Acta Sci. Math., 53 (1989), 43—50

On maximal clones of co-operations

Z. SZÉKELY

In this paper we determine all maximal clones of co-operations on a finite set, presenting a completeness criterion for co-operations in the spirit of Rosenberg's completeness theorem for operations on a finite set (cf. [3]). The result has some consequences for the theory of selective operations [2], too.

Our terminology is based on [1]. Here we present a short summary of the notions we use in this paper. For shortness, the set $\{0, 1, \dots, l-1\}$ will be denoted by $\mathbf{l}$ for every natural number l. Let A stand for the finite set $\mathbf{n}$ for $n>1$ and let $m>0$ be an integer. An *m-ary co-operation f on A* is a mapping of A into the union of m disjoint copies of A which can be given by and hence identified with a pair of mappings $\langle f_0, f_1\rangle$, where $f_0\colon A\to\mathbf{m}$ is called the *labelling* and $f_1\colon A\to A$ is called the *mapping* of f. The *i-th m-ary coprojection* $p^{m,i}$ (a special kind of co-operation) is defined by $p_0^{m,i}(a)=i$ and $p_1^{m,i}(a)=a$ for each $a\in A$ ($i\in\mathbf{m}$). The set of all co-operations and that of all m-ary co-operations on A are denoted by $\mathscr{C}_A$ and $\mathscr{C}_A^m$, respectively. The *variables* of the co-operation $f=\langle f_0, f_1\rangle\in\mathscr{C}_A^m$ are the disjoint copies of A where f maps to, indexed by the elements of $\mathbf{m}$. The i-th copy of A is an *essential variable* of f if its intersection with the range of f is nonempty, i.e. $f_0(x)=i$ for some $x\in A$. The co-operation f is called *essentially k-ary* if $|f_0(A)|=k$. Omitting all non-essential variables of f, we obtain a k-ary co-operation f_e, called the *skeleton* of f. We call a co-operation *essential* if it is injective and essentially at least binary.

Let $f\in\mathscr{C}_A^m$ and $g^{(0)}, g^{(1)}, \dots, g^{(m-1)}\in\mathscr{C}_A$. The *superposition* $h:=f(g^{(0)}, g^{(1)}, \dots, g^{(m-1)})$ of f with $g^{(0)}, g^{(1)}, \dots, g^{(m-1)}$ is the co-operation determined by the equalities $h_0(a)=g_0^{(f_0(a))}(f_1(a))$ and $h_1(a)=g_1^{(f_0(a))}(f_1(a))$ for each $a\in A$. The co-operation f is called the *main component* in this superposition. A set of co-operations on A is called a *clone* if it contains all coprojections and is closed under superposition. The least clone containing a set C of co-operations is called *the clone generated by C* and denoted by $[C]$. C is *complete* if $[C]$ equals $\mathscr{C}_A$. (A co-operation f is called *Sheffer*

Received February 28, 1986 and in revised form July 10, 1988.

if $\{f\}$ is complete.) The mappings of the set C generate a semigroup $\mathscr{S}(C)$ of self-maps of A called the *semigroup of* C. We call C *transitive* if $\mathscr{S}(C)$ is transitive. Note that $\mathscr{S}(C)\subseteq\mathscr{S}[C]$.

We remark that the lattice of clones of co-operation on A is finite. This fact can be shown in an easy way using the following remarks:

(1) The relation $\approx$ on $\mathscr{C}_A$ defined for $f, g\in\mathscr{C}_A$ by $f\approx g$ if both of the skeletons of f and g are k-ary and $g_e=f_e(p^{k,0\pi}, p^{k,1\pi}, \dots, p^{k,(k-1)\pi})$ for some permutation π of $\mathbf{k}$ is an equivalence relation with finitely many blocks. (Note that each block of the partition associated with $\approx$ can be represented by an at most $|A|$-ary co-operation and the number of these co-operations is finite.)

(2) Every subclone of $\mathscr{C}_A$ is a union of some blocks of the equivalence $\approx$ defined above. (It is trivial noting that for a clone C from $f\in C$ it follows $g\in C$ for each $g\in\mathscr{C}_A$ with $f\approx g$.)

A *maximal clone* of co-operations on A is a proper subclone C of $\mathscr{C}_A$ such that $C\subset D\subset\mathscr{C}_A$ for no clone D. Similarly to the case of algebras, a pair $\langle A, F\rangle$ with a nonempty set A and $F\subseteq\mathscr{C}_A$ is called a *coalgebra*. We say that $\langle A, F\rangle$ is a *finite coalgebra* if A is finite. $\langle A, F\rangle$ is called *primal* is F is complete. A co-operation $f\in\mathscr{C}_A$ is said to be *constant* if both f_0 and f_1 are constants. The coalgebra $\langle A, F\rangle$ is *functionally complete* if the union of F with the set of constant co-operations on A is complete.

There is a close connection between co-operations and regular selective operations, as follows. Let P and M be nonempty sets, let k be a natural number and let $f_0\colon P\to\mathbf{k}$ and $f_1\colon P\to P$. The k-ary operation f on M^P is called a *regular selective operation* if for every $p\in P$ the p-component of the result of f is the f_1-component of the f_0-th operand. Observe that the mappings f_0 and f_1 can be considered as the labelling and the mapping of a k-ary co-operation on P. Moreover, for any nontrivial M and nonempty P this natural correspondence yields a bijection between the regular selective operations on M^P and the co-operations on P. This bijection is a clone isomorphism. Hence the lattice of clones of regular selective operations on a finite power of a set is isomorphic to the lattice of clones of co-operations on a finite set and our criterion for the maximality of a clone of co-operations provides a description of all maximal proper subclones of the clone of all regular selective operations on a set M^P with P finite (cf. [1], [2]).

Consider a nonempty subset T of A. We say that a co-operation $f\in\mathscr{C}_A$ *preserves* T if T is closed under the mapping f_1. Let π be a partition of A. f *preserves* π if the labelling f_0 is constant on each block of π and f_1 preserves π in the usual sense (i.e. $f_1(a)\equiv_\pi f_1(b)$ holds for every $a, b\in A$ with $a\equiv_\pi b$, where $\equiv_\pi$ is the equivalence associated with π).

We call a co-operation $f\in\mathscr{C}_A$ (x, y)-*gluing* for some distinct $x, y\in A$ if $f(x)=$ $=f(y)$ (i.e. $f_i(x)=f_i(y)$ for $i\in\mathbf{2}$). Note that an arbitrary superposition with an

(x, y)-gluing main component is also (x, y)-gluing. We say that f *glues* in $T \subseteq A$ if f is (x, y)-gluing for some $x, y \in T$. We write $f \| T$ for "f does not glue in T" (i.e., $f|_T$ is injective on T). Let M be a family of subsets of A. M is called *disjoint* if its members are pairwise disjoint and called *uniform* if all its members have the same cardinality. M is *regular* if it is nonempty, disjoint, uniform and distinct from $A^* := \{\{a\}: a \in A\}$. The set of regular families of subsets of A will be denoted by $\mathrm{Rf}(A)$. The family M determines the following relation $\sim_M$ on A: $x \sim_M y$ if $x, y \in S$ for some $S \in M$. If M is disjoint, then $\sim_M$ is an equivalence on the set $\bigcup M := \bigcup_{S \in M} S$. We remark that every member of $\mathrm{Rf}(A)$ can also be considered as a partial equivalence on A.

Let $M \in \mathrm{Rf}(A)$ and $S \in M$ be arbitrary. The co-operation f *preserves S in M* if f_0 is constant on S and f_1 maps S into a member of M, i.e. $f_0(x) = f_0(y)$ and $f_1(x) \sim_M f_1(y)$ for all $x, y \in S$. (Note that the property "f preserves S in M" is not equivalent to the simple property "f preserves S" even in the case of M singleton!) Further, (i) f *weakly preserves S in M* if either f preserves S in M or f glues in S, (ii) f *(weakly) preserves M* if f (weakly) preserves each $S \in M$ in M, and (iii) a subset C of $\mathscr{C}_A$ *(weakly) preserves M* if each $f \in C$ (weakly) preserves M. Denote by C_M the set of co-operations weakly preserving M.

Let $f \in \mathscr{C}_A^m$, $T \subseteq A$ and $|f_0(T)| = k$. We put $\mathrm{ess}_T(f) := k$ and $\mathrm{ess}(f) := \mathrm{ess}_A(f)$. Let $g^{(0)}, g^{(1)}, \ldots, g^{(m-1)} \in \mathscr{C}_A$. The superposition $h = f(g^{(0)}, g^{(1)}, \ldots, g^{(m-1)})$ is called *disjoint* if the ranges of $g_0^{(0)}, g_0^{(1)}, \ldots, g_0^{(m-1)}$ are pairwise disjoint. The following fact is obvious:

Lemma 1. *Let $h = f(g^{(0)}, g^{(1)}, \ldots, g^{(m-1)})$ be a disjoint superposition, let $T \subseteq A$ and for $i \in \mathbf{m}$ put $T_i := T \cap f_0^{-1}(i) = \{x \in T: f_0(x) = i\}$. If $f \| T$ and $g^{(i)} \| f_1(T_i)$ (in particular, if $g^{(i)}$ is non-gluing) for each $i \in m$, then $h \| T$ and* $\mathrm{ess}(h) \geqq \mathrm{ess}(f)$.

A disjoint superposition of form

$$h = f(p^{k,0}, \ldots, p^{k,j-1}, g(p^{k,j}, \ldots, p^{k,j+m'-1}), p^{k,j+m'}, \ldots, p^{k,k-1})$$

will be denoted shortly by $h = f(\ldots, g, \ldots)_j$. Here $h \in \mathscr{C}_A^k$ where $k = m + m' - 1$ for $f \in \mathscr{C}_A^m$ and $g \in \mathscr{C}_A^{m'}$. Obviously we have:

Lemma 2. *Let $T \subseteq A$, $f \in \mathscr{C}_A^m$ and $g \in \mathscr{C}_A^{m'}$. If both f and g preserve T then $f(\ldots, g, \ldots)_j$ preserves T.*

We shall also use the following trivial facts:

Lemma 3. *Let T and T' be proper distinct subsets of A and let $C = \{f \in \mathscr{C}_A: f \text{ preserves } T\}$. Then there is an $f \in C$ not preserving T'.*

Lemma 4. *Let C_1 be a set of selfmaps of A. The semigroup generated by C_1 is transitive if and only if no non-trivial subset of A is preserved by C_1.*

We need some other preparations, as follows:

Lemma 5. *For arbitrary* $M \in \mathrm{Rf}(A)$ *the set* C_M *is a proper subclone of* $\mathscr{C}_A$.

Proof. First observe that $C_M \neq \mathscr{C}_A$ (indeed, there is some $f \in \mathscr{C}_A$ not preserving weakly M). We show that C_M is a clone. Clearly the coprojections preserve M and so it is enough to show the closedness under superposition, i.e. to prove that $h = f(g^{(0)}, g^{(1)}, \dots, g^{(m-1)}) \in C_M$ for arbitrary m-ary $f \in C_M$ and $g^{(0)}, g^{(1)}, \dots, g^{(m-1)} \in C_M$.

In order to do so consider a subset $S \in M$. The definition of C_M implies that either f glues in S or f preserves S in M. If $f(x) = f(y)$ for two distinct $x, y \in S$ then $h(x) = g^{(f_0(x))}(f_1(x)) = g^{(f_0(y))}(f_1(y)) = h(y)$, i.e. h glues in S too. Thus assume $f \| S$. Then $f_0(S) = i$ for some $i \in \mathbf{m}$, f_1 is injective on S and $f_1(S) \subseteq S'$ for some $S' \in M$. However, $|S'| = |S|$, whence f_1 maps S bijectively onto S'. If $g^{(i)}$ glues in S', i.e. $g^{(i)}(u) = g^{(i)}(v)$ for two distinct $u, v \in S'$ then (as f maps S onto S') $f_1(x) = u$ and $f_1(y) = v$ for some $x, y \in S$ and so $h(x) = g^{(f_0(x))}(f_1(x)) = g^{(i)}(u) = g^{(i)}(v) = g^{(f_0(y))}(f_1(y)) = h(y)$, i.e. h glues in S too. Thus assume $g^{(i)} \| S'$. Then $g^{(i)}$ preserves S' in M, i.e. $g_0^{(i)}$ is constant on S' and $g_1^{(i)}$ maps S' onto some $S'' \in M$. Since for all $x \in S$, $h_0(x) = g_0^{(i)}(f_1(x))$, we see that h_0 is constant on S and, similarly, $h_1(x) = g_1^{(i)}(f_1(x))$ for all $x \in S$ shows $h_1(S) \subseteq S''$, i.e. h preserves S in M. Therefore, h weakly preserves S in M.

Lemma 6. *Let* $M \in \mathrm{Rf}(A)$ *and suppose that the common cardinality of the members of* M *equals* $k > 1$. *Consider the* m-*ary co-operation* $f \in \mathscr{C}_A \setminus C_M$ *and put* $D := [C_M \cup \{f\}]$. *Let* S *be an arbitrary member of* M *which is not weakly preserved by* f. *Then for every* $\{u, v\} \subseteq S$ *there is a co-operation* $f^* \in D$ *such that* f^* *preserves* S, $f^* \| S$ *and* $f_0^*(u) \neq f_0^*(v)$.

Proof. It will be done in several steps.

Claim 0. For every permutation $\bar{h}$ of S there exists a unary co-operation $h' \in C_M$ preserving the set S, such that h_1' extends $\bar{h}$.

Indeed, put $h_0'(x) = 0$ for all $x \in A$, $h_1'(x) = \bar{h}(x)$ for $x \in S$ and $h_1'(x) = x$ on $A \setminus S$. Then h' obviously preserves M.

Claim 1. There are $\{x, y\} \subseteq S$ and $f' \in D$ such that $f' \| S$ and $f_0'(x) \neq f_0'(y)$.

Indeed, from the choice of S it follows $f \| S$. Furthermore, clearly it suffices to consider the case of f_0 constant on S, i.e. $f_0(S) = j$ for $j \in \mathbf{m}$ and $f_1(x) \not\sim_M f_1(y)$ for some $x, y \in S$. Consider the co-operation h defined as follows:

Suppose $M = \{S_0, S_1, \dots, S_{q-1}\}$ and $A \setminus \bigcup M = \{w_0, w_1, \dots, w_{r-1}\}$ where $0 \leqq r \leqq n - qk \leqq n - k$. Let $h \in C_M$ from $\mathscr{C}_A^{q+k}$ defined by

$$(*) \qquad h_0(x) = \begin{cases} i & \text{if } x \in S_i \quad (i \in q) \\ q + j & \text{if } x = w_j \quad (j \in r) \end{cases} \quad \text{and} \quad h_1 = \mathrm{id}_A.$$

Obviously $h_0(f_1(x)) \neq h_0(f_1(y))$. Put $f' = f(\ldots, h, \ldots)_j$. According to Lemma 1 $f' \| S$ and $f_0'(x) = h_0(f_1(x)) \neq h_0(f_1(y)) = f_0'(y)$.

Claim 2. There are $x, y \in S$ and $f'' \in D$ such that f'' preserves the set S, $f'' \| S$ and $f_0''(x) \neq f_0''(y)$.

Indeed, consider x, y and f' from Claim 1. Suppose $f' \in \mathscr{C}_A^{m'}$, put $J := f_0'(S)$ and, for each $j \in J$, put $R_j := \{u \in A : f_0'(z) = j \text{ and } f_1'(z) = u \text{ for some } z \in S\}$. Further, for each $j \in J$ let $h^{(j)} \in C_M$ be a unary co-operation such that $h_0^j(x) = j$ for all $x \in R_j$, $h^{(j)} \| R_j$ and $h_1^{(j)}(A) \subseteq S$. Such an $h^{(j)}$ exists, because $|R_j| \leqq |S|$. Form the disjoint superposition $f'' = f'(g^{(0)}, g^{(1)}, \ldots, g^{(m'-1)})$, where $g^{(j)} = h^{(j)}(p^{m',j})$ for $j \in J$ and $g^{(j)} = p^{m',j}$ otherwise. Lemma 1 implies $f'' \| S$. As $h^{(j)}$ preserves S for each $j \in J$, from the definition of R_j it follows that f'' also preserves S. Furthermore, $f_0'' = f_0'$, hence $f_0''(x) \neq f_0''(y)$ holds too.

To prove the assertion of the lemma consider two arbitrary distinct elements $u, v \in S$. Let x, y and f'' satisfy Claim 2. As $x \neq y$, Claim 0 implies that there exists a unary $h' \in C_M$ with $h_1'(u) = x$ and $h_1'(v) = y$. Put $f^* := h'(f'')$. Since it is a disjoint superposition, f^* preserves S and $f^* \| S$ by virtue of Lemmas 1 and 2. Furthermore, $f_0^*(u) = f_0''(x) \neq f_0''(y) = f_0^*(v)$, as needed.

Lemma 7. *Let D and S be the same as in Lemma 6. For every $i = 1, 2, \ldots, k$ there are an i-element subset H of S and $g \in D$ such that g preserves the set S, $g \| S$ and g_0 is injective on H.*

Proof. We proceed by induction on $i = 1, 2, \ldots, k$. The assertion is trivial for $i = 1$.

Let $1 \leqq i < k$. Assume the statement is valid for H_i and $g^{(i)} \in \mathscr{C}_A^{m_i}$. Choose an arbitrary element $x \in S \setminus H_i$ and let $H_{i+1} := H_i \cup \{x\}$. If $g_0^{(i)}$ is injective on H_{i+1}, we can put $g^{(i+1)} := g^{(i)}$.

Assume $g_0^{(i)}(y) = g_0^{(i)}(x) = j (\in \mathbf{m}_i)$ for some $y \in H_i$. As $g^{(i)} \| S$, the elements $u = g_1^{(i)}(x)$ and $v = g_1^{(i)}(y)$ are distinct. Hence by Lemma 6 there exists an m^*-ary co-operation $f^* \in D$ such that f^* preserves the set S, $f^* \| S$ and $f^*(u) \neq f^*(v)$. Now put $g^{(i+1)} = g^{(i)}(\ldots, f^*, \ldots)_j$, where $g^{(i+1)} \in \mathscr{C}_A^{m_{i+1}}$ for $m_{i+1} = m_i + m^* - 1$. Lemma 1 and 2 imply that $g^{(i+1)}$ preserves S and $g^{(i+1)} \| S$. The definition of $g^{(i+1)}$ yields that $g_0^{(i+1)}(x) = f_0^*(u) \neq f_0^*(v) = g_0^{(i+1)}(y)$. As $g^{(i+1)}$ is a disjoint superposition and, for $z_1, z_2 \in H_i$, $g_0^{(i)}(z_1) \neq g_0^{(i)}(z_2)$ implies $g_0^{(i+1)}(z_1) \neq g_0^{(i+1)}(z_2)$, we conclude that $g_0^{(i+1)}$ is injective on H_{i+1} and the lemma is proved.

Corollary 8. *Let the conditions of Lemma 6 be satisfied. Then there exists a co-operation $g \in D$ such that g_0 is injective on S.*

The promised Rosenberg-type criterion for completeness of sets of co-operations is the following.

Theorem. *A set C of co-operations on a finite set A is complete if and only if no regular family of subsets of A is weakly preserved by C.*

Proof. We shall prove the following claim, which is equivalent to the theorem: A set $C \subseteq \mathscr{C}_A$ is a maximal clone if and only if $C = C_M$ for some $M \in \mathrm{Rf}(A)$.

1. Sufficiency. Let $M \in \mathrm{Rf}(A)$. In accordance with Lemma 5, C_M is a proper subclone of $\mathscr{C}_A$. We verify that C_M is maximal by showing that for arbitrary $f \in \mathscr{C}_A \setminus C_M$ the clone $D := [C_M \cup \{f\}]$ equals $\mathscr{C}_A$. This will be done in two parts.

(i) Suppose that $M \neq A^*$ consists of singletons. Put $\overline{M} := \bigcup M$. Then $h \in \mathscr{C}_A$ weakly preserves M iff it preserves $\overline{M}$. If H is a proper subset of A distinct from $\overline{M}$, then in accordance with Lemma 3 there is a $g \in C_M$ not preserving H. Clearly f does not preserve $\overline{M}$, thus $C_M \cup \{f\}$ preserves no proper subset of A. Then $C_M \cup \{f\}$ is transitive as a consequence of Lemma 4. Further, C_M obviously contains an essentially n-ary co-operation and thus applying Proposition 2 from [1] we obtain that $C_M \cup \{f\}$ is complete, as required.

(ii) Now suppose that the common cardinality of the members of M equals $k > 1$. Then C_M is transitive as C_M contains all the constants in $\mathscr{C}_A$ (as each of them glues in every $S \in M$). We shall construct an essentially n-ary co-operation in D. Let S be an arbitrary member of M being not weakly preserved by f (there is such an S as $f \notin C_M$), and let $\tilde{f}$ be a selfmap of A, which maps each member of M bijectively onto S. Consider the unary co-operation $\tilde{f}$ with mapping $\tilde{f}_1$, equal to $\tilde{f}$ on $\bigcup M$ and to the identity map otherwise. Clearly $\tilde{f} \in C_M$. Take the co-operation h defined by $(*)$ and the co-operation g from Corollary 8. Form the disjoint superposition

$$g^* := h(\tilde{f}(g(p^{n,0}, p^{n,1}, \dots, p^{n,k-1})), \dots$$
$$\dots, \tilde{f}(g(p^{n,(q-1)k}, p^{n,(q-1)k+1}, \dots, p^{n,qk-1})), p^{n,qk}, p^{n,qk+1}, \dots, p^{n,qk+r-1}) \in C_M,$$

where q and r are the same as in $(*)$. From the properties of $h, \tilde{f}$ and g it follows $\mathrm{ess}_{S'}(g^*) = |S'| = k$ for each $S' \in M$. Also we see that $\mathrm{ess}_{A \setminus \cup M}(g^*) = \mathrm{ess}_{A \setminus \cup M}(h) = |A \setminus \bigcup M| = r$. As g^* is a disjoint superposition, its essential arity can be obtained additively: $\mathrm{ess}(g^*) = \mathrm{ess}_{(\cup M) \cup (A \setminus \cup M)}(g^*) = \sum_{S' \in M} \mathrm{ess}_{S'}(g^*) + \mathrm{ess}_{A \setminus \cup M}(g^*) = \sum_{S' \in M} |S'| + |A \setminus \bigcup M| = kq + r = n$. This completes the proof of the sufficiency.

Remark. For $M = \{A\}$ the clone C_M is called the Słupecki clone of co-operations on A. It consists of all non-essential co-operations. We see that it is a maximal clone, which occurs in the coalgebraic counterpart of Słupecki's completeness criterion for operations (Proposition 4 in [1]).

2. Necessity. Consider an arbitrary maximal clone C in $\mathscr{C}_A$. We verify that there exists a family $M \in \mathrm{Rf}(A)$ weakly preserved by C. This is enough, since then $C \subseteq C_M \subset \mathscr{C}_A$ from Lemma 5 and thus C has to equal the clone C_M.

(i) If C is not transitive, then in virtue of Lemma 4 there is a nonempty subset $T\subset A$ preserved by C. However, then $M:=\{\{a\}\subset A\colon a\in T\}\in \mathrm{Rf}(A)$ is preserved by C too.

(ii) Assume in the sequel that C is transitive. Observe that the clone of all gluing co-operations on A is a proper subset of the Słupecki clone on A. Thus C being maximal, it contains a non-gluing co-operation, for else C would be complete according to Proposition 2 in [1].

Consider an (m-ary) non-gluing co-operation $f\in C$ with maximal essential arity for the set of non-gluing co-operations of C. Denote by π the partition of A induced by f_0 and let M_1 be the set of blocks of π with maximal number of elements. The members of M_1 are not singletons, else π would be trivial and hence f essentially n-ary. It follows that $M_1\in \mathrm{Rf}(A)$.

Claim 0. For arbitrary $T\in M_1$, the restriction of f_1 to T is a bijection from T onto some $T'\in M_1$.

Let $j:=f_0(T)(\in \mathrm{m})$ and put $f':=f(\dots,f,\dots)_j\in C$. Obviously, for any $z\in A, f_1'(z)$ equals $f_1(f_1(z))$ if $f_0(z)=j$ and $f_1(z)$ otherwise. Lemma 1 implies $f'\|A$ and $\mathrm{ess}(f')\geqq \mathrm{ess}(f)$. It is easy to realize that $\mathrm{ess}(f')>\mathrm{ess}(f)$ iff there are $x,y\in A$ such that $f_0(x)=f_0(y)=j$ and $f_0(f_1(x))\neq f_0(f_1(y))$, i.e. $f_1(x)\equiv_\pi f_1(y)$ does not hold for some $x,y\in T$. Then it follows from the choice of f that $f_1(x)\equiv_\pi f_1(y)$ for each $x,y\in T$. Further, f is injective, thus f_1 is $1-1$ on T, whence $|f_1(T)|=|T|$. Then $T':=f_1(T)\in M_1$, as needed.

Put the set $M_2:=\{T\in M_1\colon f_1(\bigcup M_1)\cap T\neq\emptyset\}$ and let $M:=\{S\in M_1\colon$ there is $g\in C$ and $S'\in M_2$ such that the restriction of g_1 to S' is a bijection from S' onto $S\}$.

Due to Claim 0, M_2 is nonempty. On the other hand, $M_2\subseteq M$; thus M is also nonempty and $M\in \mathrm{Rf}(A)$.

We show that M is weakly preserved by C. This property will be obtained as a result of two claims. Let $S\in M$ be arbitrary and let $g\in C$ and $S'\in M_2$ be associated with S in the definition of M. Note that g can be chosen to be unary. Now Claim 0 guarantees that a suitable restriction of f_1 is a bijection onto S' from some $S''\in M_1$. Let $k:=f_0(S'')$.

Claim 1. If $h\in C$ and $h\|S$, then h_0 is constant on S.

Indeed, put $f^*:=f(\dots,g(h),\dots)_k\in C$. Then, for arbitrary $z\in A$, $f_1^*(z)$ equals $h_1(g_1(f_1(z)))$ if $f_0(z)=k$ and $f_1(z)$ otherwise. From Lemma 1 it follows $f^*\|A$ and $\mathrm{ess}(f^*)\geqq \mathrm{ess}(f)$. Similarly to the discussion of f' above, $\mathrm{ess}(f^*)>\mathrm{ess}(f)$ iff $h_0(g_1(f_1(x)))=h_0(g_1(f_1(y)))$ does not hold for some $x,y\in S''$. As f_1 and g_1 are $1-1$ when restricted to S'' resp. S', this condition is equivalent to $h_0(u)\neq h_0(v)$ for some $u,v\in S$. However, the choice of f implies that this condition does not hold, as asserted.

Claim 2. If $h \in C$ and $h \| S$ then the restriction of h_1 to S is a bijection from S onto some $S_0 \in M$.

Indeed, assume $h \in \mathscr{C}_A^{m_0}$ and let $k_0 := h_0(S) \in \mathbf{m}_0$. Put $h' := h(\dots, f, \dots)_{k_0} \in C$. Obviously $h'_0(z) = f_0(h_1(z)) + k_0$ and $h'_1(z) = f_1(h_1(z))$ for $z \in S$. Lemma 1 implies $h' \| S$, thus it follows from Claim 1 that h'_0 is constant on S, whence for each $x, y \in S$ we have $f_0(h_1(x)) = f_0(h_1(y))$, i.e. $h_1(x) \equiv_\pi h_1(y)$. Note that h_1 is injective on S, since $h \| S$ and h_0 is constant on S. Then, as S is a block of maximal size in π, the restriction of h_1 to S is a bijection from S onto some $S_0 \in M_1$. Now consider $S' \in M_2$. The restriction of the mapping of the co-operation $g^* := g(h) \in C$ to S' is the product of the bijections $g_1|_{S'}$ and $h_1|_S$, hence $g_1^*|_{S'}$ is a bijection from S' to S_0. Thus $S_0 \in M$, as required.

This completes the proof of the theorem.

We list some easy consequences of the Theorem (we omit their proofs).

Corollary 9. (Proposition 3 in [1].) *A co-operation on* $\mathbf{n}$ *is Sheffer if and only if it preserves neither non-least partitions nor nonempty proper subsets of* $\mathbf{n}$.

Corollary 10. *A finite coalgebra* $\langle A, F \rangle$ *is*

(i) *primal if and only if no regular family of subsets of A is weakly preserved by F;*

(ii) *functionally complete if and only if no regular family of nonsingleton subsets of A is weakly preserved by F.*

Corollary 11. *No distinct maximal clones of co-operations on a finite set have the same semigroups.*

The last corollary is the coalgebraic counterpart of the well-known fact that maximal clones of operations on a finite set are uniquely determined by the (semigroup of) unary operations they contain.

Acknowledgements. I am deeply indebted to professor Béla Csákány who firstly studied this topic. I have enjoyed his inspiring encouragement and help in my work. I am also grateful to the referee for his helpful comments.

References

[1] B. Csákány, Completeness in coalgebras, *Acta Sci. Math.*, **48** (1985), 75—84.

[2] B. Csákány, Selective algebras and compatible varieties, *Studia Sci. Math. Hungar.*, **19** (1984), 431—436.

[3] I. G. Rosenberg, Über die funktionale Vollständigkeit in den mehrwertigen Logiken, *Rozpr. ČSAV Řada Mat. Přír. Věd*, **80**, 4 (1970), 3—93.

BOLYAI INSTITUTE
ARADI VÉRTANÚK TERE 1
6720 SZEGED, HUNGARY