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Reflexive lattices of operator ranges with more than one generator

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Introduction. A linear submanifold (subspace, not necessarily closed) in a Hilbert space $\mathfrak{H}$ is an operator range (paraclosed subspace) if it is the range of some bounded operator on $\mathfrak{H}$—some member of $B(\mathfrak{H})$ (the algebra of bounded linear operators on $\mathfrak{H}$). We refer the interested readers to the article [2] of FILLMORE and WILLIAMS for detailed discussions of operator ranges. Since the publication of the pioneering work of FOIAŞ [3] on operator ranges invariant under algebras of operators, much progress has been made by many authors in this direction. However, there are few concrete examples of reflexive lattices of operator ranges have been explicitly described. A reflexive lattice of operator ranges is the lattice of all operator ranges invariant under an algebra of operators. The extreme case of singly generated lattices are described in [6] in terms of the generators. In [1] a description of the operator range lattice invariant under a reflexive algebra with commutative invariant subspace lattice is given. Here we describe the reflexive lattice of operator ranges in terms of the generators. All lattices here will be lattices of operator ranges.

Main results. For fixed positive operators $P_1, P_2, \ldots, P_n$, the reflexive lattice generated by (the ranges of) $P_1, P_2, \ldots, P_n$ will be denoted by $\mathrm{RL}(P_1, P_2, \ldots, P_n)$. This is the lattice invariant under the algebra of operators leaving the ranges $P_1\mathfrak{H}, P_2\mathfrak{H}, \ldots, P_n\mathfrak{H}$ invariant. We wish to represent this lattice as ranges of functions of the generators as in [5] for the case of single generator. For $a>0$, the set of all continuous concave nonnegative nondecreasing functions on $[0, a]$ will be denoted by $\mathbf{K}[0, a]$.

Theorem. *Let $P_1, P_2, \ldots, P_n$ be commuting positive operators on $\mathfrak{H}$ such that there is an orthogonal decomposition $\sum_{j=1}^{n} \oplus \mathfrak{H}_j$ of $\mathfrak{H}$, reducing for every P_i, such that*

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the restriction of P_j to the orthogonal complement $\sum_{j\neq i}\oplus\mathfrak{H}_j$ *of* $\mathfrak{H}_i$ *is invertible,* $i=1,2,\dots,n$. *Then the following conditions on an operator range* $\mathfrak{R}$ *are equivalent:*

(i) $\mathfrak{R}\in\mathrm{RL}(P_1,P_2,\dots,P_n)$,

(ii) $\mathfrak{R}=(\prod_j \varphi_j(P_j))\mathfrak{H}$, *for some* $\varphi_j\in\mathrm{K}[0,\|P_j\|]$, $j=1,2,\dots,n$.

Proof. Let $\mathscr{A}$ be the algebra of operators on $\mathfrak{H}$ that leave the ranges of $P_1,P_2,\dots,P_n$ invariant, and let $\mathfrak{R}$ be an $\mathscr{A}$-invariant operator range. Since the von Neumann algebra generated by $P_1,\dots,P_n$ and the projections onto $\mathfrak{H}_1,\mathfrak{H}_2,\dots,\mathfrak{H}_n$ is commutative and is contained in $\mathscr{A}$, $\mathfrak{R}$ is the range of some operator in the commutant of this commutative von Neumann algebra by a result of FOIAŞ [3, Lemma 8, p. 890]. It follows that $\mathfrak{R}=(\mathfrak{R}\cap\mathfrak{H}_1)+(\mathfrak{R}\cap\mathfrak{H}_2)+\dots+(\mathfrak{R}\cap\mathfrak{H}_n)$ and each $\mathfrak{R}\cap\mathfrak{H}_i$ is an operator range in the reflexive lattice generated by the range of $P_i|\mathfrak{H}_i$. By [6] Theorem 8, $\mathfrak{R}\cap\mathfrak{H}_i$ is the range of some operator of the form $\varphi_i(P_i|\mathfrak{H}_i)$, where φ_i is in $\mathrm{K}[0,\|P_i|\mathfrak{H}_i\|]$. Extend φ_i to all of $[0,\|P_i\|]$ by defining $\varphi_i(t)=\varphi_i(\|P_i|\mathfrak{H}_i\|)$ for all $t\in(\|P_i|\mathfrak{H}_i\|,\|P_i\|]$. Then φ_i is still a concave function and $\varphi_i(P_i)$ is defined. We claim that $\mathfrak{R}=(\varphi_1(P_1)\dots\varphi_n(P_n))\mathfrak{H}$.

To see the inclusion $\mathfrak{R}\subseteq(\varphi_1(P_1)\dots\varphi_n(P_n))\mathfrak{H}$, let $x\in\mathfrak{R}$. Then $x=x_1+x_2+\dots+x_n$, where $x_i\in\mathfrak{R}\cap\mathfrak{H}_i$, $i=1,2,\dots,n$. We note that if one of $\mathfrak{R}\cap\mathfrak{H}_i\neq\{0\}$, then $\mathfrak{R}\cap\mathfrak{H}_j\neq\{0\}$ for all j. Indeed, let $x_i\in\mathfrak{R}\cap\mathfrak{H}_i$, $x_i\neq 0$. For each j, let $x_j\in P_j\mathfrak{H}_j$, $x_j\neq 0$ (assuming $P_j\mathfrak{H}_j\neq\{0\}$, otherwise we can omit P_j from the discussion at the beginning). Define the operator $Ax=(x,x_i)x_j$ for $x\in\mathfrak{H}$. Then $A\mathfrak{H}\subseteq P_j\mathfrak{H}_j$. Thus $AP_k\mathfrak{H}\subseteq P_k\mathfrak{H}$, $k=1,2,\dots,n$ (since $P_k\mathfrak{H}_j=\mathfrak{H}_j$ for $j\neq k$). Therefore $A\in\mathscr{A}$. Thus, $A\mathfrak{R}\subseteq\mathfrak{R}$ and hence $P_j\mathfrak{H}_j\subseteq\mathfrak{R}$ for all $j=1,2,\dots,n$. In particular, $\mathfrak{R}\cap\mathfrak{H}_j\neq\{0\}$. For a fixed $i=1,2,\dots,n$, it is easy to see that $\varphi_i(P_i)|\mathfrak{H}_j$ is invertible for all $j\neq i$; and $(\prod_{j\neq i}(\varphi_j(P_j)|\mathfrak{H}_i)^{-1})x_i\in\mathfrak{R}\cap\mathfrak{H}_i$. Thus, there is a $y_i\in\mathfrak{H}_i$ such that $\varphi_i(P_i)y_i=(\prod_{j\neq i}(\varphi_j(P_j)|\mathfrak{H}_i)^{-1})x_i$. Let $y=y_1+\dots+y_n$. Then $(\prod_{j=1}^n\varphi_j(P_j))y=x$. Therefore $\mathfrak{R}\subseteq(\varphi_1(P_1)\dots\varphi_n(P_n))\mathfrak{H}$.

To see the opposite inclusion, let $y=(\varphi_1(P_1)\dots\varphi_n(P_n))x$, for some $x\in\mathfrak{H}$. Write $x=x_1+x_2+\dots+x_n$ where $x_i\in\mathfrak{H}_i$. Let $z_i=(\prod_{j\neq i}\varphi_j(P_j))x_i\in\mathfrak{H}_i$.

Then obviously, $y=\sum_{i=1}^n\varphi_i(P_i)z_i$ is an element of $\varphi_1(P_1)\mathfrak{H}_1+\dots+\varphi_n(P_n)\mathfrak{H}_n=(\mathfrak{R}\cap\mathfrak{H}_1)+\dots+(\mathfrak{R}\cap\mathfrak{H}_n)=\mathfrak{R}$. So $\mathfrak{R}\supseteq(\varphi_1(P_1)\dots\varphi_n(P_n))\mathfrak{H}$. Thus equality holds. This proves the implication (i)⇒(ii). For the converse we note that $(\prod_{j=1}^n\varphi_j(P_j))\mathfrak{H}=\bigcap_{j=1}^n(\varphi_j(P_j))\mathfrak{H}$, and each $\varphi_j(P_j)\mathfrak{H}$ is $\mathscr{A}$-invariant. The proof is thus complete.

In the special case of $\mathfrak{H}=L^2[0,1]$, we have more definite conclusions when the generators are some special multiplication operators. To simplify the statement, we introduce some notations. For a function f on $[0, 1]$, $Z(f)$ denotes the set of zeros of f. For a sequence of functions $f_1, f_2, \dots, f_n$, $Z(f_1, f_2, \dots, f_n) = Z(f_1) \cup \cup Z(f_2) \cup \dots \cup Z(f_n)$. If G is an open set relative to $[0, 1]$, G is a disjoint union of open intervals together with perhaps one or both of $[0, \alpha)$ and $(\beta, 1]$ for some $\alpha, \beta \in (0, 1)$. A nonnegative continuous function is concave on G if the restriction to each component of G is concave (chords below graph). The set of all such functions will be denoted by $\mathbf{C}(G)$. For each $\varphi \in L^\infty[0, 1]$, the multiplication operator on $L^2[0, 1]$ induced by φ will be denoted by M_φ. The symbol x denotes the identity function on $[0,1]$, and 1 the constant function sending every $t \in [0, 1]$ to 1.

Corollary 1. $\mathrm{RL}(M_x, M_{1-x}) = \{M_\varphi \mathfrak{H}:\ \varphi \in \mathbf{C}([0, 1]),\ Z(\varphi) \subseteqq \{0, 1\}\}$.

Proof. Let $\mathfrak{R} \in \mathrm{RL}(M_x, M_{1-x})$. Then by the proof of the above theorem $\mathfrak{R} = (M_\varphi M_{\psi^\sim})\mathfrak{H}$, where $\psi^\sim(t) = \psi(1-t)$, and φ, ψ are nonnegative, nondecreasing concave functions on $[0, 1]$. Since the functions φ and ψ are nonzero (assuming $\mathfrak{R} \neq \{0\}$) the restrictions $\varphi|[1/2, 1]$ and $\psi|[0, 1/2]$ are bounded from below, we may replace them by a constant functions, viz: the functions taking the constant values $\varphi(1/2)$ and $\psi(1/2)$ on $[1/2, 1]$ and $[0, 1/2]$ respectively. Then it is obvious that $M_\varphi M_{\psi^\sim} = M_{\varphi\psi^\sim}$ and $\varphi\psi^\sim$ is concave near the points 0 and 1. By replacing the restriction of $\varphi\psi^\sim$ to an interval $[\alpha, \beta]$, $\alpha, \beta \in (0, 1)$ by a suitable linear function, we may assume that $\varphi\psi^\sim$ is a concave function on all of $[0, 1]$. Thus, the inclusion $\subseteqq$ of the sets in the corollary holds. The opposite inclusion follows from a result of [4] (see [5, Theorem B]).

With a suitable modification, the above proof can be adapted to a proof of the following

Corollary 2. *Let* $f_1, f_2, \dots, f_n$ *be nonnegative continuous functions on* $[0, 1]$ *such that* $Z(f_1), \dots, Z(f_n)$ *are pairwise disjoint. Then* $\mathrm{RL}(M_{f_1}, \dots, M_{f_n}) = \{M_\varphi \mathfrak{H}:\ \varphi \in \mathbf{C}([0; 1]/Z(f_1, \dots, f_n)),\ Z(\varphi) \subseteqq Z(f_1, \dots, f_n)\}$.

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