on V whose restriction to S is strongly torsion free is of this type.*

Proof. To show that υ is an equivalence relation it suffices to verify that for $p\in P$, $s, t\in S$, $p\upsilon s$ and $p\upsilon t$ imply that $s\sigma t$. However, $p\upsilon s$ and $p\upsilon t$ imply the existence of some $x\in S$ σ-linked to p with $x\sigma s$ and $x\sigma t$. Thus $s\sigma t$.

We next verify that υ is a left congruence. The "right" case is dual.

Case 1. Let $t, t'\in T\setminus P$, $c\in V$ and $t\upsilon t'$. Hence $t\tau t'$. If $ct\in T\setminus P$ then $ct'\in T\setminus P$ since τ is 0-restricted. Hence $ct\tau ct'$ or $ct\upsilon ct'$. Next let $ct\in P^*\cup S$, then $ct'\in P^*\cup S$. We consider several subcases.

(a) $ct, ct'\in P^*$: By (1), for some $x, y\in S$, $A\in\Sigma$ $ct\upsilon x$ and $ct'\upsilon y$ so that $cta\sigma xa$, $ct'a\sigma ya$ for all $a\in A$. By (2), $t\tau t'$ implies $cta\sigma ct'a$ which implies $x\sigma y$ since σ is strongly torsion free. Thus $ct\upsilon ct'$ by definition of υ.

(b) $ct\in S$, $ct'\in P^*$: For some $x\in S$ and $A\in\Sigma$, $ct'a\sigma xa$ for all $a\in A$. Hence by (2), $cta\sigma xa$ for all $a\in A$ so $ct\sigma x$ since σ is strongly torsion free. Hence $ct\upsilon ct'$.

(c) The other cases $ct\in P^*$, $ct'\in S$ and $ct, ct'\in S$ are treated similarly.

Case 2. Let $p, p'\in P^*$, $p\upsilon p'$ and $c\in V$. Then $p\upsilon s$, $p'\upsilon s$ for some $s\in S$ by the definition of υ. By (1), for some $A\in\Sigma$, $cpa\sigma csa$ for all $a\in A$. Similarly $cp'a\sigma csa$ for all $a\in A$. Again we consider several cases.

(a) $cp, cp'\in P^*$: Then $cp\upsilon x$ and $cp'\upsilon y$ for some $x, y\in S$. Thus $xa\sigma cpa\sigma cp'a\sigma ya$ for all $a\in A$ and so $x\sigma y$ and $cp\upsilon cp'$.

(b) $cp\in S, cp'\in P^*$: Then $cp'\upsilon x$ for some $x\in S$. Thus for all $a\in A$, $cpa\sigma cp'a\sigma xa$ so $cp\sigma x$ and $cp\upsilon cp'$.

The verification of the remaining cases is either similar to some case considered above or follows immediately from (1) or (2).

Conversely let μ be semigroup congruence on V whose restriction to S is strongly torsion free. Let $P=\{t\in T^*|t\mu s$ for some $s\in S\}\cup\{0\}$, then P is an (S, S)-subsystem of T. Let $\sigma=\mu|_S$ and define τ on T/P by:

$$t\tau t' \text{ if and only if } t\mu t' \quad (t, t'\in T\setminus P); \quad 0\tau 0.$$

Then clearly σ is a strongly torsion free semigroup congruence on S, every element of P^* is σ-linked to an element of S, and τ is a partial multiplication preserving 0-restricted (S, S)-congruence on T/P and conditions (1) and (2) are satisfied. Clearly $\mu\subseteqq\upsilon=(\sigma, P, \tau)$. To see the converse we need to consider two cases.

Case 1. $p\upsilon p'$; $p, p' \in P^*$: Then there are $s, s' \in S$ and $A \in \Sigma$ with $pa\sigma sa$, $p'a\sigma s'a$ for all $a \in A$, and $s\sigma s'$. On the other hand there are $x, x' \in S$ with $p\mu x$, $p'\mu x'$. Thus for all $a \in A$, $xa\sigma pa\sigma sa$ and $x'a\sigma p'a\sigma s'a$ so $x\sigma s\sigma s'\sigma x'$. Thus $p\mu s$, $p'\mu s'$ and $s\mu s'$ so $p\mu p'$.

Case 2. $p\upsilon s$; $p \in P^*$, $s \in S$: There is $x \in S$, $A \in \Sigma$ with $pa\sigma xa$ for all $a \in A$ and $x\sigma s$. However $p\mu x'$ for some $x' \in S$ so $x'a\sigma xa$ for all $a \in A$ or $x\sigma x'$. Thus $p\mu x'\mu x\mu s$ or $p\mu s$.

Corollary 2. *A relation μ on V is a semigroup congruence whose restriction to S is strongly torsion free if and only if μ is of the form (σ, P, τ) for some strongly torsion free semigroup congruence σ on S.*

If P is a nonzero (S, S)-subsystem of T such that $P^* \cup S$ is a strict extension of S (i.e. for all $p \in P^* \cup S$ there is some $s \in S$, $A \in \Sigma$ with $xa = sa$ for all $a \in A$ [4]), then P can be used in (σ, P, τ). In this case condition (1) is automatically satisfied but condition (2) must still hold.

Definition. A semigroup extension V of S by $T = V/S$ is determined by the *partial homomorphism* $\omega\colon T^* \to S$ if (1) ω preserves the partial multiplication and the (S, S)-system multiplication on T, and (2) the multiplication of $a, b \in V$ is given by

$$a * b = \begin{cases} (\omega a)(\omega b) & \text{if} \quad a, b \in T^*, ab \text{ undefined} \\ (\omega a)b & \text{if} \quad a \in T^*, b \in S, ab = 0 \\ a(\omega b) & \text{if} \quad b \in T^*, a \in S, ab = 0 \\ ab & \text{otherwise.} \end{cases}$$

Recall from [4] that if S is strongly torsion free, a semigroup extension V of S by torsion $T = V/S$ is strict if and only if V is determined by a partial homomorphism $\omega\colon T^* \to S$.

When V is determined by a partial homomorphism, we have the following result:

Proposition 3. *Let V be an extension of S determined by a partial homomorphism $\omega\colon T^* \to S$ where $T = V/S$ is torsion, σ be a strongly torsion free semigroup congruence on S, and P be any (S, S)-subsystem of T. Then there exists a multiplication preserving (S, S)-congruence τ on T/P for which $\upsilon = (\sigma, P, \tau)$ is a semigroup congruence on V. Moreover, condition* (2) *on σ and τ is equivalent to*

$$\omega t \sigma \omega t' \text{ if } t\upsilon t' \tag{3}$$

while condition (1) *holds automatically.*

Proof. Let τ be the identity congruence on T/P, then the first statement follows from the remarks preceding the statement of the proposition. If (2) holds, and $t\tau t'$, then for some $A\in\Sigma$ and all $a\in A$, $ta\sigma t'a$ hence $(\omega t)a\sigma(\omega t')a$ but since S is strongly torsion free, $\omega t\sigma\omega t'$. If (3) holds, then $t\tau t'$ and $x\sigma y$ implies $\omega t\sigma\omega t'$ from which $(\omega t)x\sigma(\omega t')y$ and so $tx\sigma t'y$ if tx, $t'y=0$, otherwise $tx\tau t'y$ since $tx\upsilon t'y$ (υ is a congruence) and tx, $t'y$ are both nonzero.

Remark. If T has no nontrivial (S, S)-subsystems then $P=\{0\}$ or $P=T$. Consequently for any semigroup congruence on V, either S is saturated by υ ($P=\{0\}$) or every υ-class intersecting T^* also intersects S; in these cases both conditions (1) and (2) are vacuous.

3. Homomorphic images of V. In this section we describe the homomorphic image of V induced by a congruence $\upsilon=(\sigma, P, \tau)$, where σ is strongly torsion free. Recall that for any semigroup congruence σ on S, $\sigma^\#$ denotes the natural mapping of S onto S/σ.

Theorem 1. *Let V be a semigroup extension of S by torsion $T=V/S$ determined by the partial homomorphism $\theta\colon T^*\to Q$. Let $\upsilon=(\sigma, P, \tau)$ where σ is strongly torsion free. Then υ is a semigroup congruence on V and one of the following two cases occurs:*

(i) $P=T$; *then* $V/\upsilon\cong S/\sigma$; *or*

(ii) $P\neq T$; *then* V/υ *is an extension of* S/σ, *by* $(V/\upsilon)/(S/\sigma)\cong(T/P)/\tau$ *determined by the partial homomorphism* $\beta\colon((T/P)/\tau)^*\to Q(S/\sigma)$ *where* β *is defined by* $\beta(\tau^\# t)=\langle\tau^\# t\rangle$, *where* $\langle\tau^\# t\rangle$ *is the equivalence class in* $Q(S/\sigma)$ *of the mapping*

$$\lambda_{\tau^\# t}\colon \sigma^\#(t^{-1}S)\to S/\sigma \quad \textit{defined by} \quad \lambda_{\tau^\# t}(\sigma^\# a)=\sigma^\#(ta).$$

Proof. That υ is a congruence follows from Theorem 2.1. If $P=T$, the mapping $\varrho(\sigma^\# x)=\upsilon^\# x$ for all $x\in S$ is a semigroup isomorphism from S/σ onto V/υ.

Suppose $P\neq T$. Let $K=T/P$, $V'=V/\upsilon$, and $S'=S/\sigma$. V' is a semigroup extension of $(P^*\cup S)/\upsilon$ by K/υ (by an obvious abuse of notation). From the construction of (σ, P, τ) it is clear that $(P^*\cup S)/\upsilon\cong S/\upsilon\cong S/\sigma$, and $K/\upsilon\cong K/\tau$. Hence we may consider V' as an extension of S' by $K'=K/\tau$. Here S' is strongly torsion free so we may describe this extension by means of a partial homomorphism β defined above. Let $\circ$ be the multiplication in V, $*$ the multiplication in V', and denote the multiplication in T, K and S' by juxtaposition. It remains to show that $*$ satisfies the conditions of Theorem 1.1 in $V'=K'^*\cup S'$.

For any $a', b'\in K'^*$ $(a'=\upsilon^\# a=\tau^\# a)$,

$$a'*b'=(a\circ b)'=\begin{cases}(ab)' & \text{if } ab\in T\setminus P\\ s' & \text{if } ab\in P^*,\ ab\upsilon s\\ [\theta a\theta b]' & \text{if } ab \text{ is undefined}\end{cases}=\begin{cases}a'b' & \text{if } a'b'\neq 0\\ (\beta a')(\beta b') & \text{if } a'b' \text{ is undefined.}\end{cases}$$

If $a' \in S'$, $b' \in K'^*$ then

$$a' * b' = (a \circ b)' = \begin{cases} (ab)' & \text{if } ab \in T \setminus P \\ s' & \text{if } ab \in P^*, ab \upsilon s \\ (a\theta b)' & \text{if } ab = 0 \end{cases} = \begin{cases} a'b' & \text{if } a'b' \neq 0' \\ a'\beta b' & \text{if } a'b' = 0'. \end{cases}$$

The case $a' \in K'^*$, $b' \in S'$ is similar to the above case and if $a', b' \in S'$ then $a' * b' = (a \circ b)' = (ab)' = a'b'$.

Corollary 2. *Under the same hypothesis and notation as in the theorem, if V is also a strict extension of S and $P \neq T$, then V/υ is an extension of S/σ by $(T/P)/\tau$ determined by the partial homomorphism ϱ: $((T/P)/\tau)^* \to S/\sigma$ defined by $\varrho(\tau^\# x) = \sigma^\# s$ where for some $A \in \Sigma$, $xa = sa$ for all $a \in A$.*

4. Extensions determined by a partial homomorphism. Let V be a semigroup extension of S by torsion T determined by a partial homomorphism ω: $T^* \to S$, σ be a semigroup congruence on S, P be an (S, S)-subsystem of T, τ be a 0-restricted partial multiplication preserving (S, S)-congruence on T/P, and suppose $\omega a \sigma \omega b$ if $a \tau b$ where $a, b \in P$. On V define the relation υ by

$$\begin{array}{ll} a, b \in T \setminus P: & a \upsilon b \text{ iff } a \tau b \\ a, b \in P^*: & a \upsilon b \text{ iff } \omega a \sigma \omega b \\ a \in P^*,\ b \in S: & a \upsilon b \text{ iff } b \upsilon a \text{ iff } \omega a \sigma b, \text{ and} \\ a, b \in S: & a \upsilon b \text{ iff } a \sigma b. \end{array}$$

We write $\upsilon = [\sigma, P, \tau]$.

Theorem 1. *The following statements hold:*

(i) $\upsilon = [\sigma, P, \tau]$ *is a semigroup congruence on* V;

(ii) *if σ is strongly torsion free then* $[\sigma, P, \tau] = (\sigma, P, \tau)$;

(iii) *every semigroup congruence μ on V whose restriction to S is strongly torsion free is of the form* $[\sigma, P, \tau]$;

(iv) *if $P = T$, then* $V/\upsilon \cong S/\sigma$;

(v) *if $P \neq T$ and σ is strongly torsion free then V/υ is an extension of S/σ by $(T/P)/\tau$ determined by the partial homomorphism ω' defined by*

$$\omega'(\tau^\# a) = \sigma^\# (\omega a), \quad a \in T \setminus P, \tag{4}$$

and

(vi) *condition* (3) *is equivalent to the existence of the function* ω': $((T/P)/\tau)^* \to S/\sigma$ *satisfying* (4).

Proof. (i) υ is clearly reflexive and symmetric. Let $p \in P^*$ and $s, t \in S$ with $p \upsilon s$, $p \upsilon t$. Then $\omega p \sigma s$, $\omega p \sigma t$ and $s \sigma t$ or $s \upsilon t$. Let $a, b, c \in T^* \cup S$ with $a, b, c \in T^*$, $a \upsilon b$. If $a, b, c \in T^*$ with $ac \in T^*$ then $bc \in T^*$ and $ac \upsilon bc$. If $ac \in P^*$ and $bc \in P^*$ then since $\omega a\sigma\omega b$, $\omega(ac)=\omega a\omega c\sigma\omega b\omega c=\omega(bc)$ or $ac\upsilon bc$. If $ac\in P^*$, $bc\in S$ then $\omega a\sigma\omega b\Rightarrow\omega(ac)\sigma\omega b\omega c\Rightarrow\omega(ac)\sigma bc$. The other cases are either obvious, or follow easily by arguments similar to the above.

(ii) Since in the definition of (σ, P, τ), for any $p\in P^*$, $\omega p\in S$ is σ-linked to p, $[\sigma, P, \tau]\subseteq(\sigma, P, \tau)$. Conversely suppose $p_1, p_2\in P^*$ and $p_1(\sigma, P, \tau)p_2$. Then p_i is σ-linked to $s_i\in S$ $(i=1, 2)$ by $A\in\Sigma$ and $s_1\sigma s_2$. Hence $(\omega p_1)a\sigma s_1 a\sigma s_2 a\sigma(\omega p_2)a$ for all $a\in A$ and since $A\in\Sigma$ and σ is strongly torsion free, $\omega p_1\sigma\omega p_2$ or $p_1[\sigma, P, \tau]p_2$. The cases $p_1\in P^*$, $p_2\in S$ and $p_1\in S$, $p_2\in P^*$ are obtained by similar arguments.

(iii) This follows from ii) and Theorem 2.1.

(iv) This is obvious.

(v) Using the notation in the proof of Theorem 3.1, for $t_1, t_2\in T\setminus P$ we obtain:

$$t_1' = t_2' \Rightarrow t_1\tau t_2 \Rightarrow \omega t_1\sigma\omega t_2 \Rightarrow (\omega t_1)' = (\omega t_2)' \Rightarrow \omega' t_1' = \omega' t_2$$

and so ω' is single-valued. If $t_1' t_2'$ is defined, then $t_1 t_2\in T\setminus P$ and

$$\omega'(t_1' t_2') = \omega'(t_1 t_2)' = [\omega(t_1 t_2)]' = (\omega t_1\omega t_2)' = (\omega t_1)'(\omega t_2)' = (\omega' t_1')(\omega' t_2')$$

and ω' is a partial homomorphism. If $t_1' t_2'$ is undefined and $t_1, t_2\in T\setminus P$ then

$$t_1' * t_2' = (t_1\circ t_2)' = \begin{cases}[\omega(t_1 t_2)]' & \text{if } t_1 t_2 \text{ is defined}\\ ((\omega t_1)(\omega t_2))' & \text{if } t_1 t_2 \text{ is undefined}\end{cases}$$
$$= [(\omega t_1)(\omega t_2)]' = (\omega t_1)'(\omega t_2)' = (\omega' t_1')(\omega' t_2').$$

If $t_1\in T\setminus P$ and $s\in S$, we have

$$t_1' * s' = (t_1\circ s)' = ((\omega t_1)s)' = (\omega t_1)' s' = (\omega' t_1')s'$$

and dually $s' * t_1' = s'(\omega' t_1')$.

(vi) By (v), (3) implies the existence of ω' satisfying (4). Conversely if (4) holds, then

$$t_1\tau t_2 \Rightarrow t_1' = t_2' \Rightarrow \omega' t_1' = \omega' t_2' \Rightarrow (\omega t_1)' = (\omega t_2)' \Rightarrow \omega t_1\sigma\omega t_2$$

and (3) holds.

Condition (4) can be expressed by saying that the following diagram commutes:

$$\begin{tikzcd}
(T/P)^* \arrow[r, "\omega^*"] \arrow[d, "\tau'"'] & S \arrow[d, "\sigma^{\#}"] \\
((T/P)/\tau)^* \arrow[r, "\omega'"] & S/\sigma
\end{tikzcd}$$

where $\omega^*=\omega|_{T\setminus P}$ and $\tau'=\tau^{\#}|_{T\setminus P}$.

Comparing Theorem 4.1 with Theorem 3.1, we see that condition (3) in the definition of $\upsilon=[\sigma, P, \tau]$ implies that υ is a semigroup congruence on V, while in Theorem 3.1, we had to suppose that σ is strongly torsion free to prove that (σ, P, τ) is a semigroup congruence on V. In Theorem 4.1, we obtain all semigroup congruences μ on V whose restriction to S is strongly torsion free; if $\mu|_S$ is not strongly torsion free, condition (3) need not hold.

Corollary 2. *Let T be a zero (left zero, right zero) (S, S)-system, then all semigroup congruences $\upsilon=[\sigma, P, \tau]$ on V can be constructed as follows: let σ be a semigroup congruence on S, and on T^* define σ' by*

$$t_1 \sigma' t_2 \Leftrightarrow \omega t_1 \sigma \omega t_2. \tag{5}$$

Let P be any (S, S)-subsystem of T, τ be a 0-restricted multiplication preserving equivalence relation (right S-congruence, left S-congruence) on T/P for which $\tau|_{T\setminus P} \subseteqq \sigma'|_{T\setminus P}$. Then (3) holds and $[\sigma, P, \tau]$ is a semigroup congruence on V. Conversely any semigroup congruence $[\sigma, P, \tau]$ on V can be constructed in this fashion. In particular, we obtain all semigroup congruences on V whose restriction to S is strongly torsion free.

Proof. On zero (left zero, right zero) (S, S)-systems all 0-restricted multiplication preserving equivalence relations (right S-congruences, left S-congruences) are (S, S)-congruences. From (5) and $\tau|_{T\setminus P} \subseteqq \sigma'|_{T\setminus P}$ it follows that (3) holds. Hence $[\sigma, P, \tau]$ is a semigroup congruence on V by Theorem 4.1.

Conversely, if $[\sigma, P, \tau]$ is a semigroup congruence on V, then (3) is satisfied and so $\tau|_{T\setminus P} \subseteqq \sigma'|_{T\setminus P}$.

The last statement of the corollary follows from part (iii) of Theorem 4.1.

When T is a zero (S, S)-system, every subset of T containing 0 is an (S, S)-subsystem, while (S, S)-subsystems of the other two types are 0-simple. Thus it is possible to characterize in a simple way a large class of semigroup congruences on V when T is of one of these types of (S, S)-system. Moreover, the extension is determined by a partial homomorphism.

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