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Orthogonal polynomials and their zeros

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Let $d\mu$ be a finite positive Borel measure on the interval $[0, 2\pi)$ such that its support is an infinite set, and let $\{\varphi_n\}_{n=0}^{\infty}$, $\varphi_n(z)=\varphi_n(d\mu, z)=\varkappa_n z^n+\ldots$, $\varkappa_n= =\varkappa_n(d\mu)>0$, denote the system of orthonormal polynomials associated with $d\mu$, that is,

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi_m(z)\overline{\varphi_n(z)}\, d\mu(\theta) = \delta_{nm}, \quad z = e^{i\theta}.$$

The corresponding monic orthogonal polynomials $\varkappa_n^{-1}\varphi_n$ will be denoted by Φ_n. For an nth degree polynomial P the reverse polynomial P^* is defined by $P^*(z)= =z^n\overline{P(1/\bar{z})}$. Let $z_{kn}=z_{kn}(d\mu)$ be the zeros of φ_n ordered in such a way that

$$(1) \qquad |z_{nn}| \leqq |z_{n-1,n}| \leqq \ldots \leqq |z_{1n}| < 1$$

(cf. [7, p. 292]).

P. ALFARO and L. VIGIL (cf. [1, Proposition 1] and [2, Theorem 1]) proved that for every sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ with $|z_n|<1$, $n=1, 2, \ldots$, there is a unique measure $d\mu$ (modulo an arbitrary positive constant factor) such that $\varphi_n(d\mu, z_n)=0$ for $n=1, 2, \ldots$. This result can be obtained from the recurrence formula

$$(2) \qquad \Phi_n(z) = z\Phi_{n-1}(z)+\Phi_n(0)\Phi_{n-1}^*(z)$$

(cf. [7, formula (11.4.7), p. 293]) as follows. By (2) the recurrence coefficients $\Phi_n(0)$

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can be expressed as

$$\Phi_n(0) = -z_{kn}\Phi_{n-1}(z_{kn})/\Phi^*_{n-1}(z_{kn}),$$

and thus one can successively define the monic polynomials ψ_n by $\psi_0=1$ and

$$\psi_n(z) = z\psi_{n-1}(z)-[z_n\psi_{n-1}(z_n)/\psi^*_{n-1}(z_n)]\psi^*_{n-1}(z),$$

$n=1, 2, \ldots$. It is a matter of simple induction to show that $|\psi_n(0)|<1$, $n=1, 2, \ldots$, and then $\{\psi_n\}_{n=0}^{\infty}$ is orthogonal with respect to some $d\mu$ (cf. [4, Theorem 8.1, p. 156]). Since $\psi_n(z_n)=0$, this is the measure $d\mu$ we were looking for.

We point out that P. Alfaro and L. Vigil's result solves the following problem proposed by P. Turán: is there a measure $d\mu$ such that the set $\{z_{kn}(d\mu)\}$ is dense in the unit disk (cf. [9, Problem 67, p. 69]). Namely, the above measure $d\mu$ associated with any sequence $\{z_n\}$ which is dense in the unit disk provides such an example.

In view of this result by P. Alfaro and L. Vigil (and also because of the relation $\Phi_n(0)=\Pi z_{kn}$), one would want to seek for connections between orthogonal polynomials, their zeros and their recurrence coefficients. In spite of the great variety of results of such nature for orthogonal polynomials on the real line, and in spite of the intimate connection between real and complex orthogonal polynomials, there is only a very limited amount of research performed in this direction (cf. J. Szabados [6] and R. Askey's comment to paper [34—2] in [8, Vol. 2, p. 542]).

The main purpose of this note is to find a relationship between the quantities r_1, r_2, r_3 and r_4 which are defined as follows:

$$r_1(d\mu) = \limsup_{n\to\infty} |\Phi_n(d\mu, 0)|^{1/n},$$

$$r_2(d\mu) = \inf_k \limsup_{n\to\infty} |z_{kn}(d\mu)|,$$

$$r_3(d\mu) = \{\inf r\colon \sup_n \max_{|z|=r^{-1}} |\Phi^*_n(d\mu, z)| <\infty\}$$

and

$$r_4(d\mu) = \{\inf r\colon D(d\mu, z)^{-1} \text{ is analytic for } |z| < r^{-1}\}$$

where for $|z|<1$ the Szegő function $D(d\mu)$ is given by

$$D(d\mu, z) = \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \log\mu'(t)\frac{u+z}{u-z}\,dt\right\}, \quad u = e^{it},$$

if $\log\mu'$ is integrable, and $D(d\mu)\equiv 0$ otherwise (cf. [3], [4], [5] and [7]).

Theorem 1. *For every measure $d\mu$ we have $r_1(d\mu)=r_2(d\mu)$. If there is $j\in\{1, 2, 3, 4\}$ such that $r_j(d\mu)<1$ then $r_1(d\mu)=r_3(d\mu)=r_4(d\mu)$.*

Proof.

Step 1. $r_1\leq r_2$. Since $\Phi_n(0)=\Pi z_{kn}$ and $|z_{kn}|<1$, we have $|\Phi_n(0)|\leq|z_{kn}|^{n-k+1}$ for $k=1, 2, \ldots, n$ (cf. (1)), and thus $r_1\leq r_2$ follows.

Step 2. $r_1=1 \Rightarrow r_1=r_2$. This is obvious in view of Step 1 and $r_2 \leqq 1$.

Step 3. $r_1<1 \Rightarrow r_2, r_4 \leqq r_3 \leqq r_1$. By a result of YA. L. GERONIMUS [4, Theorem 8.3, p. 160] the sequence $\{|\Phi_n|\}$ is uniformly bounded on the unit circle. Thus by the maximum principle $\{|z^{-n}\Phi_n(z)|\}$ is uniformly bounded for $|z| \geqq 1$. Repeated application of the recurrence formula (2) leads to

$$\Phi_n^*(z) = 1+z\sum_0^{n-1} \overline{\Phi_{k+1}(0)}\,\Phi_k(z).$$

Therefore $\lim \Phi_n^*=\Phi^*$ exists uniformly on every disk with radius less than r_1^{-1} which implies $r_3 \leqq r_1$. By formula (8.6) in [4, p. 156]

$$\varkappa_n^2 = \varkappa_0^2 \prod_{l=1}^{n} [1-|\Phi_l(0)|^2]^{-1} \tag{3}$$

so that $r_1<1$ implies the boundedness of the sequence $\{\varkappa_n\}$ which by a theorem of YA. L. GERONIMUS [4, Section 1.2 (15), p. 14] guarantees the integrability of $\log \mu'$. But then by the Szegő theory (cf. [7, Theorem 12.1.1, p. 297]) $\lim \Phi_n^*=D(0)D^{-1}$ holds uniformly on compact subsets of the open unit disk where D denotes the Szegő function. Applying Vitali's theorem we can conclude that $\lim \Phi_n^*(z)=\Phi^*(z)$ exists for every $|z|<r_3^{-1}$ and obtain $\Phi^*=D(0)D^{-1}$, and thus $r_4 \leqq r_3$. In addition, since Φ^* possesses at most a finite number of zeros inside every disk with radius $r<r_3^{-1}$, the number of elements of the sets $\{z\colon |z| \leqq r,\ \Phi_n^*(z)=0\}$ is bounded for every $r<r_3^{-1}$. This follows from Rouche's theorem. In other words, $\{|\{z_{kn}\}_{k=0}^n \cap \{z\colon |z| \geqq r\}|\}_{n=0}^{\infty}$ is bounded for every $r>r_3$. Thus $r_2 \leqq r_3$.

Step 4. $r_1 \leqq r_3$. We may assume $r_3<\infty$. Then by Cauchy's formula

$$\overline{\Phi_n(0)} = \frac{1}{2\pi i} \oint_{|z|=r^{-1}} z^{-n-1}\Phi_n^*(z)\,dz = O(r^n)$$

holds for every $r>r_3$. Hence $r_1 \leqq r_3$.

Step 5. $r_1 \leqq r_4$. We may assume $r_4<1$. Then $\log \mu'$ is apriori integrable, and thus we have the Szegő theory at our disposition. Applying formula (5.1.18) in [3, p. 195] and using $\lim \varphi_n^*=D^{-1}$ in $L_2(d\mu)$ (cf. [3, p. 219]) we obtain

$$\overline{\Phi_n(0)} = \frac{1}{2\pi} \int_0^{2\pi} D^{-1}(z)\overline{\Phi_n(z)}\,d\mu(\theta), \quad z=e^{i\theta}. \tag{4}$$

Let us denote the Taylor expansion of D^{-1} by $\Sigma c_k z^k$. Then $\limsup |c_k|^{1/k}=r_4$ and by orthogonality

$$\overline{\Phi_n(0)} = \frac{1}{2\pi}\int_0^{2\pi}\Big[\sum_{k=n}^{\infty} c_k z^k\Big]\overline{\Phi_n(z)}\,d\mu(\theta), \quad z=e^{i\theta}.$$

Now using Cauchy's inequality and $\varkappa_0 \leqq \varkappa_n$ (cf. (3)) we obtain $r_1 \leqq r_4$.

Combining the inequalities proved in Steps 1 through 5, we get immediately Theorem 1.

Corollary. *The following assertions are pairwise equivalent:*

(a) $\limsup_{n\to\infty} |z_{1n}(d\mu)| < 1.$

(b) $\limsup_{n\to\infty} |\Phi_n(d\mu, 0)|^{1/n} < 1.$

(c) *$d\mu$ is absolutely continuous and $\mu'(\theta) = g(\theta)$ a.e. where g is a positive analytic function.*

Remark. 1. Note that this corollary characterizes the measures for which all zeros of the corresponding orthogonal polynomials lie in a smaller circle inside the unit circle.

2. There are many other statements equivalent to (a) above. Here are a few of them:

(d) There is $0<r<1$ such that $\Phi_n(d\mu, z) = O(r^n)$ for $|z|=r$.

(e) $\limsup_{n\to\infty} \max_{|z|=1} |\Phi_{n+1}(d\mu, z) - z\Phi_n(d\mu, z)|^{1/n} < 1.$

(f) $\limsup_{n\to\infty} \operatorname{ess\,sup}_{|z|=1} |\Phi_n(d\mu, z)z^{-n} - D^{-1}(d\mu, z)| < 1.$

Using the considerations below it is a fairly simple exercise to prove that any of (d)—(f) is equivalent to any of (a)—(c).

Proof of the Corollary. (a)⇒(b) by Theorem 1. That (c) implies (b) follows from the formula

$$\bar{\Phi}_n(0) = \frac{1}{2\pi}\int_0^{2\pi} \big(D^{-1}(z) - T_{n-1}(\theta)\big)\overline{\Phi_n(z)}\,d\mu(\theta) =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \big((\mu'(\theta))^{-1/2} - T_{n-1}(\theta)\big)\overline{\Phi_n(e^{i\theta})}\,d\mu(\theta) \quad (z=e^{i\theta})$$

(cf. (4)) where T_{n-1} is any trigonometric polynomial of degree at most $n-1$, if we take into account that the Φ_n's are uniformly bounded on $|z|=1$ (see [4, Theorem 4.5]) and that, by the analiticity of $(\mu')^{-1/2}$, we can choose a $0<q<1$ and $\{T_{n-1}\}$ such that

$$\big|(\mu'(\theta))^{-1/2} - T_{n-1}(\theta)\big| \leqq Kq^n \quad (\theta\in[0, 2\pi]).$$

Finally, both (b)⇒(a) and (b)⇒(c) follows if we can show (see Step 3 above) that

(5) $$\Phi^*(z) \neq 0 \quad \text{if} \quad |z| = 1.$$

In fact, (b) implies that $d\mu$ is absolutely continuous (see [4, Theorem 8.5]) and $\mu'(\theta)=D^{-2}(e^{i\theta})$ a.e., hence (b)⇒(c) is an immediate consequence of Step 3; while (b)⇒(a) can be derived from Rouché's theorem, namely there is a neighbourhood U of the unit circumference such that Φ_n^* does *not* have a zero in U (and hence Φ_n does not have a zero in U^{-1}) for large n. (5) follows from

$$\Phi^*(z) = D(0)D^{-1}(z) = (\mu'(\theta))^{-1/2}, \quad z = e^{i\theta},$$

and the analiticity of Φ^* on $|z|=1$ (which was proved above under the assumption (b)), namely $\Phi^*(e^{i\theta_0})=0$ would imply that $\mu'(\theta)\sim(\theta-\theta_0)^{-2}$ in a neighborhood of θ_0 except on a set of measure zero and this contradicts $\mu'\in L^1[\theta_0-\pi, \theta_0+\pi]$. The proof is complete.

Example. Let $1<R\leqq\infty$. Let f be analytic in the open (but not in the closed) disk U_R with radius R centered at 0, and assume $f(0)=1$ and $f(z)\neq 0$ for $|z|\leqq 1$. Let $1<|z_1|\leqq|z_2|\leqq\dots$ be the zeros of f in U_R. Define the measure $d\mu$ by $d\mu(\theta)= =|f(e^{i\theta})|^{-2}\,d\theta$. Then $\lim \Phi_n^*(z)=f(z)$ uniformly for $|z|\leqq r<R$. Hence

(6) $$\lim_{n\to\infty} z_{kn}(d\mu) = (\bar{z}_k)^{-1}$$

holds for every k if f has infinitely many zeros in U_R. If f has finitely many zeros there, say N, then (6) is satisfied for $k=1, 2, \dots, N$. In the former case we have

$$R^{-1} = \limsup_{n\to\infty} |\Phi_n(d\mu, 0)|^{1/n} < \lim_{n\to\infty} |z_{kn}(d\mu)| = |z_k|^{-1},$$

$k=1, 2, \dots$. If, in addition, f is a polynomial of degree, say, m then by the Bernstein—Szegő formula (cf. [3, Theorem 5.4.5, p. 224]) $\Phi_n^*=f$ for $n\geqq m$, and thus $z_{kn}=(\bar{z}_k)^{-1}$ $k=1, 2, \dots, m$, $z_{kn}=0$, $k=m+1, \dots, n$ and $\Phi_n(0)=0$ holds for $n\geqq m$.

We conclude this paper by observing that similarly to P. Alfaro and L. Vigil's result in [1, 2], orthogonal polynomials on the real line are also completely determined by some of their zeros.

Theorem 2. *Let* $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ *be given sequences of real numbers such that*

$$\dots < x_3 < x_2 < x_1 = y_1 < y_2 < y_3 < \dots.$$

Then there exists a unique system of monic polynomials $\{P_n\}_{n=0}^{\infty}$ *orthogonal with respect to a positive measure on the real line such that* $P_n(x_n)=P_n(y_n)=0$ *and* $P_n(t)\neq 0$ *for* $t\notin[x_n, y_n]$, $n=1, 2, \dots$.

Proof. Set $P_0=1$, $A_0=0$ and $b_0=x_1$. Define $\{P_n\}_{n=1}^{\infty}$, $\{A_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ by

$$P_n(x)=(x-b_{n-1})P_{n-1}(x)-A_{n-1}P_{n-2}(x), \tag{7}$$

$$A_n=(x_{n+1}-y_{n+1})\left[\frac{P_{n-1}(x_{n+1})}{P_n(x_{n+1})}-\frac{P_{n-1}(y_{n+1})}{P_n(y_{n+1})}\right]^{-1} \quad \text{and} \quad b_n=x_{n+1}-A_n\frac{P_{n-1}(x_{n+1})}{P_n(x_{n+1})}.$$

(The latter two formulae come from (7) and from the requirement $P_{n+1}(x_{n+1})=$ $=P_{n+1}(y_{n+1})=0$.) Using induction one can show that $P_n(x)\neq 0$ if $x\notin[x_n, y_n]$, $P_n(x_n)=P_n(y_n)=0$ and $A_n>0$ for $n=1,2,\ldots$. Hence by Favard's theorem (cf. [3, Theorem 2.1.5, p. 60]) $\{P_n\}_{n=1}^{\infty}$ is an orthogonal polynomial system.

If $x_n=-y_n$ for $n=1,2,\ldots$, then the formula for A_n and b_n above reduces to

$$A_n=x_{n+1}\frac{P_n(x_{n+1})}{P_{n-1}(x_{n+1})} \quad \text{and} \quad b_n=0.$$

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