Acta Sci. Math., 53 (1989), 147—152

On the reflexivity of contractions with isometric parts

KATSUTOSHI TAKAHASHI

For a bounded linear operator T on a Hilbert space, let $\operatorname{Alg} T$ denote the weakly closed algebra generated by T and the identity. Also let $\operatorname{Lat} T$ and $\operatorname{Alg}\operatorname{Lat} T$ denote the lattice of all invariant subspaces of T and the algebra of all operators A such that $\operatorname{Lat} T \subseteqq \operatorname{Lat} A$, respectively. An operator T is said to be *reflexive* if $\operatorname{Alg}\operatorname{Lat} T = \operatorname{Alg} T$. (Note that we always have $\operatorname{Alg} T \subseteqq \operatorname{Alg}\operatorname{Lat} T$.) The first examples of reflexive operators were given by SARASON [7], that is, he proved that normal operators and analytic Toeplitz operators are reflexive. Subsequently DEDDENS [4] proved the reflexivity of isometries, and now various classes of operators are known to be reflexive.

In [9] and [10], WU considered the generalizations of Deddens' result. In [9] the reflexivity was proved for contractions T on $\mathfrak{H}$ such that $T|\mathfrak{M}$ and $T^*|\mathfrak{H}\ominus\mathfrak{M}$ are isometries for some $\mathfrak{M}\in\operatorname{Lat} T$, and in [10] for contractions which have parts similar to the adjoints of unilateral shifts, in particular, for contractions with a unilateral shift summand. The results of [10] were generalized in [2] as conjectured by Wu, that is, it was proved that if T is a contraction and there exists a nonzero operator X such that $XT=SX$ where S is a unilateral shift, then T is reflexive. In this note we prove the reflexivity of a contraction with a unilateral shift part. This result contains the main theorem of [9] as a special case. As an application, we obtain the reflexivity result for a contraction T on a separable Hilbert space such that $u\Theta_T^*$ is an operator-valued H^∞-function for some nonzero scalar H^∞-function u, where Θ_T is the characteristic function of T and $\Theta_T^*(e^{it})=(\Theta_T(e^{it}))^*$ for almost every t, in particular, for a contraction T such that Θ_T is a polynomial. Our proof needs the reflexivity result of [2] stated above. We will extensively use the theory of contractions developed by SZ.-NAGY and FOIAŞ [8].

Theorem 1. *If T is a contraction on a Hilbert space $\mathfrak{H}$ and there exists a nonzero $\mathfrak{M}\in\operatorname{Lat} T$ such that $T|\mathfrak{M}$ is a unilateral shift, then T is reflexive.*

Received December 3, 1985.

First let us prove the following lemma.

Lemma 2. *If T is a contraction on $\mathfrak{H}$ and there exists a nonzero $\mathfrak{M}\in \mathrm{Lat}\, T$ such that $T|\mathfrak{M}$ is a unilateral shift, then there exists a nonzero operator $Y\colon \mathfrak{H}\to L^2$ satisfying the following conditions* (i) *and* (ii); (i) $YT=WY$ *where W is the bilateral shift on L^2 defined by* $(Wf)(e^{it})=e^{it}f(e^{it})$ *a.e. t,* $f\in L^2$, (ii) *there exists a linear manifold $\mathfrak{L}$ dense in $\mathfrak{H}\ominus \ker Y$ such that $W|\mathfrak{N}_{Yx}$ is a unilateral shift for all $0\neq x\in\mathfrak{L}$, where $\mathfrak{N}_{Yx}=\bigvee\{W^nYx\colon n\geqq 0\}$ (a cyclic subspace for W).*

Proof. By assumption, if $\mathfrak{M}_1$ is a cyclic subspace for T included in $\mathfrak{M}$, then $T|\mathfrak{M}_1$ is unitarily equivalent to the unilateral shift $S=W|H^2$ (cf. [6, Theorem 3.33]), hence there exists an isometry $Z\colon H^2\mapsto\mathfrak{H}$ such that $TZ=ZS$. Let U be the minimal unitary dilation of T acting on $\mathfrak{G}$, thus U is a unitary operator such that $PU|\mathfrak{H}=T$ where P is the orthogonal projection of $\mathfrak{G}$ onto $\mathfrak{H}$, and if $\mathfrak{G}_+=\bigvee_{n\geqq 0}U^n\mathfrak{H}$, then $\mathfrak{G}_+\ominus\mathfrak{H}\in\mathrm{Lat}\,U$ (cf. [8, Theorem I.4.1 and 4.2]). By the lifting theorem of Sz.-Nagy and Foiaş (cf. [8, Theorem II.2.3] and [5, Corollary 5.1]) there exists an operator $\tilde{Z}\colon L^2\mapsto\mathfrak{G}$ satisfying the conditions (a) $U\tilde{Z}=\tilde{Z}W$, (b) $P\tilde{Z}|H^2=Z$ and (c) $\|\tilde{Z}\|=\|Z\|=1$. Let us show that the operator $Y=\tilde{Z}^*|\mathfrak{H}\colon\mathfrak{H}\mapsto L^2$ is a required one.

Since the condition (a) implies $\tilde{Z}^*U=W\tilde{Z}^*$, to prove $YT=WY$, it suffices to show that $\mathfrak{G}_+\ominus\mathfrak{H}\subseteqq\ker\tilde{Z}^*$. Since $\mathfrak{G}_+\ominus\mathfrak{H}\in\mathrm{Lat}\,U$, $\mathfrak{G}_+\ominus\mathfrak{H}$ is orthogonal to $\bigvee_{n\geqq 0}U^{*n}\mathfrak{H}$. On the other hand, since Z is isometric, it follows from (b) and (c) that $\tilde{Z}|H^2=Z$, and since $\tilde{Z}W^{*n}=U^{*n}\tilde{Z}$ $(n=1,2,\ldots)$ by (a), we see that $\tilde{Z}$ is an isometry and $\mathrm{ran}\,\tilde{Z}\subseteqq\bigvee_{n\geqq 0}U^{*n}\mathfrak{H}$. Therefore it follows that $\mathfrak{G}_+\ominus\mathfrak{H}\subseteqq\ker\tilde{Z}^*$. Next to see (ii), let $\mathfrak{M}_0=\{Zp;\ p$ is an analytic polynomial$\}$. Clearly $\mathfrak{M}_0$ is linear and dense in ZH^2. Also since $\tilde{Z}|H^2=Z$, we have $ZH^2\subseteqq\mathfrak{H}\ominus\ker Y$. We consider $\mathfrak{L}=\mathfrak{M}_0\oplus\big((\mathfrak{H}\ominus\ker Y)\ominus ZH^2\big)$, which is linear and dense in $\mathfrak{H}\ominus\ker Y$. If $0\neq x=Zp+x_1\in\mathfrak{L}$ where p is a polynomial of degree n and $x_1\in(\mathfrak{H}\ominus\ker Y)\ominus ZH^2$, then $Yx=p+Yx_1$ because $\tilde{Z}|H^2=Z$ and $\tilde{Z}$ is an isometry. Since x_1 is orthogonal to ZH^2, or equivalently Yx_1 is orthogonal to H^2, it follows that $\chi^{-(n+1)}Yx$, where $\chi(e^{it})=e^{it}$, is orthogonal to H^2, so that $Yx=qg$ $(Yx\neq 0)$, where q is a function in L^∞ such that $|q(e^{it})|=1$ a.e. t and g is an outer function in H^2 (cf. [3, Chapter IV, Theorem 6.1 and Corollary 6.4]). This shows $\mathfrak{N}_{Yx}=qH^2$, hence the isometry $W|\mathfrak{N}_{Yx}$ is a unilateral shift. Thus the condition (ii) holds.

Any contraction T can be decomposed uniquely as $T=U\oplus T_1$ where U is a unitary operator and T_1 is a *completely non-unitary (c.n.u.)* contraction, that is, T_1 has no nontrivial unitary direct summand. The operators U and T_1 are called the *unitary part* and the *c.n.u. part* of T, respectively. For a contraction T whose unitary part is absolutely continuous, the H^∞-functional calculus defines a weak*-weak continuous algebra homomorphism, $u\mapsto u(T)$, from H^∞ to $\mathrm{Alg}\,T$, and T is said to be of class C_0 if $u(T)=0$ for some nonzero $u\in H^\infty$ (cf. [8, Chapter III]).

Proof of Theorem 1. Let $T=U_s\oplus T_1$ on $\mathfrak{H}=\mathfrak{H}_s\oplus\mathfrak{H}_1$ where U_s is a singular unitary operator and T_1 is a contraction whose unitary part is absolutely continuous. It is known that the reflexivity of T is equivalent to that of T_1 (cf. the proof of [9, Theorem 4.1]). Since T has a unilateral shift part, as in the proof of Lemma 2, we have an isometry Z such that $TZ=ZS$ where S is the unilateral shift on H^2. If P_s is the orthogonal projection onto $\mathfrak{H}_s$, then $U_s(P_sZ)=(P_sZ)S$ and it follows from [5, Corollary 5.1 and Theorem 3] that $P_sZ=0$, hence $\operatorname{ran} Z\subseteq\mathfrak{H}_1$. This shows that T_1 has a unilateral shift part. Thus we may assume that the unitary part of T is absolutely continuous and it suffices to show that for each $A\in\operatorname{Alg}\operatorname{Lat} T$, there exists $f\in H^\infty$ such that $A=f(T)$.

Let Y, W and $\mathfrak{L}$ be as in Lemma 2, and let $\tilde{\mathfrak{L}}$ be the set $\{x_1+x_2\colon x_1\in\ker Y$ and $0\neq x_2\in\mathfrak{L}\}$ that is dense in $\mathfrak{H}$. If $x\in\tilde{\mathfrak{L}}$, that is, $x=x_1+x_2$ where $x_1\in\ker Y$ and $0\neq x_2\in\mathfrak{L}$, then since $Yx=Yx_2(\neq 0)$, by Lemma 2 the isometry $W|\mathfrak{N}_{Yx}$ is a unilateral shift and $(W|\mathfrak{N}_{Yx})(Y|\mathfrak{M}_x)=(Y|\mathfrak{M}_x)(T|\mathfrak{M}_x)$ with $Y|\mathfrak{M}_x\neq 0$, where $\mathfrak{M}_x=\bigvee\{T^nx\colon n\geqq 0\}$, so it follows from [2, Theorem 4] that

$$\operatorname{Alg}\operatorname{Lat}(T|\mathfrak{M}_x)=\{f(T)|\mathfrak{M}_x\colon f\in H^\infty\}.$$

Here note that the unitary parts of T and $T|\mathfrak{M}_x$ are absolutely continuous. Take $A\in\operatorname{Alg}\operatorname{Lat} T$. For each $x\in\tilde{\mathfrak{L}}$, since $\mathfrak{M}_x\in\operatorname{Lat} T\subseteq\operatorname{Lat} A$ and $A|\mathfrak{M}_x\in\operatorname{Alg}\operatorname{Lat}(T|\mathfrak{M}_x)$, by the above fact there is $f_x\in H^\infty$ such that $A|\mathfrak{M}_x=f_x(T)|\mathfrak{M}_x$, in particular, $Ax=f_x(T)x$. Here note that it follows from the identity $WY=YT$ with $Yx\neq 0$ that $T|\mathfrak{M}_x$ is not of class C_0 (cf. [8, Proposition III.4.1]), so that the function f_x is determined uniquely by x. Since $\tilde{\mathfrak{L}}$ is dense in $\mathfrak{H}$, in order to show $A=f(T)$ for some $f\in H^\infty$, it suffices to prove that $f_x=f_y$ for all $x,y\in\tilde{\mathfrak{L}}$. First suppose $x-y\in\ker Y$. Then since $Yx=Yy$ and $\ker Y\in\operatorname{Lat} T\subseteq\operatorname{Lat} A$, we have

$$(f_x-f_y)(W)Yx=Yf_x(T)x-Yf_y(T)y=YAx-YAy=YA(x-y)=0,$$

and since $Yx\neq 0$, it follows that $f_x=f_y$. Next assume that $x-y\notin\ker Y$. Then since clearly $x-y\in\tilde{\mathfrak{L}}$, there is $f_{x-y}\in H^\infty$ such that

$$f_{x-y}(T)x-f_{x-y}(T)y=f_{x-y}(T)(x-y)=A(x-y)=Ax-Ay=f_x(T)x-f_y(T)y,$$

hence $(f_{x-y}-f_x)(T)x=(f_{x-y}-f_y)(T)y\in\mathfrak{M}_x\cap\mathfrak{M}_y$. Therefore we have

$$f_x(T)(f_{x-y}-f_x)(T)x=A(f_{x-y}-f_x)(T)x=f_y(T)(f_{x-y}-f_x)(T)x,$$

and since $T|\mathfrak{M}_x$ is not of class C_0, $(f_x-f_y)(f_{x-y}-f_x)=0$. Similarly we have $(f_x-f_y)(f_{x-y}-f_y)=0$. This shows $f_x=f_y$ and completes the proof.

Let T be a contraction on a separable Hilbert space. The *characteristic function* Θ_T of T is defined by

$$\Theta_T(\lambda)=[-T+\lambda D_{T^*}(I-\lambda T^*)^{-1}D_T]|\mathfrak{D}_T\quad(|\lambda|<1),$$

where $D_T=(I-T^*T)^{1/2}$, $D_{T^*}=(I-TT^*)^{1/2}$ and $\mathfrak{D}_T=(\operatorname{ran} D_T)^-$. The function Θ_T is an operator-valued H^∞-function whose values are contractions from $\mathfrak{D}_T$ to $\mathfrak{D}_{T^*}:=(\operatorname{ran} D_{T^*})^-$ (cf. [8, Chapter VI]). If T is c.n.u., then it follows from [8, Theorem VII.4.7] that there exists a nonzero $\mathfrak{M}\in\operatorname{Lat} T$ such that $T|\mathfrak{M}$ is a unilateral shift if and only if there exists a nonzero $h\in H^2(\mathfrak{D}_{T^*})$ such that $\Theta_T^* h\in\Delta_T L^2(\mathfrak{D}_T)$, where $H^2(\mathfrak{D}_{T^*})$ (resp. $L^2(\mathfrak{D}_T)$) is the space of $\mathfrak{D}_{T^*}$-valued H^2-functions (resp. $\mathfrak{D}_T$-valued L^2-functions), $\Theta_T^*(e^{it})=(\Theta_T(e^{it}))^*$ a.e. t and $\Delta_T(e^{it})=(I-\Theta_T(e^{it})^*\Theta_T(e^{it}))^{1/2}$ a.e. t.

Now we obtain the reflexivity result for a contraction T such that $u\Theta_T^*$ is an operator-valued H^∞-function for some nonzero scalar function $u\in H^\infty$. If such a contraction T is of class C_{00}, that is, $T^n\to 0$ and $T^{*n}\to 0$ strongly as $n\to\infty$, then since $\Theta_T(e^{it})$ is unitary a.e. t (cf. [8, Proposition VI.3.5]), the condition that $u\Theta_T^*$ is an operator-valued H^∞-function with a nonzero $u\in H^\infty$ means that $u(T)=0$ and so T is of class C_0 (cf. [8, Theorem VI.5.1]). Reflexive contractions of class C_0 were characterized in terms of their Jordan models [1].

Theorem 3. *Let T be a contraction on a separable Hilbert space such that $u\Theta_T^*$ is an operator-valued H^∞-function for some nonzero $u\in H^\infty$. If the c.n.u. part of T is not of class C_{00}, then T is reflexive.*

Proof. By Theorem 1 it suffices to show that T or T^* has a unilateral shift part. Since the characteristic function of a contraction is equal to the one of its c.n.u. part, we may assume that T is a c.n.u. contraction. Since $\Theta_T^*(I-\Theta_T\Theta_T^*)=\Delta_T^2\Theta_T^*$ and by the assumption for Θ_T the function $u(I-\Theta_T\Theta_T^*)$ is an operator-valued H^∞-function, if $\lim\|T^n x\|\neq 0$ for some x, or equivalently $\Theta_T(e^{it})$ is not coisometric on a set of t's of positive Lebesgue measure (cf. [8, Proposition VI.3.5]), then there is a nonzero $h\in H^2(\mathfrak{D}_{T^*})$ such that $\Theta_T^* h\in\Delta_T L^2(\mathfrak{D}_T)$, and so T has a unilateral shift part by the fact remarked above. Also since $\Theta_{T^*}(e^{it})=(\Theta_T(e^{-it}))^*$ a.e. t for the characteristic function Θ_{T^*} of T^* (cf. [8, p. 239]), the contraction T^* satisfies the same condition as T, that is, $\tilde{u}\Theta_{T^*}^*$ is an operator-valued H^∞-function where $\tilde{u}$ is a function in H^∞ defined by $\tilde{u}(e^{it})=\overline{u(e^{-it})}$ a.e. t. Thus if $\lim\|T^{*n}x\|\neq 0$ for some x, then it follows that T^* has a unilateral shift part.

The following theorem gives a complement of Theorem 3.

Theorem 4. *Let $T=U\oplus T_1$ where U is a unitary operator and T_1 is a contraction of class C_0. Then T is reflexive if and only if the following condition* (i) *or* (ii) *holds:*

(i) *U has a (nontrivial) bilateral shift summand;*

(ii) *T_1 is reflexive.*

Proof. Again we may assume that U is absolutely continuous (cf. the proof of [9, Theorem 4.1]). If U has a bilateral shift summand, then by Theorem 1 T is reflexive. If U has no bilateral shift summand, then by Lemma 5 below we have $\mathrm{Alg}\, T = \mathrm{Alg}\, U \oplus \mathrm{Alg}\, T_1$ and $\mathrm{Lat}\, T = \mathrm{Lat}\, U \oplus \mathrm{Lat}\, T_1$, so $\mathrm{Alg\,Lat}\, T = \mathrm{Alg\,Lat}\, U \oplus \mathrm{Alg\,Lat}\, T_1$. Therefore it follows from the reflexivity of the unitary operator U (cf. [7]) that T is reflexive if and only if T_1 is. This shows Theorem 4.

The implication (2)⇒(1) in the following lemma was pointed out by P. Y. Wu.

Lemma 5. *Let* $T = U \oplus T_1$ *on* $\mathfrak{H} = \mathfrak{H}_0 \oplus \mathfrak{H}_1$ *where* U *is an absolutely continuous unitary operator and* T_1 *is a contraction of class* C_0. *Then the following conditions are equivalent:*

(1) *U has no bilateral shift summand;*

(2) $\mathrm{Lat}\, T = \mathrm{Lat}\, U \oplus \mathrm{Lat}\, T_1$;

(3) $\mathrm{Alg}\, T = \mathrm{Alg}\, U \oplus \mathrm{Alg}\, T_1$.

Proof. (1)⇒(2): Since the inclusion $\mathrm{Lat}\, U \oplus \mathrm{Lat}\, T_1 \subseteq \mathrm{Lat}\, T$ is obvious, we have to show that any $\mathfrak{M} \in \mathrm{Lat}\, T$ is decomposed into $\mathfrak{M} = \mathfrak{L} \oplus \mathfrak{N}$ where $\mathfrak{L} \in \mathrm{Lat}\, U$ and $\mathfrak{N} \in \mathrm{Lat}\, T_1$. Suppose $\mathfrak{M} \in \mathrm{Lat}\, T$. Since T_1 is of class C_0, there is a nonzero function $f \in H^\infty$ such that $f(T_1) = 0$. We set $\mathfrak{L} = (f(T)\mathfrak{M})^- \subseteq \mathfrak{M}$. Then clearly $\mathfrak{L} \in \mathrm{Lat}\, T$ and $\mathfrak{L} \subseteq (\mathrm{ran}\, f(T))^- = (\mathrm{ran}\, f(U))^- \subseteq \mathfrak{H}_0$, so $\mathfrak{L}$ is an invariant subspace of U. But since U has no bilateral shift summand, $\mathfrak{L}$ reduces U (cf. [3, Chapter VII, Proposition 5.2]), hence $\mathfrak{L}$ also reduces T. Then the subspace $\mathfrak{N} = \mathfrak{M} \ominus \mathfrak{L}$ is invariant for T and since $f(T)\mathfrak{N} \subseteq \mathfrak{N}$ and $f(T)\mathfrak{N} \subseteq f(T)\mathfrak{M} \subseteq \mathfrak{L}$, we have $f(T)\mathfrak{N} = \{0\}$. But since $f(T) = f(U) \oplus 0$ and obviously $f(U)$ is injective, we conclude $\mathfrak{N} \subseteq \mathfrak{H}_1$, and therefore $\mathfrak{N} \in \mathrm{Lat}\, T_1$. This shows (2).

(1)⇒(3): For $n = 1, 2, \ldots$, $T^{(n)} = U^{(n)} \oplus T_1^{(n)}$ satisfies the same condition as T, where for an operator A, $A^{(n)}$ denotes the direct sum of n copies of A. Therefore, using the implication (1)⇒(2) proved already, we have $\mathrm{Lat}\, T^{(n)} = \mathrm{Lat}\, U^{(n)} \oplus \mathrm{Lat}\, T_1^{(n)}$. If $A \in \mathrm{Alg}\, U$ and $B \in \mathrm{Alg}\, T_1$, then clearly $\mathrm{Lat}\, U^{(n)} \oplus \mathrm{Lat}\, T_1^{(n)} \subseteq \mathrm{Lat}\, (A \oplus B)^{(n)}$, so that $\mathrm{Lat}\, T^{(n)} \subseteq \mathrm{Lat}\, (A \oplus B)^{(n)}$ for $n = 1, 2, \ldots$, hence it follows from Sarason's lemma (cf. [6, Theorem 7.1]) that $A \oplus B \in \mathrm{Alg}\, T$. This shows $\mathrm{Alg}\, U \oplus \mathrm{Alg}\, T_1 \subseteq \mathrm{Alg}\, T$. Since the converse inclusion is obvious, we conclude $\mathrm{Alg}\, T = \mathrm{Alg}\, U \oplus \mathrm{Alg}\, T_1$.

(3)⇒(2) is obvious. (2)⇒(1): If U has a bilateral shift summand, then by the proof of Theorem 1 $\mathrm{Alg\,Lat}\, T = \{f(T) : f \in H^\infty\}$. Since the condition (2) implies the inclusion $\mathrm{Alg\,Lat}\, U \oplus \mathrm{Alg\,Lat}\, T_1 \subseteq \mathrm{Alg\,Lat}\, T$, we have $0 \oplus I \in \mathrm{Alg\,Lat}\, T$, so that there is $f \in H^\infty$ such that $f(U) = 0$ and $f(T_1) = I$, but this is impossible because $f(U) = 0$ implies $f = 0$. This shows (2)⇒(1).

References

[1] H. Bercovici, C. Foiaş and B. Sz.-Nagy, Reflexive and hyperreflexive operators of class C_0, *Acta Sci. Math.*, **43** (1981), 5—13.

[2] H. Bercovici and K. Takahashi, On the reflexivity of contractions on Hilbert space, *J. London Math. Soc.*, (2) **32** (1985), 149—156.

[3] J. B. Conway, *Subnormal operators*, Research Notes in Math. **51**, Pitman (Boston, 1981).

[4] J. A. Deddens, Every isometry is reflexive, *Proc. Amer. Math. Soc.*, **28** (1971), 509—512.

[5] R. G. Douglas, On the operator equation $S^*XT=X$ and related topics, *Acta Sci. Math.*, **30** (1969), 19—32.

[6] H. Radjavi and P. Rosenthal, *Invariant subspaces*, Springer-Verlag (Berlin, 1973).

[7] D. Sarason, Invariant subspaces and unstarred operator algebras, *Pacific J. Math.*, **17** (1966), 511—517.

[8] B. Sz.-Nagy and C. Foiaş, *Harmonic analysis of operators on Hilbert spaces*, North-Holland (Amsterdam, 1970).

[9] P. Y. Wu, Contractions with constant characteristic functions are reflexive, *J. London Math. Soc.*, (2) **29** (1984), 533—544.

[10] P. Y. Wu, Contractions with a unilateral shift summand are reflexive, *Integral Equations Operator Theory*, **7** (1984), 899—904.

DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE (GENERAL EDUCATION)
HOKKAIDO UNIVERSITY
SAPPORO 060
JAPAN