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Noncyclic vectors for the backward Bergman shift

SHELLEY WALSH*)

§ 1. Introduction and notation. The Bergman space $\mathscr{A}^2$ is the Hilbert space of analytic functions f on the unit disk D such that

$$\|f\|^2 = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 |f(re^{i\theta})|^2 r \, dr \, d\theta < \infty.$$

The Bergman shift is the operator S on $\mathscr{A}^2$ defined by $(Sf)(z)=zf(z)$. If we let $e_n=(n+1)^{1/2} z^n$ then $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis for $\mathscr{A}^2$ and $Se_n=\left(\dfrac{n+1}{n+2}\right)^{1/2} e_{n+1}$, so S is a weighted shift. The Bergman shift is a subnormal operator so in particular it is hyponormal, so by Theorem 2 in [5], the functions which are contained in finite dimensional S^*-invariant subspaces are the finite linear combinations of the functions of the form $K_{\alpha,n}$ for some $\alpha \in D$ and n a nonnegative integer. In this paper I will give some examples of noncyclic vectors for S^*, which are not contained in finite dimensional S^*-invariant subspaces. I will do this by giving two sufficient conditions for the smallest invariant subspace containing the function $\sum_{k=1}^{\infty} c_k K_{\alpha_k}$ to be the orthogonal complement of $\{f\colon f(\alpha_k)=0 \text{ for all } k\}$. This is done in § 2.

The theorem in [2] which Theorem 1 in [5] follows from for the special case of the unweighted shift (Theorem 2.1.1) has as one of its consequences that the sum of two noncyclic vectors is noncyclic. In § 3 I will use the second condition given in § 2 to show that this is not true for S^*.

Throughout this paper cyclic will mean cyclic for S^*. If $f \in \mathscr{A}^2$, then $[f]_*$ will be the smallest S^*-invariant subspace containing f. If $\alpha \in D$ and n is a nonnega-

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tive integer then $K_{\alpha,n}$ will be the function in $\mathscr{A}^2$ such that $\langle f, K_{\alpha,n}\rangle = f^{(n)}(\alpha)$ and $K_{\alpha,0}$ will be written K_α when it is convenient.

Since

$$K_{\alpha,n}(z) = \sum_{j=n}^{\infty}(j+1)j\dots(j-n+1)\bar{\alpha}^{j-n}z^j = \frac{(n+1)!\,z^n}{(1-\bar{\alpha}z)^{n+2}},$$

Theorem 1′ in [5] can be stated for the Bergman shift as follows.

Theorem 0. *If f is analytic in a neighborhood of D, then f is either cyclic or a rational function with zero residue at each pole.*

Proof. It suffices to show that the rational functions with zero residue at each pole are the linear combinations of the $K_{\alpha,n}$'s. The residue of $K_{\alpha,n}$ at its only pole $\frac{1}{\bar{\alpha}}$ is

$$\left[(n+1)\left(\frac{-1}{\bar{\alpha}}\right)^{n+2} z^n\right]^{(n+1)}\left(\frac{1}{\bar{\alpha}}\right) = 0,$$

so any lineary combination of the $K_{\alpha,n}$'s has zero residue at all its poles. Conversely, to show that every rational function with zero residue at each pole is a linear combination of the $K_{\alpha,n}$'s it suffices to show that the function $\frac{1}{(1-\bar{\alpha}z)^{n+2}}$ is a linear combination of them, for any $\alpha\in D$ and nonnegative integer n. This is true because

$$\frac{1}{(1-\bar{\alpha}z)^{n+2}} = \sum_{j=0}^{n}\frac{\binom{n}{j}\bar{\alpha}^j z^j}{(1-\bar{\alpha}z)^{j+2}}.$$

§ 2. Some infinite dimensional cyclic invariant subspaces for S^*.

Theorem 1. *If $\{\alpha_k\}_{k=1}^{\infty}$ is a Blaschke sequence of distinct points in D and $\{c_k\}_{k=1}^{\infty}$ is a sequence of nonzero complex numbers such that $f=\sum_{k=1}^{\infty} c_k K_{\alpha_k}\in\mathscr{A}^2$, then $[f]_*=\{g\in\mathscr{A}^2\colon g(\alpha_k)=0$ for all $k\}^{\perp}$.*

Proof. If $g(\alpha_k)=0$ for all k then

$$\langle g, S^{*n}f\rangle = \langle z^n g, f\rangle = \sum_{k=1}^{\infty}\bar{c}_k\alpha_k^n g(\alpha_k) = 0, \quad \text{so} \quad g\in[f]_*^{\perp}.$$

If $h\in H^\infty$ then if $h^*(z)=\overline{h(\bar{z})}$, there is a uniformly bounded sequence of polynomials $\{q_n\}$ with $\|q_n-h^*\|\to 0$. Then $\|q_n(S^*)f-P(\bar{h}f)\|=\left\|P(q_n(\bar{z})f-\bar{h}f)\right\|\leq \leq\|q_n(\bar{z})f-\bar{h}f\|$ which tends to zero by the Lebesgue dominated convergence theorem so $P(\bar{h}f)\in[f]_*$. Hence if $g\perp[f]_*$ then $0=\langle g, P(\bar{h}f)\rangle=\langle hg, f\rangle=\sum_{k=1}^{\infty}\overline{c_k}h(\alpha_k)g(\alpha_k)$ for any h in H^∞. Fix m and let h be an H^∞ function such that $h(\alpha_m)=1$ and $h(\alpha_k)=0$ for $k\neq m$. Then $c_m g(\alpha_m)=0$. Since $c_m\neq 0$, it follows that $g(\alpha_m)=0$.

The next result uses a result of L. BROWN, A. SHIELDS, and K. ZELLER [1] concerning dominating sequences.

Definition. *If $\{\alpha_k\}$ is a sequence of distinct points in D, then $\{\alpha_k\}$ is dominating if for any function h in H^∞, we have $\|h\|_\infty=\sup_k|h(\alpha_k)|$.*

The following is contained in Theorem 3 of [1].

Lemma 1. *If $\{\alpha_k\}_{k=1}^{\infty}$ is a sequence of distinct points in D with all its limit points on ∂D, then the following are equivalent.*

(i) *There exists $\{a_k\}_{k=1}^{\infty}$ such that $0<\sum_{k=1}^{\infty}|a_k|<\infty$ and $\sum_{k=1}^{\infty}a_k\alpha_k^n=0$ for all non-negative integers n.*

(ii) *$\{\alpha_k\}$ is a dominating sequence.*

(iii) *Almost every boundary point $p=e^{i\theta}$ may be approached nontangentially by points of $\{\alpha_k\}$.*

Theorem 2. *Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of distinct points in D which has all its limit points on ∂D and is not a dominating sequence, and let $\{c_k\}_{k=1}^{\infty}$ be a sequence of nonzero complex numbers such that $\sum_{k=1}^{\infty}\frac{|c_k|}{1-|\alpha_k|^2}<\infty$. If $f=\sum_{k=1}^{\infty}c_kK_{\alpha_k}$, then $[f]_*=\{g\in\mathscr{A}^2\colon g(\alpha_k)=0 \text{ for all } k\}^\perp$.*

Proof. If $g(\alpha_k)=0$ for all k, then for any n, we have

$$\langle g, S^{*n}f\rangle=\sum_{k=1}^{\infty}\overline{c_k}\alpha_k^n g(\alpha_k)=0$$

so $g\in[f]_*^\perp$. If $g\in[f]_*^\perp$ then $\sum_{k=1}^{\infty}\overline{c_k}\alpha_k^n g(\alpha_k)=0$, for any n. For any k, we have

$$|g(\alpha_k)|=|\langle g, K_{\alpha_k}\rangle|\leqq\|g\|\,\|K_{\alpha_k}\|=\frac{\|g\|}{1-|\alpha_k|^2}.$$

So since $\sum_{k=1}^{\infty}\frac{|c_k|}{1-|\alpha_k|^2}<\infty$, the sum $\sum_{k=1}^{\infty}|\overline{c_k}g(\alpha_k)|$ is finite. Thus by Lemma 1, we have $\overline{c_k}g(\alpha_k)=0$ for all k. Since $c_k\neq 0$, it follows that $g(\alpha_k)=0$ for all k.

§ 3. Two noncyclic vectors whose sum is cyclic. In this section I will use Theorem 2 and the results and methods in [3] concerning zero sets for $\mathscr{A}^2$ to give an example of two noncyclic vectors whose sum is cyclic.

Definition. *A set E of points in D is a zero set for $\mathscr{A}^2$ if there exists a function $f\not\equiv 0$ in $\mathscr{A}^2$ with $f(z)=0$ (where $z\in D$) if and only if z is in E.*

The following lemmas are proved in [3].

Lemma 2. *If* $\mu > 1$ *and* β *is a positive integer with* $\beta > \mu^2+1$, *then*

$$f(z) = \prod_{j=1}^{\infty} (1+\mu z^{\beta^j}) \in \mathcal{A}^2.$$

Lemma 3. *If* $f \in \mathcal{A}^2$, $f(0) \neq 0$ *and* $\{\alpha_1, \alpha_2, \ldots\}$ *are the zeros of* f *indexed so that* $|\alpha_k| \leqq |\alpha_{k+1}|$, *then*

$$\prod_{k=1}^{N} \frac{1}{|\alpha_k|} = O(N^{1/2}).$$

Lemma 4. *Let* $f(z) = \prod_{j=0}^{\infty} (1+\mu z^{\beta^j})$ *where* $\mu > 1$ *and* $\beta \geqq 2$ *is an integer. If* $a = \frac{\log \mu}{\log \beta}$ *and* $\{\alpha_1, \alpha_2, \ldots\}$ *are the zeros of* f *indexed so that* $|\alpha_k| \leqq |\alpha_{k+1}|$, *for all* k, *then* $\prod_{k=1}^{N} \frac{1}{|\alpha_k|} > \text{Const} \cdot N^a$.

Lemma 5. *A subset of a zero set for* $\mathcal{A}^2$ *is a zero set for* $\mathcal{A}^2$.

Example 1. Let β be even and $\mu^2+1 < \beta < \mu^3$. Then the function $f(z) = \prod_{j=2}^{\infty} (1+\mu z^{\beta^j})$ belongs to $\mathcal{A}^2$. Let E be its zero set and $E_1 = \{re^{i\theta} \in E \colon \pi/2 \leqq \theta < 2\pi\}$. Then E_1 is a zero set by Lemma 5. The set E has β^j equally spaced points on the circle $|z| = \mu^{-\beta^j}$. On the same circle, the set E_1 has $\frac{3}{4}\beta^j$ points. Let $\{z_1, z_2, \ldots\}$ be the points of E and $\{\alpha_1, \alpha_2, \ldots\}$ be the points of E_1 indeed so that $|z_k| \leqq |z_{k+1}|$ and $|\alpha_k| \leqq |\alpha_{k+1}|$ for all k. By Lemma 4, if $a = \frac{\log \mu}{\log \beta}$, then for any N, we have $\prod_{k=1}^{N} \frac{1}{|z_k|} \geqq \text{Const} \cdot N^a$. Thus if $j \geqq 2$ and $N = \beta^2 + \ldots + \beta^j$, then

$$\prod_{k=1}^{3N/4} \frac{1}{|\alpha_k|} = \left(\prod_{k=1}^{N} \frac{1}{|z_k|} \right)^{3/4} \geqq \text{Const} \cdot N^{3a/4} = \text{Const} \cdot (3N/4)^{3a/4}.$$

Choose $0 < \varphi < \pi/2$ such that $e^{i\varphi} E_1$ is disjoint from E_1 and let $E_2 = e^{i\varphi} E_1$. Then E_2 is also a zero set for $\mathcal{A}^2$. If $0 < \theta < \pi/2$ then $e^{i\theta}$ is not a nontangential limit point of E_1 and if $\varphi < \theta < \pi/2 + \varphi$ then $e^{i\theta}$ is not a nontangential limit point for E_2, so, by Lemma 1, E_1 and E_2 are not dominating.

Let $\{c_k\}$ be a sequence of nonzero complex numbers such that $\sum_{k=1}^{\infty} \frac{|c_k|}{1-|\alpha_k|^2} < \infty$. Let $f_1 = \sum_{k=1}^{\infty} c_k K_{\alpha_k}$ and $f_2 = \sum_{k=1}^{\infty} c_k K_{e^{i\varphi}\alpha_k}$. Then by Theorem 2,

$$[f_i]_*^{\perp} = \{g \in \mathcal{A}^2 \colon g(z) = 0 \text{ for all } z \in E_i\}$$

for $i=1, 2$. If $\varphi<\theta<\pi/2$, then $e^{i\theta}$ is not a nontangential limit point of $E_1\cup E_2$, so, by Lemma 1, $E_1\cup E_2$ is not dominating. Therefore by Theorem 2,

$$[f_1+f_2]_*^\perp = \{g\in\mathscr{A}^2:\ g(z)=0 \ \text{ for all } \ z\in E_1\cup E_2\}.$$

If $\{\gamma_1, \gamma_2, \ldots\}$ are the members of $E_1\cup E_2$ indexed so that $|\gamma_k|\leqq|\gamma_{k+1}|$ for all k, then since

$$\prod_{k=1}^{3N/4}\frac{1}{|\alpha_k|}\geqq \mathrm{Const}\cdot(3N/4)^{3a/N},$$

for $N=\beta^2+\ldots+\beta^j$, we have $\prod_{k=1}^{N}\frac{1}{|\gamma_k|}\geqq\mathrm{Const}\cdot N^{3a/2}$, for infinitely many N's. Since $\beta<\mu^3$, we have $a=\frac{\log\mu}{\log\beta}>1/3$, so $3a/2>1/2$. Thus by Lemma 3, $E_1\cup E_2$ is not a zero set for $\mathscr{A}^2$, so $[f_1+f_2]_*^\perp=\{0\}$ and thus f_1+f_2 is cyclic.

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UHLANDSTRASSE 6
6781 HOHFRÖSCHEN
WEST GERMANY