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Generalized projections for hyponormal and subnormal operators

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0. Sufficient conditions for the existence of certain invariant subspaces of a pure hyponormal operator, T, are obtained. In case T is also subnormal these subspaces are even reducing. In particular, a pure subnormal operator T is shown to be reducible in case $\sigma(T)$ is bisected by the imaginary axis and if, in addition, that part of $\sigma(T)$, which has a projection onto the real axis lying in the absolutely continuous support of $\operatorname{Re}(T)$, is sufficiently sparse near the imaginary axis.

1. Let T be a pure hyponormal operator on the separable Hilbert space $\mathscr{H}$. Thus, $T^*T \geqq TT^*$ and there is no nontrivial reducing subspace of T on which T is normal. In particular, $\sigma_p(T)$ is empty. Let C be a rectifiable, positively oriented, simple closed curve separating the spectrum $\sigma(T)$; thus, $\sigma(T)$ intersects both $\operatorname{int} C$ and $\operatorname{ext} C$, the interior and exterior, respectively, of C. It may be noted that, in general, the set $C \cap \sigma(T)$ may have positive (arc length on C) measure. There will be proved the following

Theorem 1. *Let T be purely hyponormal on $\mathscr{H}$ and satisfy*

$$\int_C \|(T-t)^{-1}x\|\,|dt| < \infty, \quad x \in \mathscr{X}, \tag{1.1}$$

where $\mathscr{X}$ is a set dense in $\mathscr{H}$. Then there exists a linearly independent pair of invariant subspaces $\mathscr{M}_i$ and $\mathscr{M}_e$ of T for which $\mathscr{H} = \mathscr{M}_i \vee \mathscr{M}_e$ and

$$\sigma(T|M_i) = (\sigma(T) \cap \operatorname{int} C)^- \quad \textit{and} \quad \sigma(T|M_e) = (\sigma(T) \cap \operatorname{ext} C)^-. \tag{1.2}$$

Further, in case T is also subnormal, $\mathscr{M}_i$ and $\mathscr{M}_e$ are reducing subspaces of T on $\mathscr{H} = \mathscr{M}_i \oplus \mathscr{M}_e$.

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Proof of Theorem 1. Define the "projection" P_C by

$$P_C x = -(2\pi i)^{-1} \int_C (T-t)^{-1} x \, dt, \quad x \in \mathscr{X}, \tag{1.3}$$

so that, by (1.1), $(P_C x, y) = -(2\pi i)^{-1} \int_C ((T-t)^{-1}x, y) \, dt$ is defined as a Lebesgue integral for any x in $\mathscr{X}$ and y in $\mathscr{H}$. Clearly, it may be assumed that $\mathscr{X}$ is a linear manifold. If $\mathscr{M}_i$ and $\mathscr{M}_e$ are the respective closures of the linear manifolds $P_C\mathscr{X}$ and $(I-P_C)\mathscr{X}$, then, in particular, $\mathscr{M}_i$ and $\mathscr{M}_e$ are hyperinvariant subspaces of T. Relation (1.2) now follows from a proof analogous to that of [5], pp. 13—14, and will be omitted. (The set L and the curve C_R of [5] correspond to the present $\mathscr{X}$ and C.) A crucial part of the argument in [5] is that the set $\{x: \sigma_T(x) \subset \sigma\}$ is a subspace whenever σ is any nonempty compact subset of the plane and $\sigma_T(x)$ is the local spectrum of any vector x in $\mathscr{H}$. This result is due to STAMPFLI [7] (p. 288, see also p. 295) in case T^* has no point spectrum and to RADJABALIPOUR [6] in the general case.

Also, $\mathscr{M} \equiv \mathscr{M}_i \cap \mathscr{M}_e = \{0\}$. For if $\mathscr{M} \neq \{0\}$, then $\sigma(T|\mathscr{M}) \subset C$ and hence $\sigma(T|\mathscr{M})$ has (area) measure zero. Consequently (cf. [3]), $\mathscr{M} \neq \{0\}$ is a reducing space of T on which T is normal, in contradiction to the hypothesis that T is purely hyponormal.

Before completing the proof of the remainder of Theorem 1 when T is subnormal, there will be proved the following

Lemma. *If T is a pure hyponormal operator satisfying* (1.1) *then*

$$\bar{t} \notin \sigma_p(T^*) \quad \text{for} \quad t \in C - Z, \tag{1.4}$$

where Z is a subset of C of (arc length) measure zero. In case T is also subnormal on $\mathscr{H}$ with the minimal normal extension $N = \int z \, dE_z$ on $\mathscr{K} \supset \mathscr{H}$, then

$$E(C) = 0. \tag{1.5}$$

Proof of Lemma. As noted above, since T is purely hyponormal, $\sigma_p(T)$ is empty. Further, by (1.1), for x fixed in $\mathscr{X}$ and for almost all t on C, $y_t = (T-t)^{-1}x$ is defined. Thus, for each x in $\mathscr{X}$, there exists a set $Z(x)$ on C of arc length measure zero and with the property that $x \in R(T-t)$ for $t \in C - Z(x)$. If $\{x_1, x_2, \ldots\}$ is a countable subset of $\mathscr{X}$ which is dense in $\mathscr{H}$ then $Z = \bigcup_{k=1}^{\infty} Z(x_k)$ is also a zero set. Thus, $\mathscr{R}(T-t)$ is dense in $\mathscr{H}$ for all t in $C - Z$ and, in particular, relation (1.4) follows.

Next, relation (1.5) will be established when T is also subnormal. Let x be any vector in $\mathscr{X}$. For t in $C - Z(x)$ one has $y_t = (T-t)^{-1}x$, hence $x = (T-t)y_t = (N-t)y_t$, and so $(T-t)^{-1}x = \int_{\sigma(N)} (z-t)^{-1} dE_z x$. (Note that $E(\{t\})x = 0$.) Consequently, for any u in $\mathscr{K}$, an application of the Schwarz inequality and (1.1) yields

$$\int_C \Big(\int_{\sigma(N)} |z-t|^{-1} |d(E_z x, u)| \Big) |dt| \le \int_C \Big(\int_{\sigma(N)} |z-t|^{-2}\, d\|E_z x\|^2 \Big)^{1/2} \Big(\int_{\sigma(N)} d\|E_z u\|^2 \Big)^{1/2} |dt| =$$

$$= \Big(\int_C \|(T-t)^{-1} x\| \, |dt| \Big) \|u\| < \infty.$$

$\Big($Note that $\int_C = \int_{C-Z(x)}$.$\Big)$ Consequently, in view of Fubini's theorem,

$$\int_{\sigma(N)} \Big(\int_C |t-z|^{-1} |dt| \Big) |d(E_z x, u)| < \infty. \tag{1.6}$$

However, $\int_C |t-z|^{-1}\,|dt| = \infty$ for all z on C. (In fact, otherwise, there would exist some z^* on C for which $\int_C |t-z^*|^{-1}\,|dt| < \infty$. However, z^* is not an atom of the measure $|dt|$ on C and so $1 \le \int_{t^*}^{z^*} |t-z^*|^{-1}\,|dt| \to 0$ as $t^* \to z^*$, a contradiction.) Hence, by (1.6), $(E(C)x, u) = 0$ for u arbitrary in $\mathscr{K}$ and x arbitrary in $\mathscr{X}$. Thus, for x in $\mathscr{X}$, $E(C)x = 0$ and hence also $0 = N^{*k}E(C)x = E(C)N^{*k}x$ ($k = 0, 1, 2, \ldots$). Since $\mathscr{X}$ is dense in $\mathscr{H}$ and N is the minimal normal extension of T, the linear span of $\{N^{*k}\mathscr{X}\}$ ($k = 0, 1, 2, \ldots$) is dense in $\mathscr{K}$ and (1.5) follows. This completes the proof of the Lemma.

The assertion of Theorem 1 when T is purely subnormal now follows from the above Lemma and Corollary 1 of [4], p. 106. In fact, only the hypothesis (5.1) of Corollary 1, corresponding to (1.1) of the present paper, is need to ensure the validity of the assertion of Corollary 1. Indeed, the remaining hypotheses there, namely, that $\{z \in C: \bar{z} \in \sigma_p(T^*)\}$ has measure zero and that $E(C) = 0$, are consequences of (1.1), in view of the Lemma. For completeness, however, an alternate proof of the assertion of Theorem 1 when T is purely subnormal will be given below.

By the Lemma, $E(C) = 0$, and so for x in $\mathscr{X}$ and u in $\mathscr{K}$, one has, by Fubini's theorem,

$$(P_C x, u) = \int_{\sigma(N)-C} \Big[-(2\pi i)^{-1} \int_C (z-t)^{-1}\, dt \Big]\, d(E_z x, u) = \int_{\sigma(N)-C} \Phi(z)\, d(E_z x, u),$$

where $\Phi(z)$ is the characteristic function of $\operatorname{int} C$. Thus, $(P_C x, u) = (E(\operatorname{int} C)x, u)$ for all u in $\mathscr{K}$ and so $P_C x = E(\operatorname{int} C)x$ for all x in $\mathscr{X}$. Let P denote the orthogonal projection $P: \mathscr{K} \to \mathscr{H}$. Since the (orthogonal projection) $E(\operatorname{int} C)$ is bounded on $\mathscr{K}$ and $E(\operatorname{int} C)x = P_C x \in \mathscr{H}$ for x in $\mathscr{X}$, then clearly $E(\operatorname{int} C)P = PE(\operatorname{int} C)P\big(= PE(\operatorname{int} C)\big)$. Thus $E' = E(\operatorname{int} C)|\mathscr{H}$ is an orthogonal projection and $E'|\mathscr{X} = P_C$. Since $TP_C x = P_C Tx$ for x in $\mathscr{X}$ and $\mathscr{X}$ is dense in $\mathscr{H}$, then T commutes with E'. Further, it is clear that $E'\mathscr{H} = \mathscr{M}_i$, and so the spaces $\mathscr{M}_i$ and $\mathscr{M}_e$ defined earlier reduce T and $\mathscr{H} = \mathscr{M}_i \oplus \mathscr{M}_e$. This completes the proof of Theorem 1.

2. For use below, note that if T is purely hyponormal then $\operatorname{Re}(T)$ is absolutely continuous; see [2], p. 46.

Theorem 2. *Let T be purely subnormal on $\mathcal{H}$ and suppose that $\sigma(T)$ intersects both the right and left open half planes $R=\{z\colon \operatorname{Re}(z)>0\}$ and $L=\{z\colon \operatorname{Re}(z)<0\}$. In addition, let*

$$\int_{\alpha} t^{-2}F(t)\,dt < 2\pi, \tag{2.1}$$

where α is the absolutely continuous support of $\operatorname{Re}(T)$ *and $F(t)$ is the linear measure of the vertical cross section $\sigma(T)\cap\{z\colon \operatorname{Re}(z)=t\}$ of $\sigma(T)$. Then there exist subspaces $\mathcal{M}_R$ and $\mathcal{M}_L$ of $\mathcal{H}$ reducing T, satisfying $\mathcal{H}=\mathcal{M}_R\oplus\mathcal{M}_L$ and*

$$\sigma(T|\mathcal{M}_R) = (\sigma(T)\cap\mathcal{R})^- \quad \textit{and} \quad \sigma(T|M_L) = (\sigma(T)\cap L)^-.$$

Theorem 2 follows from Theorem (*) and its proof in [5] and from Theorem 1 above. In fact, let C denote the positively oriented boundary of the semicircular disk $\{z\colon \operatorname{Re}(z)>0,\ |z|<r\}$, where $r>0$ is chosen so large that $\sigma(T)\subset\{z\colon |z|<r\}$. It was shown in [5] that $\mathcal{K}$ of Theorem 1 above can now be chosen so as to contain the range of $E^A(\beta)$ where $\{E^A\}$ is the spectral family of $A=\operatorname{Re}(T)$ and β is any Borel set of the real line whose closure does not contain 0. This completes the proof of Theorem 2.

For other sufficient conditions ensuring the reducibility of a subnormal operator see the references in Conway [1], pp. 299—300.

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