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A spectral dilation of some non-Dirichlet algebra

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Let X be a compact Hausdorff space, let $C(X)$ be the algebra of complex-valued continuous functions on X, and let A be a uniform algebra on X. Let $\mathfrak{H}$ be a complex Hilbert space and $L(\mathfrak{H})$ the algebra of all bounded linear operators on $\mathfrak{H}$. I is the identity operator in $\mathfrak{H}$. An algebra homomorphism $f \to T_f$ of A in $L(\mathfrak{H})$, which satisfies

$$T_1 = I \quad \text{and} \quad \|T_f\| \leqq \|f\|$$

is called a representation of A on $\mathfrak{H}$. A representation $\Phi \to U_\Phi$ of $C(X)$ on a Hilbert space $\mathfrak{K}$ is called a spectral dilation of the representation $f \to T_f$ of A on $\mathfrak{H}$ if $\mathfrak{H}$ is a Hilbert subspace of $\mathfrak{K}$ and

$$T_f x = PU_f x \quad \text{for} \quad f \in A \quad \text{and} \quad x \in \mathfrak{H}$$

where P is the orthogonal projection of $\mathfrak{K}$ on $\mathfrak{H}$.

If A is a Dirichlet algebra on X and $f \to T_f$ a representation of A on $\mathfrak{H}$, then there exists a spectral dilation. This was proved by FOIAŞ and SUCIU (cf. [3, Theorem 8.7]). However, it is unknown whether any representation of a non-Dirichlet algebra has a spectral dilation. In this paper we give an example of a uniform algebra which has a spectral dilation for any operator representation and is a subalgebra of a disc algebra, of codimension one.

If $f \to T_f$ is a representation of A on a Hilbert space $\mathfrak{H}$ with the inner product (x, y) $(x, y \in \mathfrak{H})$, then there are measures $\mu_{x,y}$ $(x, y \in \mathfrak{H})$ such that $\|\mu_{x,y}\| \leqq \|x\| \|y\|$ for $x, y \in \mathfrak{H}$ and

$$(T_f x, y) = \int f \, d\mu_{x,y} \quad \text{for} \quad f \in A \quad \text{and} \quad x, y \in \mathfrak{H}$$

(see [3, p. 173]). Let τ be in the maximal ideal space of A and G the Gleason part of τ. We say that the representation $f \to T_f$ of A is G-continuous (G-singular) if

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there exists a system of finite measures $\{\mu_{x,y}\}$ such that $\mu_{x,y}$ is G-absolutely continuous (G-singular) and $(T_f x, y)=\int f\, d\mu_{x,y}$ for all $f\in A$ and all $x, y\in \mathfrak{H}$ (cf. [2, p. 182]). We need the following three lemmas to give a theorem. The first one is a theorem of MLAK [2, Theorem 2.3] and the second one is one result of FOIAŞ and SUCIU (cf. [3, p. 173]).

Lemma 1. *Let $f\to T_f$ be a representation of A on $\mathfrak{H}$. Then $f\to T_f$ is a unique orthogonal sum $T_f=T_f^a\oplus T_f^s$ where the representation $f\to T_f^a$ ($f\to T_f^s$) of A is G-absolutely continuous (G-singular).*

Lemma 2. *Let $f\to T_f$ be a representation of A on $\mathfrak{H}$. Then there are measures $\mu_{x,y}$ ($x, y\in\mathfrak{H}$) such that $\|\mu_{x,y}\|\leqq\|x\|\,\|y\|$ for $x, y\in\mathfrak{H}$ and*

$$((T_f+T_g^*)x, y)=\int (f+\bar{g})\, d\mu_{x,y}$$

for $f, g\in A$ and $x, y\in\mathfrak{H}$.

A family $\lambda_{x,y}$ ($x, y\in\mathfrak{H}$) of measures on X is called semispectral if it satisfies the following properties:

(1) $$\lambda_{\alpha x+\beta y, z}=\alpha\lambda_{x,z}+\beta\lambda_{y,z},$$

(2) $$\int \Phi\, d\lambda_{x,y}=\overline{\int \bar{\Phi}\, d\lambda_{y,x}} \quad (\Phi\in C(X)),$$

(3) $$\lambda_{x,x}\geqq 0,$$

(4) $$\|\lambda_{x,y}\|\leqq\gamma\|x\|\,\|y\|$$

where α and β are complex numbers, and γ is a positive number.

Now we can give an example of a uniform algebra which has a spectral dilation for any operator representation and is not a Dirichlet algebra. Let $\mathbf{T}$ be the unit circle and $\mathscr{A}$ the algebra of those continuous functions on $\mathbf{T}$ which have analytic extensions $\tilde{f}$ to the interior such that $\tilde{f}(0)=f(1)$. Then $\mathscr{A}$ is a uniform algebra on $\mathbf{T}$ and $\mathbf{T}$ is the Shilov boundary of $\mathscr{A}$. The complex homomorphism τ on $\mathscr{A}$ is defined by $\tau(f)=\tilde{f}(0)=f(1)$. Both $d\theta/2\pi$ and the unit point mass δ_1 at 1 represent the same linear functional τ on $\mathscr{A}$. Therefore $\mathscr{A}$ is not a logmodular algebra and hence not a Dirichlet algebra on $\mathbf{T}$ (cf. [1, p. 38]).

Lemma 3. *If μ is an annihilating measure on $\mathbf{T}$ for $\mathscr{A}+\overline{\mathscr{A}}$ then $d\mu=c(d\theta/2\pi-d\delta_1)$ for some constant c.*

Proof. We may assume that μ is a real measure on T. If μ annihilates $\mathscr{A}$ then

$$\int z\, d\mu=\int z^2\, d\mu=\int z^3\, d\mu=\dots$$

because the functions $z-z^2, z^2-z^3, z^3-z^4, \dots$ are all in $\mathscr{A}$. Hence for any positive integer n

$$\int z^n(d\mu-c_1\, d\delta_1)=0,$$

where $c_1=\int z\,d\mu$. By a theorem of F. and M. Riesz (cf. [1, p. 45]), $d\mu-c_1 d\delta_1=$ $=h\,d\theta/2\pi$ for some h in the usual Hardy space H^1. The absolutely continuous part of μ with respect to $d\theta/2\pi$ is a real measure and coincides with $h\,d\theta/2\pi$. Since H^1 has not nonconstant real functions, h is constant. Thus $d\mu=c\,d\theta/2\pi+c_1 d\delta_1$ and $c=-c_1$ because $\int 1\,d\mu=0$.

Theorem. *Let $f\to T_f$ be a representation of $\mathscr{A}$ on a Hilbert space $\mathfrak{H}$. There exists a spectral dilation $\Phi\to U_\Phi$ of $f\to T_f$.*

Proof. By Lemma 1 we may assume that the representation $f\to T_f$ of $\mathscr{A}$ is G-continuous or G-singular, where G is the Gleason part of τ in the maximal ideal space of $\mathscr{A}$. Suppose the representation is G-continuous. By Lemma 2 there are measures $\mu_{x,y}$ $(x, y\in\mathfrak{H})$ such that $\|\mu_{x,y}\|\leqq\|x\|\,\|y\|$ and $((T_f+T_g^*)x, y)=\int (f+\bar{g})\,d\mu_{x,y}$ for $f, g\in\mathscr{A}$ and $x, y\in\mathfrak{H}$. Since the representation of $\mathscr{A}$ is G-continuous, by the definition $\mu_{x,y}$ is absolutely continuous with respect to $d\theta/2\pi+d\delta_1$. Hence

$$d\mu_{x,y} = h_{x,y}\,d\theta/2\pi+c_{x,y}\,d\delta_1$$

where $h_{x,y}$ is in the usual Lebesgue space $L^1(d\theta/2\pi)$ and $c_{x,y}$ is constant.

Put

$$d\lambda_{x,y} = (h_{x,y}+c_{x,y})\,d\theta/2\pi.$$

We shall prove that the family $\lambda_{x,y}$ $(x, y\in\mathfrak{H})$ of measures on $\mathbf{T}$ is semispectral, that is, it satisfies (1)—(4). (4) is clear. $d\mu_{\alpha x+\beta y,z}-(\alpha\,d\mu_{x,z}+\beta\,d\mu_{y,z})$ annihilates $\mathscr{A}+\overline{\mathscr{A}}$. Therefore by Lemma 3 for some constant $a_{x,y,z}$

$$d\mu_{\alpha x+\beta y,z}-(\alpha\,d\mu_{x,z}+\beta\,d\mu_{y,z}) = a_{x,y,z}(d\theta/2\pi-d\delta_1),$$

consequently

$$h_{\alpha x+\beta y,z}-(\alpha h_{x,z}+\beta h_{y,z}) = a_{x,y,z}$$

and

$$c_{\alpha x+\beta y,z}-(\alpha c_{x,z}+\beta c_{y,z}) = -a_{x,y,z}.$$

This implies (1). $d\mu_{x,y}-d\bar{\mu}_{y,x}$ annihilates $\mathscr{A}+\overline{\mathscr{A}}$.

Therefore by Lemma 3 for some constant $b_{x,y}$

$$d\mu_{x,y}-d\bar{\mu}_{y,x} = b_{x,y}(d\theta/2\pi-d\delta_1)$$

consequently

$$h_{x,y}-\bar{h}_{y,x} = b_{x,y} \quad\text{and}\quad c_{x,y}-\bar{c}_{y,x} = -b_{x,y}.$$

This implies (2). By Proposition 7.8 in [3], if $f\in\mathscr{A}$ and $\operatorname{Re} f\geqq 0$ then $\operatorname{Re} T_f\geqq 0$. Hence if $u\in\mathscr{A}+\overline{\mathscr{A}}$ and $u\geqq 0$ then $\int u\,d\mu_{x,x}\geqq 0$. Thus for $u\in\mathscr{A}+\overline{\mathscr{A}}$ with $u\geqq 0$

$$\int u\,d\lambda_{x,x} = \int u(h_{x,x}+c_{x,x})\,d\theta/2\pi = \int uh_{x,x}\,d\theta/2\pi+c_{x,x}\int u\,d\theta/2\pi =$$
$$= \int uh_{x,x}\,d\theta/2\pi+c_{x,x}\int u\,d\delta_1 = \int u\,d\mu_{x,x} \geqq 0.$$

By the Riemann—Lebesgue lemma we know that $z^n \to 0$ in the weak* topology of $L^\infty(d\theta/2\pi)$. Hence the functions $z, z^2, z^3, \dots$ are all in the weak*-closure of $\mathscr{A}$ because $z^k = (z^k - z^{k-1}) + \dots + (z^n - z^{n-1}) - z^n$ for $n > k$. Therefore for $u \in C(\mathbf{T})$ with $u \geqq 0 \int u \, d\lambda_{x,x} \geqq 0$ and this implies (3).

Since the family $\lambda_{x,y}$ $(x, y \in \mathfrak{H})$ of measures on $\mathbf{T}$ is semispectral, there is a positive definite map $\Phi \to T'_\Phi$ of $C(\mathbf{T})$ in $L(\mathfrak{H})$ (cf. [3, Theorem 7.1]). By a dilation theorem of Naimark (cf. [3, Theorem 7.5]), we obtain a representation $\Phi \to U_\Phi$ of $C(\mathbf{T})$ on a Hilbert space $\mathfrak{K}$ which is a spectral dilation of $\Phi \to T'_\Phi$. If $f \in \mathscr{A}_0$ then $\int f \, d\theta/2\pi = \int f \, d\delta_1 = 0$ and hence

$$(T'_f x, y) = \int f \, d\lambda_{x,y} = \int f h_{x,y} \, d\theta/2\pi = \int f \, d\mu_{x,y} = (T_f x, y), \quad (x, y \in \mathfrak{H}).$$

Thus $T'_f = T_f$ if $f \in \mathscr{A}$ and the representation $\Phi \to U_\Phi$ is the spectral dilation of $f \to T_f$.

If the representation is G-singular, the family $\mu_{x,y}$ $(x, y \in \mathfrak{H})$ is singular with respect to $d\theta/2\pi + d\delta_1$. Then Lemma 3 implies that it is semispectral immediately, and the proof can be completed as above.

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