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On normal extensions of unbounded operators. II*)

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This paper continues our study of unbounded subnormal operators. The results contained here may be regarded as reviewing, extending and completing those of [21] (and also of [20] and [22]). The next paper [26] in this series will be devoted to spectral problems as well as to the question of uniqueness of normal extensions.

Subnormal operators in general aspect

1. Let S be a densely defined linear operator in a complex Hilbert space $\mathfrak{H}$. $\mathfrak{D}(S)$, $\mathfrak{N}(S)$ and $\mathfrak{R}(S)$ stands for the domain of S, the null space of S and the range of S, respectively. S is said to be *subnormal* if there is a Hilbert space $\mathfrak{K}$ containing $\mathfrak{H}$ and a normal operator N in $\mathfrak{K}$ such that

$$\mathfrak{D}(S)\subset\mathfrak{D}(N) \quad \text{and} \quad Sf=Nf \quad \text{for each} \quad f\in\mathfrak{D}(S).$$

(A densely defined linear operator N in $\mathfrak{K}$ is said to be *normal* if it is closed and $N^*N=NN^*$. This is the same as to require that $\mathfrak{D}(N)=\mathfrak{D}(N^*)$ and $\|Nf\|=\|N^*f\|$, $f\in\mathfrak{D}(N)$. A normal operator has a spectral representation on the complex plane $\mathbf{C}$.)

The first thing we have to point out is that a subnormal operator must necessarily be closable. Even more we show that $\mathfrak{D}(S)\subset\mathfrak{D}(S^*)$. To see this take $g\in\mathfrak{D}(S)$, then

$$\langle Sf, g\rangle_{\mathfrak{H}} = \langle f, N^*g\rangle_{\mathfrak{K}}, \quad f\in\mathfrak{D}(S)$$

which gives us $g\in\mathfrak{D}(S^*)$ and $S^*g=P_{\mathfrak{H}}N^*g$.

The following characterization of densely defined subnormal operators based on the spectral representation of normal extensions is due to Foiaş (cf. [8], p. 248).

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Theorem 1. *A densely defined operator S in $\mathfrak{H}$ is subnormal if and only if there is a (normalized) semispectral measure F in $\mathfrak{H}$ on the complex plane $\mathbf{C}$ such that*

$$\langle S^n f, S^m g\rangle = \int_{\mathbf{C}} \lambda^n \bar{\lambda}^m \langle F(d\lambda)f, g\rangle, \quad f, g \in \mathfrak{D}(S), \quad m, n = 0, 1.$$

This theorem seems to be the only known characterization of unbounded densely defined subnormal operators in the general case (without any additional assumption on S). It ought to be noticed that a characterization like this of Foiaş for bounded operators has appeared in [2] (cf. also [6]). However that involves all the powers of the operator S. This requirement is superfluous for bounded operators, while for unbounded ones it leads to unnecessary restriction on behavior of domains of all powers of S.

2. Now we want to discuss the relation between subnormality and quasinormality. Like in the bounded case we have two equivalent possibilities of defining quasinormal operators. Because commutativity of unbounded operators is rather a delicate matter, we wish to discuss this equivalence with more care.

A closed densely defined operator Q in a Hilbert space $\mathfrak{H}$ is said to be *quasinormal* if Q commutes with the spectral measure E of $|Q| := (Q^* Q)^{1/2}$ i.e. $E(\sigma)Q \subset \subset QE(\sigma)$, σ being a Borel subset of the non-negative part $\mathbf{R}_+$ of the real line $\mathbf{R}$.

Proposition 1. *Q is quasinormal if and only if Q is closed and U commutes with the spectral measure E of $|Q|$, where $Q = U|Q|$ is the polar decomposition of Q.*

Proof. Suppose that U commutes with E. Since E commutes with $|Q|$ (i.e. $E(\sigma)|Q| \subset |Q|E(\sigma)$) we have

$$E(\sigma)Q = E(\sigma)U|Q| = UE(\sigma)|Q| \subset U|Q|E(\sigma) = QE(\sigma).$$

Thus Q is quasinormal.

Suppose now that Q is quasinormal. Since Q commutes with E, U commutes with E on $\overline{\mathfrak{R}(|Q|)}$. Indeed, for each $f \in \mathfrak{D}(|Q|)$ we have

$$\big(UE(\sigma) - E(\sigma)U\big)|Q|f = UE(\sigma)|Q|f - E(\sigma)U|Q|f = U|Q|E(\sigma)f - E(\sigma)Qf = \\ = QE(\sigma)f - E(\sigma)Qf = 0.$$

Since $E(\{0\})$ is the orthogonal projection onto $\mathfrak{N}(|Q|)$ and $\overline{\mathfrak{R}(|Q|)}^\perp = \mathfrak{N}(|Q|) = \mathfrak{N}(U)$, $\big(UE(\sigma) - E(\sigma)U\big)f = UE(\sigma)f = UE(\sigma)E(\{0\})f = UE(\{0\})E(\sigma)f = 0$ for each $f \in \overline{\mathfrak{R}(|Q|)}^\perp$. Thus U commutes with E. This completes the proof.

The following result as well as its proof is patterned upon that for bounded operators ([6], Prop. 1.7, p. 115) however technically more involved.

Theorem 2. *Every quasinormal operator is subnormal.*

Proof. Let $S=U|S|$ be the polar decomposition of S and let $|S|=\int_0^\infty tE(dt)$ be the spectral representation of $|S|$. Denote by $P_{\mathfrak{N}(|S|)}$ and $P_{\mathfrak{N}(S^*)}$ the orthogonal projections onto $\mathfrak{N}(|S|)$ and $\mathfrak{N}(S^*)$, respectively. Define in $\mathfrak{H}\oplus\mathfrak{H}$ two operators R and $\tilde{U}$ as $R=|S|\oplus|S|$ and

$$\tilde{U}=\begin{bmatrix} U & (I-UU^*)^{1/2} \\ -(I-U^*U)^{1/2} & U^* \end{bmatrix}.$$

It is easy to see [11] that $\tilde{U}$ is a unitary operator which dilates U (the Halmos dilation) and R is a self-adjoint extension of $|S|$. Since U is a partial isometry, $\tilde{U}$ is in fact of the form

$$\tilde{U}=\begin{bmatrix} U & P_{\mathfrak{N}(S^*)} \\ -P_{\mathfrak{N}(|S|)} & U^* \end{bmatrix}.$$

Due to Proposition 1, U and U^* commute with E. Since $I-UU^*=P_{\mathfrak{N}(S^*)}$ and $I-U^*U=P_{\mathfrak{N}(|S|)}$, $P_{\mathfrak{N}(S^*)}$ and $P_{\mathfrak{N}(|S|)}$ commute with E. Consequently $\tilde{U}$ commutes with $E\oplus E$ which is the spectral measure of R. Therefore $\tilde{U}R\subset R\tilde{U}$. This implies that $R\tilde{U}=\tilde{U}\tilde{U}^*R\tilde{U}\subset\tilde{U}(R\tilde{U})^*\tilde{U}\subset\tilde{U}(\tilde{U}R)^*\tilde{U}=\tilde{U}R\tilde{U}^*\tilde{U}=\tilde{U}R$ and $\tilde{U}R=R\tilde{U}$ in consequence. Denote by N the operator $\tilde{U}R$. Since $N^*N=R\tilde{U}^*\tilde{U}R=R^2$ and $NN^*=(R\tilde{U})(R\tilde{U})^*=R\tilde{U}\tilde{U}^*R=R^2$, $N^*N=NN^*$. This means that N is normal.

Let now $f\in\mathfrak{D}(S)=\mathfrak{D}(|S|)$. Then $f\oplus 0\in\mathfrak{D}(R)$. Since $P_{\mathfrak{N}(|S|)}$ commutes with E, $P_{\mathfrak{N}(|S|)}|S|\subset|S|P_{\mathfrak{N}(|S|)}=0$. Thus

$$N(f\oplus 0)=\tilde{U}R(f\oplus 0)=\tilde{U}(|S|f\oplus 0)=U|S|f\oplus(-P_{\mathfrak{N}(|S|)}|S|f)=(U|S|f)\oplus 0=Sf\oplus 0$$

which means that N extends S. This completes the proof.

Corollary 1. *An operator is subnormal if and only if it has a quasinormal extension.*

Proof. We have only to prove that each normal operator N is quasinormal. Indeed, if $N=\int_{\mathbf{C}} zE(dz)$ then $|N|=\int_{\mathbf{C}}|z|E(dz)=\int_0^\infty tF(dt)$, where $F(\sigma)=E(\{z\in\mathbf{C}:\ |z|\in\sigma\})$, σ being a Borel subset of $\mathbf{R}_+$. Since $EN\subset NE$, $FN\subset NF$. This means that N is quasinormal.

Subnormal operators and the complex moment problem

3. The following condition, introduced by HALMOS [11], characterizes [2] bounded subnormal operators in a Hilbert space $\mathfrak{H}$. This is

$$\text{(H)} \qquad \sum_{j,k=0}^{n} \langle S^k f_j, S^j f_k \rangle \geqq 0$$

for all finite sequences $f_0, \ldots, f_n \in \mathfrak{H}$. To consider the same condition in unbounded case one needs the linear subspace $\mathfrak{D}^\infty(S)$ of $\mathfrak{H}$

$$\mathfrak{D}^\infty(S) = \bigcap_{n=0}^{\infty} \mathfrak{D}(S^n)$$

(members of $\mathfrak{D}^\infty(S)$ are customarily refered to as *C^∞-vectors*). In this paper we will require that $\mathfrak{D}^\infty(S)$ is big enough (in most cases dense in $\mathfrak{H}$). This requirement makes serious (comparing with Section 1) restriction on subnormal operators because there are symmetric operators (even semi-bounded [4]) with trivial domains of their squares. Moreover the condition (H) considered for $f_0, \ldots, f_n \in \mathfrak{D}^\infty(S)$, which is the only possibility to do, is not sufficient for subnormality for S even if $\mathfrak{D}^\infty(S)$ is dense in $\mathfrak{H}$. Let us discuss the following.

Example 1. Take a sequence of real numbers $\{a_{m,n}\}_{m,n=0}^{\infty}$ which is positive definite in the following sense:

$$\sum a_{m+p,n+q} \lambda_{m,n} \bar{\lambda}_{p,q} \geqq 0$$

for each finite sequence $\{\lambda_{m,n}\} \subset \mathbf{C}$, and which is not a two parameter moment sequence (see [1] and [9]). There are two densely defined symmetric operators A and B in some Hilbert space $\mathfrak{H}$ with a common domain $\mathfrak{D} = \mathfrak{D}(A) = \mathfrak{D}(B)$, having a vector $f_0 \in \mathfrak{D}$ such that all the powers $A^m B^n f_0$, $m, n \geqq 0$, span $\mathfrak{D}$, and such that

$$(1) \qquad a_{m,n} = \langle A^m B^n f_0, f_0 \rangle, \quad m, n \geqq 0$$

(cf. again [9]). Moreover A and B commute i.e. $ABf = BAf$ for each $f \in \mathfrak{D}$. Define $T = A + iB$. T satisfies (H) for all finite sequences $f_0, \ldots, f_n \in \mathfrak{D} = \mathfrak{D}^\infty(S)$ (even more, $\|Tf\| = \|T^* f\|$, $f \in \mathfrak{D}$, because A and B commute).

Define S as a restriction of T to the linear span of $\{T^n f_0 \colon n \geqq 0\}$.

Neither T nor S is subnormal. If T would be subnormal (then S would be too), then there existed a measure μ on $\mathbf{C}$ (constructed via the spectral measure of a normal extension of T) such that

$$\langle T^n f_0, T^m f_0 \rangle = \int_{\mathbf{C}} z^n \bar{z}^m \, d\mu(z), \quad m, n \geqq 0.$$

Then, due to (1),

$$a_{m,n} = \int_{\mathbf{C}} (\operatorname{Re} z)^m (\operatorname{Im} z)^n \, d\mu(z), \quad m, n \geqq 0.$$

This would mean that $\{a_{m,n}\}_{m,n=0}^{\infty}$ was a two parameter moment sequence, which gives a contradiction.

Thus we have got an example of an operator which has a cyclic vector (an operator S in $\mathfrak{H}$ is said to be *cyclic* with a *cyclic vector* f_0 if $f_0 \in \mathfrak{D}^{\infty}(S)$ and $\mathfrak{D}(S)$ is a linear span of $\{S^n f_0: n \geqq 0\}$) satisfies (H) on $\mathfrak{D}(S)$ but is not subnormal.

If one would be interested in an example of a non-cyclic operator, one could take a Nelson pair (cf. [17], [5]) to get an operator satisfying (H) on $\mathfrak{D}(S)$ with no normal extension.

As the following proposition shows the condition (H) is satisfied on $\mathfrak{D}(S)$ if and only if S has a formally normal extension (with dense "reducing" domain). Here by a *formally normal* operator in $\mathfrak{H}$ we mean a densely defined operator N in $\mathfrak{H}$ such that $\mathfrak{D}(N) \subset \mathfrak{D}(N^*)$ and $\|Nf\| = \|N^* f\|$ for each $f \in \mathfrak{D}(N)$.

Proposition 2. *Let S be a densely defined operator in $\mathfrak{H}$ such that $S\mathfrak{D}(S) \subset \subset \mathfrak{D}(S)$. Then S satisfies* (H) *for all finite sequences $f_0, \ldots, f_n \in \mathfrak{D}(S)$ if and only if there is a formally normal operator N in some Hilbert space $\mathfrak{K} \supset \mathfrak{H}$ such that*

(i) *$N\mathfrak{D}(N) \subset \mathfrak{D}(N)$ and $N^*\mathfrak{D}(N) \subset \mathfrak{D}(N)$,*

(ii) *$\mathfrak{D}(S) \subset \mathfrak{D}(N)$ and $S \subset N$,*

(iii) *$\mathfrak{D}(N)$ is a linear span of the set*

$$\{N^{*n} f: \; n \geqq 0, \; f \in \mathfrak{D}(S)\}.$$

Proof. The proof of the "if" part of Proposition 2 follows from the equality $N^*Nf = NN^*f$, $f \in \mathfrak{D}(N)$, via direct computation.

To prove the converse, suppose that S satisfies (H) for all finite sequences $f_0, \ldots, f_n \in \mathfrak{D}(S)$. The set $\mathfrak{S} = \mathbf{N} \times \mathbf{N}$ $(\mathbf{N} = \{0, 1, \ldots\})$ equiped with the coordinatewise defined addition and the involution $(m, n)^* = (n, m)$ becomes a $*$-semigroup. Define the form φ over $(\mathfrak{S}, \mathfrak{D}(S))$ (cf. [23])

$$\varphi((m, n); f, g) = \langle S^m f, S^n g \rangle, \quad f, g \in \mathfrak{D}(S), \quad m, n \in \mathbf{N}.$$

Then like in [24, par. 10], one can show that φ is positive definite i.e.

$$\text{(2)} \qquad \sum_{j,k=1}^{n} \varphi(s_k^* + s_j; f_j, f_k) \geqq 0, \quad f_1, \ldots, f_n \in \mathfrak{D}(S) \quad \text{and} \quad s_1, \ldots, s_n \in \mathfrak{S} \quad (n \geqq 1)$$

($S\mathfrak{D}(S) \subset \mathfrak{D}(S)$ is important here). It follows from Proposition in [23] that there is a family $\{\Phi(s): s \in \mathfrak{S}\}$ of densely defined operators in some Hilbert space $\mathfrak{K}$ with common dense domain $\mathfrak{D}$, a linear operator $V: \mathfrak{D}(S) \to \mathfrak{D}$ such that

$\mathfrak{D}\subset\bigcup_{s\in\mathfrak{S}}\mathfrak{D}(\Phi(s)^*)$ and

$$\varphi(s;\ f,g)=\langle\Phi(s)Vf, Vg\rangle,\quad s\in\mathfrak{S},\quad f,g\in\mathfrak{D}(S),$$

$$\Phi(s)\mathfrak{D}\subset\mathfrak{D}\quad\text{and}\quad\Phi(s)^*\mathfrak{D}\subset\mathfrak{D},\quad s\in\mathfrak{S},$$

$$\Phi(s)\Phi(t)f=\Phi(s+t)f,\quad s,t\in\mathfrak{S},\quad f\in\mathfrak{D},$$

$$\Phi(s^*)\subset\Phi(s)^*,\quad s\in\mathfrak{S},$$

$$\mathfrak{D}\ \text{is a linear span of}\ \{\Phi(s)Vf\colon s\in\mathfrak{S}, f\in\mathfrak{D}(S)\}.$$

Set $N=\Phi(1,0)$. Since $(1,0)^*+(1,0)=(1,0)+(1,0)^*$, N is a formally normal operator which satisfies the condition (i) of Proposition 2. Moreover we have

$$\langle S^m f, S^n g\rangle=\langle\Phi(n(1,0)^*+m(1,0))Vf, Vg\rangle=\langle N^m Vf, N^n Vg\rangle,$$
$$m,n\in\mathbf{N}\quad\text{and}\quad f,g\in\mathfrak{D}(S).$$

This implies that V is an isometry from $\mathfrak{D}(S)$ into $\mathfrak{D}$. Identifying $\mathfrak{D}(S)$ with $V\mathfrak{D}(S)$ one can easily check the conditions (ii) and (iii). This completes the proof.

Remark 1. In [21] and in this paper we consider exclusively the operators with invariant domains. If $\mathfrak{D}(S)$ is not invariant for S, we have to replace the condition (H) on $\mathfrak{D}(S)$ by the condition (2).

4. Example 1 shows that the condition (H) itself is *not* sufficient for subnormality even of cyclic operators (however it *is* for weighted shifts — cf. Section 6).

If f_0 is a cyclic vector for S and S satisfies (H) on $\mathfrak{D}^\infty(S)$ then the sequence $\{c_{m,n}\}_{m,n=0}^\infty$ defined by

$$c_{m,n}=\langle S^m f_0, S^n f_0\rangle\quad m,n\in\mathbf{N}$$

is positive definite in the sense that

$$\sum_{\substack{m,n\geqq 0\\ p,q\geqq 0}} c_{m+q,n+p}\lambda_{m,n}\overline{\lambda_{p,q}}\geqq 0$$

for all finite sequences $\{\lambda_{m,n}\}\subset\mathbf{C}$. Unfortunately positive-definiteness of $\{c_{m,n}\}_{m,n=0}^\infty$ does not imply that $\{c_{m,n}\}_{m,n=0}^\infty$ is a complex moment sequence (this is a substance of Example 1). However this gives a hope that subnormality of S (still being cyclic) may be forced by the fact that $\{c_{m,n}\}_{m,n=0}^\infty$ is a complex moment sequence. There is a characterization ([14]) of complex moment sequences in terms of non-negative polynomials which has been originated by M. Riesz. Though this may be interesting rather from the theoretical point of view than applicable to concrete sequences (read: operators — in advance), we will follow this in a context of subnormal operators. It turns out even more: a result of Slinker ([19], Th. 4.2) enables us to prove a M. Riesz-like characterization for *non-cyclic* case.

Theorem 3. *Let S be a densely defined operator in a Hilbert space $\mathfrak{H}$ such that $S\mathfrak{D}(S)\subset\mathfrak{D}(S)$. Then S is subnormal if and only if the following implication holds: if*

$$\{a_{pq}^{ij};\ i, j\in\{1, \dots, m\}\ \text{ and }\ p, q\in\{0, 1, \dots, n\}\}$$

is a sequence of complex numbers such that

$$\text{(i)}\qquad \sum_{i,j=1}^{m}\sum_{p,q=0}^{n} a_{pq}^{ij}\bar{\lambda}^{q}\lambda^{p}\bar{z}_i z_j \geqq 0, \quad \text{for all}\quad \lambda, z_1, \dots, z_m\in\mathbf{C},$$

then

$$\text{(ii)}\qquad \sum_{i,j=1}^{m}\sum_{p,q=0}^{n}\sum_{k,l=0}^{r} a_{pq}^{ij}\langle S^{k+p}f_l^j, S^{l+q}f_k^i\rangle \geqq 0,$$

for each finite sequence $\{f_k^i\colon i=1, \dots, m,\ k=0, \dots, r\}\subset\mathfrak{D}(S)$.

Proof. Suppose that S is subnormal and that N is its normal extension in a Hilbert space $\mathfrak{K}\supset\mathfrak{H}$. Notice that

$$(3)\qquad \begin{cases} \mathfrak{D}(S) = \mathfrak{D}^{\infty}(S)\subset\mathfrak{D}^{\infty}(N) \quad \text{and} \\ N(\mathfrak{D}^{\infty}(N))\subset\mathfrak{D}^{\infty}(N), \quad N^*(\mathfrak{D}^{\infty}(N))\subset\mathfrak{D}^{\infty}(N) \quad \text{and} \\ NN^*f = N^*Nf \quad \text{for each}\quad f\in\mathfrak{D}^{\infty}(N). \end{cases}$$

Define the polynomials p^{ij} $(i, j\in\{1, \dots, m\})$ of two complex variables λ and $\bar{\lambda}$ by

$$(4)\qquad p^{ij}(\lambda, \bar{\lambda}) = \sum_{p,q=0}^{n} a_{pq}^{ij}\bar{\lambda}^{q}\lambda^{p}, \quad \lambda\in\mathbf{C}.$$

Then, since $S\subset N$ and (3), we have

$$(5)\qquad \sum_{p,q=0}^{n}\sum_{k,l=0}^{r} a_{pq}^{ij}\langle S^{k+p}f_l^j, S^{l+q}f_k^i\rangle = \sum_{p,q=0}^{n}\sum_{k,l=0}^{r} a_{pq}^{ij}\langle N^{p}N^{*l}f_l^j, N^{q}N^{*k}f_k^i\rangle =$$

$$= \sum_{p,q=0}^{n} a_{pq}^{ij}\langle N^{p}h_j, N^{q}h_i\rangle = \langle p^{ij}(N, N^*)h_j, h_i\rangle,$$

where

$$(6)\qquad h_i = \sum_{k=0}^{r} N^{*k}f_k^i, \quad i = 1, \dots, m.$$

Thus we have to show that

$$(7)\qquad \sum_{i,j=1}^{m} \langle p^{ij}(N, N^*)h_j, h_i\rangle \geqq 0$$

for all $h_1, \dots, h_m\in\mathfrak{D}^{\infty}(N)$.

Let E be the spectral measure of N. Since all the complex measures $\langle E(\cdot)h_j, h_i\rangle$, $i, j\in\{1, \ldots, m\}$, are absolutely continuous with respect to the non-negative measure $\mu=\sum_{i=1}^{m}\langle E(\cdot)h_i, h_i\rangle$, we find a matrix of summable Borel functions $\{h_{ij}\}_{i,j=1}^{m}$ such that $\langle E(\sigma)h_j, h_i\rangle=\int_\sigma h_{ij}\,d\mu$ for each Borel subset σ of $\mathbf{C}$ and for all i, j.

Let Q be a countable dense subset of $\mathbf{C}$. For $c_1, \ldots, c_n\in Q$ we have

$$\int_\sigma \sum_{i,j=1}^{m} h_{ij}(\lambda)\bar{c}_i c_j\,d\mu(\lambda) = \sum_{i,j=1}^{m} \bar{c}_i c_j\langle E(\sigma)h_j, h_i\rangle = \langle E(\sigma)(\sum_{j=1}^{m} c_j h_j), \sum_{j=1}^{m} c_j h_j\rangle \geqq 0$$

for each σ. This implies that

$$\sum_{i,j=1}^{m} h_{ij}(\lambda)\bar{c}_i c_j \geqq 0 \quad \text{a.e.} \quad [\mu]. \tag{8}$$

Since Q is countable, we can find a common Borel subset σ_0 of $\mathbf{C}$ (which does not depend on the choice of the numbers c_k) such that $\mu(\sigma_0)=\mu(\mathbf{C})$ and (8) is fulfilled, first for all $c_k\in Q$ and then, after limit passage, for all complex c_k's.

Thus we have shown that the complex matrix $[h_{ij}(\lambda)]_{i,j=1}^{m}$ is positive definite for each $\lambda\in\sigma_0$. Since, by (i) and (4), the matrix $[p^{ij}(\lambda, \bar{\lambda})]_{i,j=1}^{m}$ is also positive definite, an application of the classical Schur Lemma gives us that

$$[p^{ij}(\lambda, \bar{\lambda})h_{ij}(\lambda)]_{i,j=1}^{m} \tag{9}$$

is a positive definite matrix for each $\lambda\in\sigma_0$.

Thus

$$\sum_{i,j=1}^{m}\langle p^{ij}(N, N^*)h_j, h_i\rangle = \sum_{i,j=1}^{m}\int_{\mathbf{C}} p^{ij}(\lambda, \bar{\lambda})h_{ij}(\lambda)\,d\mu(\lambda) =$$

$$= \int_{\sigma_0}\Big(\sum_{i,j=1}^{m}[p^{ij}(\lambda, \bar{\lambda})h_{ij}(\lambda)]\Big)d\mu(\lambda) \geqq 0.$$

(The integrand is non-negative due to (9).) This shows (7).

Now suppose that the implication holds for S. Then S satisfies (H) for all finite sequences $f_0, \ldots, f_r\in\mathfrak{D}(S)$ (put $m=1$, $n=0$ and $a_{00}^{11}=1$). Thus, according to Proposition 2, there is a formally normal operator N in some Hilbert space $\mathfrak{K}\supset\mathfrak{H}$, which fulfills the conditions (i), (ii) and (iii) of Proposition 2. Due to a theorem of [19] all we have to prove now is the following implication: if for each $\lambda\in\mathbf{C}$, the polynomial matrix $[p^{ij}(\lambda, \bar{\lambda})]_{i,j=1}^{m}$ is positive definite, then (7) holds for all $h_1, \ldots, h_m\in\mathfrak{D}(N)$.

For this let $[p^{ij}]$ be such a matrix of polynomials with coefficients $\{a_{pq}^{ij}\}$ as in (4). Let $h_1, \ldots, h_m\in\mathfrak{D}(N)$. Then, by (iii) of Proposition 2, there is a sequence $\{f_k^i\colon i=1, \ldots, m,\ k=0, \ldots, r\}\subset\mathfrak{D}(S)$ which fulfills the condition (6). Since N is formally normal extension of S, which has property (i) of Proposition 2, we can rewrite all the equalities (5) to obtain

$$\sum_{i,j=1}^{m} \langle p^{ij}(N, N^*)h_j, h_i\rangle = \sum_{i,j=1}^{m} \sum_{p,q=0}^{n} \sum_{k,l=0}^{r} a_{pq}^{ij} \langle S^{k+p} f_l^j, S^{l+q} f_k^i \rangle \geqq 0.$$

This proves (7) and finishes the proof of theorem.

As we have mentioned this characterization of subnormals may be useful in proof. The following application is at hand.

Corollary 2. *Let S be a densely defined operator in H such that $S\mathfrak{D}(S)\subset\subset\mathfrak{D}(S)$. Then*

(a) *If S is subnormal, then among all the subnormal operators T in $\mathfrak{H}$ extending S and such that $T\mathfrak{D}(T)\subset\mathfrak{D}(T)$ there is a maximal one.*

(b) *Suppose that there exists S^{-1} which is densely defined and $S^{-1}\mathfrak{D}(S^{-1})\subset\subset\mathfrak{D}(S^{-1})$. Then if one of the operators S and S^{-1} is subnormal, so is the other.*

Proof. (a) If $\{T_\omega\}$ is a chain (ordered by inclusion) of subnormal operators extending S and such that $T_\omega\mathfrak{D}(T_\omega)\subset\mathfrak{D}(T_\omega)$, then $\bigcup_\omega T_\omega$ is an upper bound, which, due to Theorem 3, has the same properties as T_ω's do. Now an application of the Zorn Lemma gives the conclusion (a).

(b) Let $\{a_{pq}^{ij}\}$ satisfy (i) of Theorem 3. Set $b_{pq}^{ij}=a_{n-p,n-q}^{ij}$ (remind that $0\leqq p, q\leqq n$). Then one can check that $\{b_{pq}^{ij}\}$ satisfies (i) of Theorem 3 too.

Suppose that S is subnormal. Take a finite sequence $\{f_k^i;\ 1\leqq i\leqq m,\ 0\leqq k\leqq r\}\subset\subset\mathfrak{D}(S)$. Then, because in fact $\mathfrak{D}(S)=S\mathfrak{D}(S)$, we have

$$\sum_{i,j=1}^{m} \sum_{p,q=0}^{n} \sum_{k,l=0}^{r} a_{pq}^{ij} \langle (S^{-1})^{k+p} f_l^j, (S^{-1})^{l+q} f_k^i \rangle = \sum_{i,j=1}^{m} \sum_{p,q=0}^{n} \sum_{k,l=0}^{r} b_{pq}^{ij} \langle S^{k+p} g_l^j, S^{l+q} g_k^i \rangle$$

where $g_l^j=S^{-(n+r)}f_{r-l}^j$. Applying Theorem 3 we get the conclusion (b).

A characterization like this in Theorem 3 in a case of cyclic operators appears implicity in Kilpi [14]. What can be easily deduced from [14] is the following.

Proposition 3. *Let S be a densely defined cyclic operator in $\mathfrak{H}$ with a cyclic vector f_0. Then the following conditions are equivalent:*

(i) *S is subnormal;*

(ii) *$\{\langle S^m f_0, S^n f_0\rangle\}_{m,n=0}^{\infty}$ is a complex moment sequence, i.e. there is a non-negative measure μ on $\mathbf{C}$ such that $\langle S^m f_0, S^n f_0\rangle = \int_{\mathbf{C}} z^m \bar{z}^n \, d\mu(z)$, $m, n\in\mathbf{N}$,*

(iii) *If* $\{a_{k,l}\}_{k,l=0}^{m}$ *is a complex matrix such that* $\sum_{k,l=0}^{m} a_{k,l}\lambda^k\bar{\lambda}^l \geqq 0$ *for each* $\lambda \in \mathbf{C}$, *then*

$$\sum_{k,l=0}^{m} a_{k,l}\langle S^k f_0, S^l f_0\rangle \geqq 0.$$

Our characterization in Theorem 3 applied to cyclic operators looks more complicated than that of Kilpi. Because we are unable on this stage, to reduce directly ours to Kilpi's this is why we do not state it explicitely here; though they must necessarily be equivalent.

Subnormal operators and the Stieltjes moment problem. Weighted shifts

5. As we have already known subnormal operator S satisfies the condition (H) for any choice of vectors $f_0, \ldots, f_n \in \mathfrak{D}^\infty(S)$. Taking $g_k = S^k f_k$ and replacing $f_0, \ldots, f_n$ by $g_0, \ldots, g_n$ in (H) we get the condition:

$$\text{(E)} \qquad \sum_{j,k=0}^{n} \langle S^{j+k} f_j, S^{j+k} f_k\rangle \geqq 0$$

for all choices of vectors $f_0, \ldots, f_n$ in $\mathfrak{D}^\infty(S)$, which reminds a condition considered by EMBRY [7] in the bounded case. Going on set $f_j = c_j f$ and $f_j = c_j S f$ in (E), respectively $(f \in \mathfrak{D}^\infty(S))$ we obtain

$$\sum_{j,k=0}^{n} \|S^{j+k} f\|^2 c_j \bar{c}_k \geqq 0,$$

and

$$\sum_{j,k=0}^{n} \|S^{j+k+1} f\|^2 c_j \bar{c}_k \geqq 0,$$

for all complex numbers $c_1, \ldots, c_n$. This is precisely what is required for the sequence $\{\|S^n f\|^2\}_{n=0}^{\infty}$ to be Stieltjes moment sequence i.e. to be represented as

$$\text{(S)} \qquad \|S^n f\|^2 = \int_0^{+\infty} t^n d\mu(t), \quad n \in \mathbf{N},$$

$\mu = \mu_f$ is a finite non-negative measure.

All what has been said here can be stated as

Proposition 4. *The following implications hold true:*
S is subnormal $\Rightarrow$ S satisfies (H) *on* $\mathfrak{D}^\infty(S)$,
S satisfies (H) *on* $\mathfrak{D}^\infty(S) \Rightarrow$ *S satisfies* (E) *on* $\mathfrak{D}^\infty(S)$,
S satisfies (E) *on* $\mathfrak{D}^\infty(S) \Rightarrow$ *S satisfies* (S) *for each f in* $\mathfrak{D}^\infty(S)$.

6. It turns out that the implications in Proposition 4 can be inverted for S being unilateral weighted shift. Recall S is said to be a *unilateral weighted shift* if $Se_n \in (\mathbf{C} \setminus \{0\})\, e_{n+1}$, $n \geqq 0$, where $\{e_n\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathfrak{H}$. The domain of S is meant as the linear span of $\{e_n\}_{n=0}^{\infty}$. It is clear that S is a cyclic operator with the cyclic vector e_0.

Theorem 4. *Let S be a unilateral weighted shift. Then the following conditions are equivalent:*

(i) *S is subnormal;*

(ii) *S satisfies* (H) *for all finite sequences $f_0, \ldots, f_n \in \mathfrak{D}(S)$;*

(iii) *S satisfies* (E) *for all finite sequences $f_0, \ldots, f_n \in \mathfrak{D}(S)$;*

(iv) *S satisfies* (S) *for $f = e_0$.*

Since $\mathfrak{D}(S) = \mathfrak{D}^{\infty}(S)$, all the implication but (iv)$\Rightarrow$(i) follow from Proposition 4. To prove the implication (iv)$\Rightarrow$(i) we utilize the following result which may be interesting for itself.

Lemma 1. *Let $f_0 \in \mathfrak{H}$ be a cyclic vector for S. Then the following two conditions are equivalent:*

(a) *$S \subset U \otimes R$, where U is a unitary operator in $\mathfrak{K}_1$, R is a self-adjoint operator in $\mathfrak{K}_2$, $\mathfrak{H} \subset \mathfrak{K}_1 \otimes \mathfrak{K}_2$ and $f_0 = f_1 \otimes f_2$ with some $f_1 \in \mathfrak{K}_1$ and $f_2 \in \mathfrak{D}^{\infty}(R)$;*

(b) *there are two functions $\alpha\colon \mathbf{N} \to \mathbf{C}$ and $\beta\colon \mathbf{Z} \to \mathbf{C}$ such that*

$$\langle S^n f_0, S^m f_0 \rangle = \alpha(n+m)\beta(n-m), \quad n, m \in \mathbf{N}, \tag{10}$$

$$\sum_{m,n=0}^{r} \alpha(m+n) c_m \bar{c}_n \geqq 0, \tag{11}$$

for all finite sequences $c_0, \ldots, c_r \in \mathbf{C}$,

$$\sum_{m,n=0}^{r} \beta(n-m) c_n \bar{c}_m \geqq 0, \tag{12}$$

for all finite sequences $c_0, \ldots, c_r \in \mathbf{C}$.

Proof. Let U, R and f_1, f_2 be as in (a). Because

$$\langle S^n f_0, S^m f_0 \rangle = \langle U^{n-m} f_1, f_1 \rangle \langle R^{n+m} f_2, f_2 \rangle, \quad m, n \in \mathbf{N},$$

a direct computation shows that

$$\alpha(n) = \langle R^n f_2, f_2 \rangle, \quad n \in \mathbf{N},$$

and

$$\beta(m) = \langle U^m f_1, f_1 \rangle, \quad m \in \mathbf{Z},$$

satisfy the condition (11) and (12) respectively.

Suppose that the condition (b) is satisfied. Then $\{\alpha(n)\}_{n=0}^{\infty}$ is a Hamburger moment sequence [18] and $\{\beta(n)\}_{n\in\mathbf{Z}}$ is a trigonometric moment sequence [18]. Consequently there are two positive finite measures μ and ν defined on $\mathbf{R}$ and the unit circle $\mathbf{T}$, respectively, such that

$$\langle S^n f_0, S^m f_0\rangle = \int_{\mathbf{R}} t^{n+m}\, d\mu(t) \int_{\mathbf{T}} z^{n-m}\, d\nu(z), \quad n, m\in\mathbf{N}. \tag{13}$$

Denote by M_z and M_t the multiplication operators by z and t in $L^2(\mathbf{T}, \nu)$ and $L^2(\mathbf{R}, \mu)$, respectively. Then by (13)

$$\langle S^n f_0, S^m f_0\rangle = \langle M_t^{n+m} 1_\mu, 1_\mu\rangle_{L^2(\mu)} \langle M_z^{n-m} 1_\nu, 1_\nu\rangle_{L^2(\nu)} =$$
$$= \langle (M_z\otimes M_t)^n (1_\nu\otimes 1_\mu), (M_z\otimes M_t)^m (1_\nu\otimes 1_\mu)\rangle_{L^2(\nu\otimes\mu)}, \quad m, n\in\mathbf{N}.$$

This equality allows us to identify $S^n f_0$ with $(M_z\otimes M_t)^n(1_\nu\otimes 1_\mu)$, $n\in\mathbf{N}$, which gives us $S\subset M_z\otimes M_t$. Since M_z is unitary and M_t is self-adjoint, we set $U=M_z$, $R=M_t$, $f_1=1_\nu$ and $f_2=1_\mu$ to get the conclusion. This completes the proof.

Remark 2. If any of the equivalent conditions (a) and (b) of Lemma 1 is satisfied then S is subnormal. Moreover the operator R can be choosen to be positive if (in addition to (11))

$$\sum_{m,n=0}^{r} \alpha(n+m+1) c_n \bar{c}_m \geqq 0, \tag{11'}$$

for all finite sequences $c_0, \ldots, c_r\in\mathbf{C}$, (since then (11) and (11′) imply that $\{\alpha(n)\}_{n\in\mathbf{N}}$ is a Stieltjes moment sequence) and Lemma 1 leads then to an L^2-model of S as the multiplication by z on the complex plane $\mathbf{C}$.

Proof of (iv)⇒(i) of Theorem 4. Let us define $\delta\colon \mathbf{Z}\to\{0, 1\}$ by $\delta(0)=1$ and $\delta(n)=0$ if $n\neq 0$. So we have

(14)

$$\langle S^n e_0, S^m e_0\rangle = \delta(n-m)\|S^n e_0\|^2 = \delta(n-m)\int_0^\infty t^n\, d\mu(t) = \delta(n-m)\int_0^\infty t^{(n+m)/2}\, d\mu(t),$$
$$m, n\in\mathbf{N},$$

where $\mu=\mu_{e_0}$ is the measure given by the integral representation (S). Setting

$$\alpha(n) = \int_0^\infty t^{n/2}\, d\mu(t), \quad n\in\mathbf{N}$$

and

$$\beta(n) = \delta(n), \quad n\in\mathbf{Z},$$

in (14) we get the condition (b) of Lemma 1. An application of Remark 2 completes the proof of our theorem.

7. Now we want to show usefulness of Theorem 4.

Example 2. In [21] we have shown that the *creation* operator is subnormal. This has been ensured by the condition (H) and the presence of analytic vectors for the operator. However, since this operator is a unilateral weighted shift we can use directly the condition (iv) of Theorem 4 instead of checking condition (H) and looking for analytic vectors. To be more precise, recall that the creation operator is defined as

$$A_+ = \frac{1}{\sqrt{2}}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right)$$

with $\mathfrak{D}(A_+) = \mathfrak{S}(\mathbf{R})$, the Schwartz space. Since the Hermite functions

$$f_n(x) = e^{x^2/2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x^2}, \quad n = 0, 1, \ldots,$$

form an orthogonal basis for $L^2(\mathbf{R})$ and

$$A_+ f_m = \left(-\frac{1}{\sqrt{2}}\right) f_{m+1}, \quad m = 0, 1, \ldots,$$

A_+, when restricted to the linear span $\mathfrak{D}$ of the Hermite function is a weighted shift in $L^2(\mathbf{R})$. Denote this restriction by S. Since

$$\|S^n f_0\|^2 = n!\sqrt{\pi}, \quad n = 0, 1, 2, \ldots,$$

and $\{n!\}_{n=0}^{\infty}$ is a Stieltjes moment sequence, according to Theorem 4, S is subnormal. Since $\bar{A}_+ = (A_+|\mathfrak{D})^-$, A_+ is subnormal.

Theorem 4 allows to produce subnormal operators from simpler ones. As an illustration take a subnormal weighted shift S and define $S_k = S^{*k}S^{k+1}$, k is a positive integer. Then, after some computation — which, in a more general context, will be presented elsewhere [25] — one can show that S_k satisfies the condition (iv) of Theorem 4 and consequently it is subnormal too. In particular, if S is the creation operator then

$$S_1 = \frac{\sqrt{2}}{4}\left\{(1+x^2)x - (3+x^2)\frac{\mathrm{d}}{\mathrm{d}x} - x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^3}{\mathrm{d}x^3}\right\}.$$

8. We pass now to bilateral weighted shifts. In order to prove an analogue of Theorem 4 in this case we need an appropriate version of Lemma 1.

Lemma 2. *Let S be a densely defined operator in $\mathfrak{H}$ such that $\mathfrak{N}(S) = \{0\}$ and $S\mathfrak{D}(S) = \mathfrak{D}(S)$. Suppose there is a vector $f_0 \in \mathfrak{D}(S)$ such that $\mathfrak{D}(S)$ is the linear span of the set $\{S^n f_0 \colon n \in \mathbf{Z}\}$. Then the following conditions are equivalent:*

(a) *$S \subset U \otimes R$, where U is a unitary operator in $\mathfrak{K}_1$, R is a self-adjoint operator in $\mathfrak{K}_2$ with $0 \notin \sigma_p(R)$, $\mathfrak{H} \subset \mathfrak{K}_1 \otimes \mathfrak{K}_2$ and $f_0 = f_1 \otimes f_2$ with some $f_1 \in \mathfrak{K}_1$ and $f_2 \in \bigcap_{n \in \mathbf{Z}} \mathfrak{D}(R^n)$;*

(b) *there are two functions* $\alpha, \beta\colon \mathbf{Z}\to\mathbf{C}$ *such that*

$$\langle S^n f_0, S^m f_0\rangle = \alpha(n+m)\beta(n-m), \quad n, m\in\mathbf{Z}, \tag{15}$$

$$\sum_{-r\leq m,n\leq r} \alpha(n+m)c_n\bar{c}_m \geqq 0, \tag{16}$$

for all finite sequences $c_{-r}, \ldots, c_r\in\mathbf{C}$, *and* β *satisfies* (12).

The proof of Lemma 2 goes in the same way as that of Lemma 1. However one has to use instead of the Hamburger characterization of moment sequences the following result ([13], [1]). A sequence $\{\alpha(n)\}_{n\in\mathbf{Z}}$ of complex numbers can be represented as

$$\alpha(n) = \int_{\mathbf{R}\setminus\{0\}} t^n\, d\mu(t), \quad n\in\mathbf{Z},$$

with a finite non-negative measure μ if and only if (16) holds.

Remark 3. Each of the equivalent conditions (a) and (b) of Lemma 2 guarantees subnormality of S. If the function $\alpha\colon \mathbf{Z}\to\mathbf{C}$ satisfies the additional condition

$$\sum_{-r\leq n,m\leq r} \alpha(n+m+1)c_n\bar{c}_m \geqq 0, \tag{17}$$

for all finite sequences $c_{-r}, \ldots, c_r\in\mathbf{C}$, then the operator R can be choosen to be positive. This happens because, due to the conditions (16) and (17), the sequence $\{\alpha(n)\}_{n\in\mathbf{Z}}$ becomes (cf. [1], [12]) a *two-sided Stieltjes* moment sequence which means that there is a non-negative finite measure μ such that

$$\alpha(n) = \int_{(0,+\infty)} t^n\, d\mu(t), \quad n\in\mathbf{Z}.$$

A densely defined operator S in H is said to be a *bilateral weighted shift* if there is an orthonormal basis $\{e_n\}_{n\in\mathbf{Z}}$ of $\mathfrak{H}$ such that $Se_n\in(\mathbf{C}\setminus\{0\})e_{n+1}$ for each $n\in\mathbf{Z}$. The domain $\mathfrak{D}(S)$ of S is the linear span of $\{e_n\}_{n\in\mathbf{Z}}$.

We have an analogue of Theorem 4 for bilateral weighted shifts.

Theorem 5. *Let S be a bilateral weighted shift. Then the following conditions are equivalent:*

(i) *S is subnormal;*
(ii) *S satisfies* (H) *for all finite sequences* $f_0, \ldots, f_n$ *in* $\mathfrak{D}(S)$;
(iii) *S satisfies* (E) *for all finite sequences* $f_0, \ldots, f_n$ *in* $\mathfrak{D}(S)$;
(iv) *S satisfies* (S) *for each* $f\in\{S^{-2n}e_0\}_{n\geq 0}$;
(v) $\{\|S^n e_0\|^2\}_{n\in\mathbf{Z}}$ *is a two-sided Stieltjes moment sequence.*

Proof. The only implications which need a proof are (iv)⇒(v) and (v)⇒(i).

(iv)$\Rightarrow$(v): The operator S satisfies all the assumptions of Lemma 2 with $f_0=e_0$. Now we show that the sequence $\{\|S^n e_0\|^2\}_{n\in\mathbf{Z}}$ satisfies the conditions (16) and (17). Let $c_{-r}, \ldots, c_r$ be an arbitrary sequence of complex numbers. Then

$$\sum_{-r\le n,m\le r} \|S^{n+m}e_0\|^2 c_n\bar{c}_m = \sum_{n,m=0}^{2r} \|S^{n+m}S^{-2r}e_0\|^2 d_n\bar{d}_m$$

and

$$\sum_{-r\le n,m\le r} \|S^{n+m+1}e_0\|^2 c_n\bar{c}_m = \sum_{n,m=0}^{2r} \|S^{n+m+1}S^{-2r}e_0\|^2 d_n\bar{d}_m,$$

where $d_n=c_{n-r}$ for $n\in\{0, 1, \ldots, 2r\}$. Due to (iv), all the sums appearing in the above two equalities are nonnegative. This ensures that $\{\|S^n e_0\|^2\}_{n\in\mathbf{Z}}$ is a two-sided Stieltjes moment sequence.

(v)$\Rightarrow$(i): Like in the proof of Theorem 4 we put

$$\alpha(n) = \int_{(0,+\infty)} t^{n/2}\, d\mu(t), \quad n\in\mathbf{Z}$$

and

$$\beta(n) = \delta(n), \quad n\in\mathbf{Z}.$$

The equality (15) follows from the same argument as its analogue in the proof of Theorem 4. The application of Lemma 2 completes the proof.

Subnormal operators through C^∞-vectors

9. In the papers ([20], [21], [22]) we have studied subnormal operators by means of some of their classes of C^∞-vectors. Here we wish to review and extend these investigations. Recall the definitions.

A vector $f\in\mathfrak{D}^\infty(S)$ is said to be a *bounded vector* of S if there are positive numbers $a=a(f)$ and $c=c(f)$ such that

$$\|S^n f\| \le ac^n, \quad n = 1, 2, \ldots.$$

A vector $f\in\mathfrak{D}^\infty(S)$ is said to be an *analytic vector* of S if there is a positive number $t=t(f)$ such that

$$\sum_{n=1}^{\infty} \frac{\|S^n f\|}{n!} t^n < +\infty.$$

A vector $f\in\mathfrak{D}^\infty(S)$ is said to be a *quasi-analytic vector* of S if

$$\sum_{n=1}^{\infty} \|S^n f\|^{-1/n} = +\infty.$$

Finally $f \in \mathfrak{D}^\infty(S)$ is said to be a *Stieltjes vector* of S if

$$\sum_{n=1}^{\infty} \|S^n f\|^{-1/2n} = +\infty.$$

Denote by $\mathfrak{B}(S)$, $\mathfrak{A}(S)$, $\mathfrak{Q}(S)$ and $\mathfrak{S}(S)$ the sets of bounded, analytic, quasi-analytic and Stieltjes vectors of S, respectively. It is clear that $\mathfrak{B}(S)$ and $\mathfrak{A}(S)$ are linear subspaces of $\mathfrak{H}$ and $\mathfrak{B}(S) \subset \mathfrak{A}(S) \subset \mathfrak{Q}(S) \subset \mathfrak{S}(S)$. By direct verification we get that $S(\mathfrak{B}(S)) \subset \mathfrak{B}(S)$ and $S(\mathfrak{A}(S)) \subset \mathfrak{A}(S)$. To check that $\mathfrak{Q}(S)$ and $\mathfrak{S}(S)$ share the same property as $\mathfrak{B}(S)$ and $\mathfrak{A}(S)$, use the Carleman inequality [3]:

$$\sum_{n=2}^{r} a_n^{1-(1/n)} \leqq \sum_{n=2}^{r} a_n + 2\sqrt{\sum_{n=2}^{r} a_n} \tag{18}$$

with $a_n = \|S^n f\|^{-\frac{1}{n-1}}$ and $a_n = \|S^n f\|^{-\frac{1}{2(n-1)}}$, respectively.

In [20] we have proved the following theorem.

Theorem I. *Let S be a densely defined linear operator in $\mathfrak{H}$. Suppose that $\mathfrak{D}(S) = \mathfrak{B}(S)$. Then the following conditions are equivalent:*

(i) *S is subnormal;*

(ii) *S satisfies* (H), *for all finite sequences $f_0, \ldots, f_n \in \mathfrak{D}(S)$;*

(iii) *there is an increasing sequence $\{\mathfrak{H}_n\}_{n=1}^\infty$ of closed linear subspaces of $\mathfrak{H}$ contained in $\mathfrak{D}(\bar{S})$ such that $\bar{S}\mathfrak{H}_n \subset \mathfrak{H}_n$, each restriction of $\bar{S}$ to $\mathfrak{H}_n$ is a bounded subnormal operator in $\mathfrak{H}$ and $\bigcup_{n=1}^{\infty} \mathfrak{H}_n$ is a core for $\bar{S}$.*

Remark 4. The following comments may be usefull here. Let A be a densely defined closable operator in $\mathfrak{H}$. A linear subspace $\mathfrak{D}$ of $\mathfrak{D}(A)$ is said to be a *core* for A if $\bar{A} = (A|\mathfrak{D})^-$. A closed linear subspace $\mathfrak{G}$ of $\mathfrak{H}$ is said to be *invariant* (resp. *reducing*) for A if $PAP = AP$ (resp. $PA \subset AP$), where P is the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{G}$. If a closed linear subspace $\mathfrak{G}$ of $\mathfrak{H}$ is contained in $\mathfrak{D}(A) \cap \mathfrak{D}(A^*)$ then $\mathfrak{G}$ is reducing for A if and only if $A(\mathfrak{G}) \subset \mathfrak{G}$ and $A^*(\mathfrak{G}) \subset \mathfrak{G}$.

The example of the creation operator indicates that there are closed subnormal operators having no nontrivial bounded vectors. However, if an operator *has* a dense set of bounded vectors, Theorem I provides us with some additional information on its geometrical structure. We show that quasinormal operators we have already considered in Section 2 fall in this class and get, as a by-product, another proof of subnormality of quasinormal operators.

Proposition 5. *Suppose that S is a quasinormal operator in $\mathfrak{H}$. Then $\mathfrak{B}(S)$ is a core for S, there is an increasing sequence $\{\mathfrak{H}_n\}_{n=1}^\infty$ of closed linear subspaces of $\mathfrak{H}$ contained in $\mathfrak{D}(S)$ such that each $\mathfrak{H}_n$ reduces S, each restriction of S to $\mathfrak{H}_n$ is a bounded quasinormal operator in $\mathfrak{H}_n$ and $\bigcup_{n=1}^{\infty} \mathfrak{H}_n$ is a core for S.*

Proof. First of all we show that $\mathfrak{B}(S)$ is a core for S. Let $S=U|S|$ be the polar decomposition of S and let E be the spectral measure of $|S|$. Set $\mathfrak{H}_n=E([0,n])\mathfrak{H}$ and $\mathfrak{D}=\bigcup_{n=1}^{\infty}\mathfrak{H}_n$. Take $f\in\mathfrak{H}_m$. Then

$$|S|f=|S|E([0,m])f=E([0,m])|S|f$$

and by Proposition 1,

$$Uf=UE([0,m])f=E([0,m])Uf.$$

This means that each $\mathfrak{H}_m$ and consequently $\mathfrak{D}$ is invariant for $|S|$, U and S. Thus for $f\in\mathfrak{H}_m$ we have

$$\|S^nf\|^2=\|U^n|S|^nf\|^2\leqq\||S|^nf\|^2=$$

$$=\||S|^nE([0,m])f\|^2=\int_0^m t^{2n}\langle E(dt)f,f\rangle\leqq m^{2n}\|f\|^2,$$

so $f\in\mathfrak{B}(S)$. In other words $\mathfrak{D}\subset\mathfrak{B}(S)$. It is easy to see that the equality $|S|=(|S||_{\mathfrak{D}})^-$ implies $S=(S|\mathfrak{D})^-$. So $\mathfrak{D}$ and $\mathfrak{B}(S)$ are cores for S.

Define a bounded operator $S_n=UR_n$, where $R_n=\int_0^n tE(dt)$. Then $\mathfrak{R}(R_n^2)\subset\mathfrak{R}(R_n)\subset\mathfrak{R}(|S|E([0,n]))\subset\mathfrak{R}(|S|)$, so $\mathfrak{R}(R_n^2)\subset\mathfrak{R}(|S|)$. Since U^*U is the orthogonal projection onto $\overline{\mathfrak{R}(|S|)}$, we have $U^*UR_n^2=R_n^2$. By Proposition 1, U commutes with R_n. Therefore $S_n^*S_n=U^*UR_n^2=R_n^2$, which implies $|S_n|=R_n$. Since U commutes with R_n, S_n commutes with $R_n=|S_n|$. This means that S_n is a quasinormal operator. Denote by T_n the operator $S_n|_{\mathfrak{H}_n}$. Then

$$T_n^*T_n=E([0,n])S_n^*S_n|_{\mathfrak{H}_n}=E([0,n])R_n^2|_{\mathfrak{H}_n}=(R_n|_{\mathfrak{H}_n})^2.$$

Thus $|T_n|=|S_n||_{\mathfrak{H}_n}=R_n|_{\mathfrak{H}_n}$. Since S_n commutes with R_n, T_n commutes with $|T_n|$. This means that for each $n\geqq 1$, $S|_{\mathfrak{H}_n}=T_n$ is a bounded quasinormal operator. Since $E([0,n])S\subset SE([0,n])$, $\mathfrak{H}_n$ reduces S. This completes the proof.

Corollary 3. *S is a subnormal operator if and only if there is a subnormal extension $\tilde{S}$ of S in $\tilde{\mathfrak{H}}\supset\mathfrak{H}$ such that $\mathfrak{B}(\tilde{S})$ is a core for $\tilde{S}$.*

Proof. This is an easy consequence of Proposition 5 and Theorem 2.

Corollary 4. *An operator S in $\mathfrak{H}$ is normal if and only if S is formally normal and quasinormal. In particular S is self-adjoint if and only if S is symmetric and quasinormal.*

Proof. Since S is quasinormal, there is a sequence $\{\mathfrak{H}_n\}_{n=1}^{\infty}$ of closed linear subspace of $\mathfrak{H}$ with properties described by Proposition 5. Let $\mathfrak{D}=\bigcup_{n=1}^{\infty}\mathfrak{H}_n$. Then $\mathfrak{D}\subset\mathfrak{B}(S)$ and $\mathfrak{D}$ is a core for S. Since $\mathfrak{H}_n$ is a reducing subspace for S and $\mathfrak{H}_n\subset$

$\subset\mathfrak{D}(S)\cap\mathfrak{D}(S^*)=\mathfrak{D}(S)$, Remark 4 implies that $S\mathfrak{H}_n\subset\mathfrak{H}_n$ and $S^*\mathfrak{H}_n\subset\mathfrak{H}_n$, for each n. Thus $S\mathfrak{D}\subset\mathfrak{D}$ and $S^*\mathfrak{D}\subset\mathfrak{D}$. Since S is formally normal, the nontrivial conclusion of Corollary 4 follows from Theorem 1 of [20].

10. Another result we wish to discuss is one which bears a resemblance to a result of Embry for bounded operators [7].

Theorem 6. *Let S be a densely defined operator in $\mathfrak{H}$ such that $S\mathfrak{D}(S)\subset\mathfrak{D}(S)$. Suppose that $\mathfrak{D}(S)$ is a linear span of the set $\mathfrak{Q}(S)$. Then S is subnormal if and only if S satisfies* (E) *for all finite sequences $f_0, \ldots, f_n\in\mathfrak{D}(S)$.*

This is a stronger version of Theorem 8 of [21] where instead of (E) the condition (H) appears.

In order to prove Theorem 6 we need some lemmas. The first of them gives the full characterization of determinate moment sequences in terms of their representing measures. The proof of it can be done in the same way as that for Hamburger moment sequences (cf. [9], Theorem 8).

Lemma 3. *A Stieltjes moment sequence $\{a_n\}_{n=0}^{\infty}$ with the representing measure μ is determinate if and only if the set of all polynomials of one real variable is dense in $L^2(\mathbf{R}_+, (1+x^2)\mu)$.*

Lemma 4. *Let N be a densely defined operator in $\mathfrak{K}$ such that*

$$\mathfrak{D}=\mathfrak{D}(N)=\mathfrak{D}(N^*N),\quad N(\mathfrak{D})\subset\mathfrak{D}\quad \textit{and}\quad N^*N(\mathfrak{D})\subset\mathfrak{D}, \tag{19}$$

$$N(N^*N)f=(N^*N)Nf,\quad f\in\mathfrak{D}, \tag{20}$$

$$\mathfrak{S}((N^*N))\quad \textit{is a total set in } \mathfrak{K}. \tag{21}$$

Then N is closable and $\bar{N}$ is quasinormal.

Proof. Denote by A the symmetric operator N^*N defined on $\mathfrak{D}$. Then $\langle Nf, Ng\rangle=\langle Af, g\rangle$, $f, g\in\mathfrak{D}$. This implies that N is closable. Denote by $\mathfrak{D}_0$ the linear span of $\mathfrak{S}(A)$. Then $A(\mathfrak{D}_0)\subset\mathfrak{D}_0$, $A=N^*N\subset\bar{N}^*\bar{N}$ and, by (21), $\mathfrak{S}(A)$ is a total set in $\mathfrak{K}$. It follows from [16] that $(\bar{A}|_{\mathfrak{D}_0})^-=\bar{A}=\bar{N}^*\bar{N}$. The last equality can be written as $\bar{A}=|\bar{N}|^2$. Since $\mathfrak{D}_0\subset\mathfrak{D}(|N|^2)$ and $\mathfrak{D}_0$ is a core for $|\bar{N}|^2$, $\mathfrak{D}_0$ is a core for $|\bar{N}|$. Now an application of the polar decomposition for $\bar{N}$ gives us that

$$\mathfrak{D}_0 \text{ is a core for } \bar{N}. \tag{22}$$

Let E be the spectral measure of $\bar{A}$, i.e. $\bar{A}=\int_0^\infty tE(dt)$. Then for $f\in\mathfrak{D}$ we have

$$\langle A^n f, f\rangle=\int_0^\infty t^n\langle E(dt)f, f\rangle,\quad n\in\mathbf{N},$$

and

$$\langle A^n Nf, Nf\rangle = \int_0^\infty t^n \langle E(dt)Nf, Nf\rangle, \quad n\in\mathbf{N}.$$

Since $A=N^*N$, (19) and (20) imply

$$\langle A^n Nf, Nf\rangle = \langle A^{n+1}f, f\rangle, \quad n\in\mathbf{N}.$$

Combining these three equalities we obtain

$$(23) \qquad \langle A^{n+1}f, f\rangle = \int_0^\infty t^n \langle E(dt)Nf, Nf\rangle = \int_0^\infty t^n t\langle E(dt)f, f\rangle, \quad f\in\mathfrak{D}, \quad n\geqq 0.$$

Let $f\in\mathfrak{S}(A)$. Due to the Carleman criterion (cf. [18]) the sequence $\{\langle A^n f, f\rangle\}_{n=0}^\infty$ is a determinate Stieltjes moment sequence. Using now the Carleman inequality (18) and again the Carleman criterion we infer that $\{\langle A^{n+1}f, f\rangle\}_{n=0}^\infty$ is a determinate Stieltjes moment sequence. Consequently, due to (23), we have

$$(24) \qquad \langle E(dt)Nf, Nf\rangle = t\langle E(dt)f, f\rangle.$$

Let σ be a Borel subset of $\mathbf{R}_+$. Since $\{\langle A^n f, f\rangle\}_{n=0}^\infty$ is a determinate Stieltjes moment sequence $(f\in\mathfrak{S}(A)!)$, Lemma 3 gives us a sequence $\{p_n\}_{n=1}^\infty$ of polynomials, which converges to the indicator function 1_σ of the set σ in $L^2(\mathbf{R}_+, (1+x^2)\mu)$, where $\mu=(E(\cdot)f, f)$. One can show then that $\{p_n\}_{n=1}^\infty$ converges to 1_σ in $L^2(\mathbf{R}_+, \mu)$ as well as in $L^2(\mathbf{R}_+, x\mu)$. Since

$$\|E(\sigma)f-p_n(A)f\|^2 = \int_0^\infty |1_\sigma-p_n|^2\, d\mu$$

and, by (24),

$$\|E(\sigma)Nf-p_n(A)Nf\|^2 = \int_0^\infty |1_\sigma(x)-p_n(x)|^2\langle E(dx)Nf, Nf\rangle =$$

$$= \int_0^\infty |1_\sigma(x)-p_n(x)|^2 x\, d\mu(x),$$

we have $E(\sigma)f=\lim_{n\to\infty} p_n(A)f$ and $E(\sigma)Nf=\lim_{n\to\infty} p_n(A)Nf=\lim_{n\to\infty} Np_n(A)f$. Thus $E(\sigma)f\in\mathfrak{D}(\bar{N})$ and $\bar{N}E(\sigma)f=E(\sigma)Nf$ for each $f\in\mathfrak{S}(A)$. This implies that $E(\sigma)(N|_{\mathfrak{D}_0})\subset\subset\bar{N}E(\sigma)$ and $E(\sigma)(N|_{\mathfrak{D}_0})^-\subset\bar{N}E(\sigma)$, in consequence. Due to (22) we obtain

$$(25) \qquad E(\sigma)\bar{N}\subset\bar{N}E(\sigma), \quad \text{for each Borel subset } \sigma \text{ of } \mathbf{R}_+.$$

Since $|\bar{N}|=\bar{A}^{1/2}=\int_0^{+\infty} t^{1/2}E(dt)$, the spectral measure F of $|\bar{N}|$ is given by the following formula:

$$(26) \qquad F(\sigma) = E(\varphi^{-1}(\sigma)), \quad \text{for each Borel subset } \sigma \text{ of } \mathbf{R}_+,$$

where $\varphi\colon \mathbf{R}_+ \to \mathbf{R}_+$ is a homeomorphism defined by $\varphi(x)=x^{1/2}$, $x\in\mathbf{R}_+$. The conditions (25) and (26) show that $\bar{N}$ commutes with the spectral measure F of $|\bar{N}|$. This completes the proof of Lemma 4.

The next lemma shows that the condition (E) holds on $\mathfrak{D}^\infty(S)$ if and only if S has a "formally quasinormal" extension with "reducing" domain.

Lemma 5. *Let S be a densely defined operator in $\mathfrak{H}$ such that $S(\mathfrak{D}(S))\subset\mathfrak{D}(S)$. Then S satisfies* (E) *for all finite sequences $f_0, \dots, f_n\in\mathfrak{D}(S)$ if and only if there is a densely defined operator N in some Hilbert space $\mathfrak{K}\supset\mathfrak{H}$ such that*

(i) *N satisfies the conditions* (19) *and* (20),

(ii) *$\mathfrak{D}(S)\subset\mathfrak{D}(N)$ and $S\subset N$,*

(iii) *$\mathfrak{D}(N)$ is a linear span of the set $\{(N^*N)^n f\colon n\geqq 0, f\in\mathfrak{D}(S)\}$.*

Proof. Suppose that N satisfies (i) and (ii). Then (19) and (20) imply (via an induction procedure)

$$\langle (N^*N)^n f, g\rangle = \langle N^n f, N^n g\rangle, \quad f, g\in\mathfrak{D}(N), \quad n\geqq 0,$$

and this can be used to prove the inequality (E) for all finite sequences $f_0, \dots, f_n\in\mathfrak{D}(S)$.

To prove the converse, suppose that S satisfies (E) for all finite sequences $f_0, \dots, f_n\in\mathfrak{D}(S)$. Define the form φ over $(\mathbf{N}, \mathfrak{D}(S))$ (cf. [23]) by

$$\varphi(n, f, g) = \langle S^n f, S^n g\rangle, \quad n\in\mathbf{N}, \quad f, g\in\mathfrak{D}(S).$$

$\mathbf{N}$ is a *-semigroup with the identity map as an involution. Since S satisfies (E) for all finite sequences $f_0, \dots, f_n\in\mathfrak{D}(S)$, the form φ is positive definite. Notice also that 1 is a hermitian generator of the *-semigroup $\mathbf{N}$ and $\varphi(0, f, g)=\langle f, g\rangle$ for all $f, g\in\mathfrak{D}(S)$. Thus, by Proposition of [23], there is (under suitable unitary identification — see the proof of Prop. 2) a densely defined symmetric operator A in some Hilbert space $\mathfrak{K}\supset\mathfrak{H}$ such that $A(\mathfrak{D}(A))\subset\mathfrak{D}(A)$, $\mathfrak{D}(A)$ is the linear span of the set $\{A^n f\colon n\geqq 0, f\in\mathfrak{D}(S)\}$, $\mathfrak{D}(S)\subset\mathfrak{D}(A)$ and

$$\langle S^n f, S^n g\rangle = \varphi(n; f, g) = \langle A^n f, g\rangle, \quad n\geqq 0, \quad f, g\in\mathfrak{D}(S). \tag{27}$$

Define an operator N with $\mathfrak{D}(N)=\mathfrak{D}(A)$ by

$$N\Big(\sum_{k=0}^{n} A^k f_k\Big) = \sum_{k=0}^{n} A^k S f_k, \quad f_0, \dots, f_n\in\mathfrak{D}(S), \quad n\geqq 0.$$

It follows from (27) that for all $f_0, \dots, f_n\in\mathfrak{D}(S)$

$$\Big\|\sum_{k=0}^{n} A^k S f_k\Big\|^2 = \sum_{k,l=0}^{n} \langle A^{k+l} S f_k, S f_l\rangle = \sum_{k,l=0}^{n} \langle S^{k+l} S f_k, S^{k+l} S f_l\rangle =$$

$$= \sum_{k,l=0}^{n} \langle S^{k+l+1} f_k, S^{k+l+1} f_l\rangle = \sum_{k,l=0}^{n} \langle A^{k+l+1} f_k, f_l\rangle = \Big\langle A\Big(\sum_{k=0}^{n} A^k f_k\Big), \sum_{l=0}^{n} A^l f_l\Big\rangle.$$

This implies the correctness of the definition of N and shows that $\|Nf\|^2=\langle Af,f\rangle$, $f\in\mathfrak{D}(N)$. Consequently

$$\langle Nf, Ng\rangle = \langle Af, g\rangle,\quad f, g\in\mathfrak{D}(N).$$

This implies that $A=N^*N$. The equality (20) follows from the following ones

$$NA\Big(\sum_{k=0}^{n} A^k f_k\Big) = N\Big(\sum_{k=0}^{n} A^{k+1} f_k\Big) = \sum_{k=0}^{n} A^{k+1} S f_k =$$

$$= A\Big(\sum_{k=0}^{n} A^k S f_k\Big) = AN\Big(\sum_{k=0}^{n} A^k f_k\Big),\quad f_0, \ldots, f_n\in\mathfrak{D}(S).$$

The inclusion $S\subset N$ is obvious. This completes the proof of Lemma 5.

Now we are able to pass to the proof of Theorem 6.

Proof of Theorem 6. "Only if" part of Theorem 6 follows from Proposition 4 as well as from Theorem 3.

Conversely, suppose that S satisfies (E) for all finite sequences $f_0, \ldots, f_n\in\mathfrak{D}(S)$. Due to Lemma 5, there is a densely defined operator N in some Hilbert space $\mathfrak{K}\supset\mathfrak{H}$, which satisfies the conditions (i), (ii) and (iii) of Lemma 5. Then

(28) $$\mathfrak{Q}(S)\subset\mathfrak{S}(A),\quad A = N^*N.$$

To prove this suppose that $f\in\mathfrak{Q}(S)$. Then Proposition 4 implies that the sequence $\{a_n\}_{n=0}^{\infty}$, where $a_n=\|S^n f\|^2$ for $n\in\mathbf{N}$, is a Stieltjes moment sequence. Thus $a_n^2\leqq a_k a_l$ for $k, l\in\mathbf{N}$ such that $2n=k+l$. Due to Section 1 of [21] we obtain

$$\sum_{n=1}^{\infty} \|S^{2n} f\|^{-1/2n} = +\infty.$$

It follows from (i) and (ii) of Lemma 5 that $\|A^n f\|^2=\langle A^{2n}f,f\rangle=\|S^{2n}f\|^2$, so $\sum_{n=1}^{\infty}\|A^n f\|^{-1/2n}=\sum_{n=1}^{\infty}\|S^{2n}f\|^{-1/2n}=+\infty$. Thus $f\in\mathfrak{S}(A)$.

Now we are in position to use Lemma 4. Indeed, since $\mathfrak{D}(A)$ is a linear span of $\{A^n f\colon n\geqq 0, f\in\mathfrak{D}(S)\}$ and $\mathfrak{D}(S)$ is a linear span of $\mathfrak{Q}(S)$, an application of (28) and $A(\mathfrak{S}(A))\subset\mathfrak{S}(A)$ gives us that $\mathfrak{D}(A)$ is a linear span of $\mathfrak{S}(A)$. It follows from Lemma 4 that $\bar{N}$ is quasinormal and, by Corollary 1, S is subnormal. This completes the proof.

Corollary 5. *Let S be a closed densely defined operator in $\mathfrak{H}$. Suppose that the linear span $\mathfrak{D}$ of $\mathfrak{Q}(S)$ is a core for S and that S satisfies* (E) *for all finite sequences $f_0, \ldots, f_n\in\mathfrak{D}$. Then S is a subnormal operator.*

11. Now we show that if an operator S has a dense set of analytic vectors then, similarly as in the case of weighted shifts, the condition (S), when satisfied for all $f\in\mathfrak{D}(S)$, is sufficient for S to be subnormal. This result is an extension of Lambert theorem (cf. [15], Th. 3.1) to the case of unbounded operators. Similarly as in [22] we ask whether this theorem is true for operators having dense set of quasi-analytic vectors.

Theorem 7. *Let S be a densely defined operator in $\mathfrak{H}$ such that $\mathfrak{D}(S)=\mathfrak{A}(S)$. Then S is subnormal if and only if S satisfies* (S) *for each $f\in\mathfrak{D}(S)$.*

Proof. We have only to prove sufficiency. Suppose that S satisfies (S) for each $f\in\mathfrak{D}(S)$. Then for each $f\in\mathfrak{D}(S)$ there is a unique non-negative measure μ_f such that

$$\|S^n f\|^2 = \int_0^\infty t^n \, d\mu_f(t), \quad n = 0, 1, 2, \ldots. \tag{29}$$

Using the polarization formula we define complex measures

$$\mu(\sigma; f, g) = \frac{1}{4}\{\mu_{f+g}(\sigma)-\mu_{f-g}(\sigma)+i\mu_{f+ig}(\sigma)-i\mu_{f-ig}(\sigma)\}$$

for each Borel subset σ of $\mathbf{R}_+$. Since the measure μ_f is uniquely determined we have

$$\mu_{af} = |a|^2\mu_f, \quad a\in\mathbf{C}, \quad f\in\mathfrak{D}(S).$$

This implies that $\mu_f=\mu(\cdot\,;f,f)$, $f\in\mathfrak{D}(S)$ and that the form $\mu(\sigma;\,\cdot, -)$ is hermitian symmetric. It is easy to see that

$$\langle S^n f, S^n g\rangle = \int_0^\infty t^n \mu(dt; f, g), \quad f, g\in\mathfrak{D}(S), \quad n\in\mathbf{N}. \tag{30}$$

Now we prove that $\mu(\gamma;\,\cdot, -)$ is linear with respect to the first variable. To show it is additive we write

$$\langle S^n(f+g), S^n h\rangle = \langle S^n f, S^n h\rangle+\langle S^n g, S^n h\rangle, \quad f, g, h\in\mathfrak{D}(S), \quad n\in\mathbf{N}.$$

Using the polarization formula for the form $(f,g)\to\langle S^n f, S^n g\rangle$ and the integral representation (29) we get

$$\int_0^\infty t^n\, d\nu_1(t)-\int_0^\infty t^n\, d\nu_2(t)+i\Big(\int_0^\infty t^n\, d\nu_3(t)-\int_0^\infty t^n\, d\nu_4(t)\Big) = 0, \quad n\in\mathbf{N},$$

where

$$\nu_1 = \mu_{f+g+h}+\mu_{f-h}+\mu_{g-h}, \qquad \nu_2 = \mu_{f+g-h}+\mu_{f+h}+\mu_{g+h},$$

$$\nu_3 = \mu_{f+g+ih}+\mu_{f-ih}+\mu_{g-ih}, \quad \nu_4 = \mu_{f+g-ih}+\mu_{f+ih}+\mu_{g+ih}.$$

Since the measures ν_k, $k=1, 2, 3, 4$, are non-negative we obtain

$$\int_0^\infty t^n \, d\nu_1(t) = \int_0^\infty t^n \, d\nu_2(t), \quad n \in \mathbf{N},$$

and

$$\int_0^\infty t^n \, d\nu_3(t) = \int_0^\infty t^n \, d\nu_4(t), \quad n \in \mathbf{N}. \tag{31}$$

Each of these Stieltjes moment sequences is determinate. To see this consider the first of them

$$a_n = \int_0^\infty t^n \, d\nu_1(t) = \|S^n(f+g+h)\|^2 + \|S^n(f-h)\|^2 + \|S^n(g-h)\|^2, \quad n \in \mathbf{N}.$$

Since the vectors $f+g+h, f-h, g-h$ are analytic vectors of S, one can prove that there is a positive real number $t>0$ such that

$$\sum_{n=0}^\infty \frac{a_n^{1/2}}{n!} t^n < +\infty.$$

This implies that $\sum_{n=1}^\infty a_n^{-1/2n} = +\infty$. Due to the Carleman criterion (cf. [18]), $\{a_n\}$ is a determinate Stieltjes moment sequence. The same is true for the other sequence given by (31).

Thus $\nu_1 = \nu_2$ and $\nu_3 = \nu_4$. This in conclusion implies the required additivity $\mu(\sigma;\, f+g, h) = \mu(\sigma;\, f, h) + \mu(\sigma, g, h)$. By the same trick we can prove that $\mu(\sigma;\, af, g) = a\mu(\sigma;\, f, g)$, first for $a>0$ then for $a<0$ and finally for $a=i$ which exhausts all possibilities.

Thus for each Borel subset σ of $\mathbf{R}_+$, $\mu(\sigma;\, \cdot, -)$ is a hermitian bilinear form and $\mu(\cdot\,; f, f)$ is a non-negative finite measure on $\mathbf{R}_+$ for each $f \in \mathfrak{D}(S)$. Using the generalized Naimark dilation theorem [10] we find a Hilbert space $\mathfrak{K}$, a linear operator $V\colon \mathfrak{D}(S) \to \mathfrak{K}$ and a spectral (normalized) measure E on $\mathbf{R}_+$ in $\mathfrak{K}$ such that

$$\mu(\sigma;\, f, g) = \langle E(\sigma)Vf, Vg \rangle, \quad f, g \in \mathfrak{D}(S), \tag{32}$$

for every Borel subset σ of $\mathbf{R}_+$. According to Theorem 6, the proof of Theorem 7 will be finished if we show that S satisfies (E) for all finite sequences $f_0, \dots, f_n \in \mathfrak{D}(S)$. Let $f_0, \dots, f_n \in \mathfrak{D}(S)$. Due to (32)

$$V(\mathfrak{D}(S)) \subset \mathfrak{D}\Big(\int_0^\infty t^n E(dt)\Big), \quad n \in \mathbf{N}.$$

Using (30) and (32) we obtain

$$\sum_{j,k=0}^{n} \langle S^{j+k}f_j, S^{j+k}f_k\rangle = \sum_{j,k=0}^{n} \int_0^\infty t^{j+k}\mu(dt; f_j, f_k) = \sum_{j,k=0}^{n} \left\langle \int_0^\infty t^{j+k}E(dt)Vf_j, Vf_k\right\rangle =$$

$$= \sum_{j,k=0}^{n} \left\langle \int_0^\infty t^{j}E(dt)Vf_j, \int_0^\infty t^{k}E(dt)Vf_k\right\rangle = \left\|\sum_{k=0}^{n} \int_0^\infty t^{k}E(dt)Vf_k\right\|^2 \geqq 0.$$

This completes the proof.

The proof of Theorem 7 is similar to that of Theorem 6 in [21]. For reader's convenience we have repeated the most essential parts of it.

Corollary 6. *Let S be a closed densily defined operator in H such that $\mathfrak{A}(S)$ is a core for S. If S satisfies* (S) *for each $f\in\mathfrak{A}(S)$, then S is a subnormal operator.*

In the case when the operator S is invertible, Theorem 7 implies the following

Corollary 7. *Let S be a densely defined operator with the densely defined inverse S^{-1}. Suppose $S\mathfrak{D}(S)\subset\mathfrak{D}(S)$ and $S^{-1}\mathfrak{D}(S^{-1})\subset\mathfrak{D}(S^{-1})$ and S satisfies* (S) *for each $f\in\mathfrak{D}(S)$. Then S is subnormal provided $\mathfrak{D}(S^{-1})=\mathfrak{A}(S^{-1})$.*

Proof. Due to Corollary 2 (b), it is sufficient to show that S^{-1} is subnormal and, due to Theorem 7, it is sufficient to show that S^{-1} satisfies (S) for each $f\in\mathfrak{D}(S^{-1})=\mathfrak{D}(S)$. Take $f\in\mathfrak{D}(S)$ and $c_0, \dots, c_n\in\mathbf{C}$. Define $g=S^{-2n}f$, $h=S^{-1}f$ and $d_j=c_{n-j}$, $j=0, \dots, n$. Then

$$\sum_{j,k=0}^{n} \|(S^{-1})^{j+k}f\|^2 c_j\bar{c}_k = \sum_{j,k=0}^{n} \|S^{j+k}g\|^2 d_j\bar{d}_k \geqq 0$$

and

$$\sum_{j,k=0}^{n} \|(S^{-1})^{j+k+1}f\|^2 c_j\bar{c}_k = \sum_{j,k=0}^{n} \|(S^{-1})^{j+k}h\|^2 c_j\bar{c}_k \geqq 0$$

which means that S^{-1} satisfies (S) for each $f\in\mathfrak{D}(S)$.

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