Acta Sci. Math., 53 (1989), 233—236

Two finiteness conditions for finitely generated and periodic semigroups

GIUSEPPE PIRILLO

1. Introduction. In this paper we present two finiteness conditions for a finitely generated and periodic semigroup. The first condition requires that the function which counts the number of elements of the first i generations grows less rapidly than $i(i+3)/2$. The second one requires that the semigroup be repetitive and that there should exist a positive integer p such that each element of the semigroup has order smaller than p.

2. Notations and preliminaries. Let A be an *alphabet*, A^+ (resp. A^*) the *free semigroup* (resp. *free monoid*) on A. For any *word* $w \in A^+$, $|w|$ will be the *length* of w. A word v is a *factor* of a word w if there exist two words $u, u' \in A^*$ such that $w = uvu'$.

Let S be a *semigroup*, G a finite set of *generators* of S and $\bar{G}$ be a copy of G. Let φ: $\bar{G}^+ \to S$ be the (epi-)morphism defined by $\varphi(\bar{g}) = g$, for each $\bar{g} \in \bar{G}$. Suppose that in $\bar{G}$ a total order $<$ is given and consider the lexicographic order induced by $<$ on $\bar{G}^i$, for each positive integer i (i.e., given two words $w, w' \in \bar{G}^i$ we say that w *precedes* w' *in the lexicographic order* if there exists a positive integer j, $1 \leqq j \leqq i$, such that

$$w = ua_jv, \quad w' = ub_jv'$$

where u, v, u' are words of $\bar{G}^*$, a_j and b_j are letters of G such that $a_j < b_j$).

Definition 1. We say that a word $w \in G^+$ is the canonical word of an element $s \in S$ if:

1) $\varphi(w) = s$,

2) for any other word $w' \in \bar{G}^+$ such that $\varphi(w') = s$ we have either

a) $|w| < |w'|$, or

b) $|w| = |w'|$ and w precedes w' in the lexicographic order.

Received January 24, 1986.

Fact 1. *A factor of a canonical word is a canonical word.*

Namely, if v is a factor of w and w is the canonical word of an element $s \in S$, then there exists another element $s' \in S$ such that v is the canonical word of s'.

Now consider the following subsets of S:

$$G^i = \varphi(\bar{G}^i), \quad P_i = \bigcup_{j=1}^{i} G^j, \quad R_i = P_i - P_{i-1}$$

(where $P_0 = \emptyset$) and the functions p, r from the set of positive integers into the set of positive integers defined by

$$p(i) = \operatorname{card} P_i, \quad r(i) = p(i) - p(i-1),$$

for each positive integer i.

Definition 2. We say that a finitely generated semigroup has linear growth if there exists a positive integer k such that $p(i) \leqq ki$, for each positive integer i.

For future reference we state below without proof a theorem due to Justin [4].

Theorem 1. *For a finitely generated semigroup, the following conditions are equivalent.*

a) *There exists a finite subset F of $\bar{G}^+$ such that the canonical word of each element of the semigroup belongs to F or has a factorization $w = uv^n u'$ where $u, v, u' \in F$ and n is a positive integer.*

b) *There exists a positive integer m such that $r(i) \leqq m$, for each positive integer i.*

c) *The semigroup has linear growth.*

d) *There exists a positive integer i such that $p(i) < i(i+3)/2$.*

e) *There exists a positive integer d such that $r(d) \leqq d$.*

3. Two conditions of finiteness for finitely generated semigroup. The Burnside problem for semigroups has been recently studied by several authors (see, for example, de Luca [2], de Luca and Restivo [3], Restivo and Reutenauer [6]).

We present here two conditions which are natural in the study of repetitive semigroups (see definition below) and are necessary and sufficient conditions for the finiteness of finitely generated and periodic semigroups.

Our first result is the following proposition.

Proposition 1. *Let S be a finitely generated semigroup. The following conditions are equivalent:*

a) *S is finite.*

b) *S is periodic and has linear growth.*

Proof. (a)$\to$(b) is trivial. Using Theorem 1, the proof of (b)$\to$(a) is just a remark. In fact, the finiteness of F (see condition a) of Theorem 1 and the periodicity of S gives a suitable positive integer q such that

$$S = \varphi(F)\cup\varphi(\{Fv^nF:\ v\in F,\ n \leqq q\}),$$

that is S is finite.

So, we have proved, without much effort, that if S is a periodic semigroup such that $p(i)<i(i+3)/2$ for a suitable non-negative integer i, then S is finite.

Now, let us introduce following definition.

Definition 3. Given a (finite) alphabet A and a semigroup S, a morphism α: $A^+\to S$ is called repetitive if for each integer k there exists a positive integer $l_\alpha(k)$ such that each word $w\in A^+$ of length at least $l_\alpha(k)$ can be factorized as follows:

$$w = w_0w_1 \dots w_kw_{k+1}$$

where $w_0, w_{k+1}\in A^*$, $w_1, w_2, \dots, w_k\in A^+$, and $\alpha(w_1)=\alpha(w_2)=\dots=\alpha(w_k)$.

Definition 4. A semigroup S is called repetitive if, for each finite alphabet A, each morphism α: $A^+\to S$ is repetitive.

We can prove the following proposition.

Proposition 2. *Let S be a finitely generated semigroup. The following conditions are equivalent:*

a) *S is finite.*

b) *S is periodic, repetitive and there exists a positive integer p such that each element of S has order at most p.*

Proof. The only non-trivial part of (a)$\to$(b) is "S finite"$\to$"S repetitive". This has been proved by JUSTIN [5] (see also [7]).

(b)$\to$(a). Let G be a finite set of generators of S. Let $\bar{G}$ and φ: $\bar{G}\to S$ be as in the preceding paragraph. By way of contradiction, let S be infinite. We have that the subset of $\bar{G}^+$ of the canonical words of the elements of S is infinite and so there exists a canonical word w of length greater that $l_\varphi(p+1)$.

By the repetitivity of φ we have $w=w_0w_1\dots w_pw_{p+1}w_{p+2}$ where $w_0, w_{p+2}\in\bar{G}^*$, $w_1, \dots, w_p, w_{p+1}\in\bar{G}^+$ and $\varphi(w_1)=\dots=\varphi(w_p)=\varphi(w_{p+1})$. Now considering the property of p one easily sees that the word $w_1w_2\dots w_pw_{p+1}$ is too long to be a canonical word of an element of S. This is in contradiction with Fact 1.

Remark. In the proof of Proposition 2 we can make use only of the repetitivity of the epimorphism φ.

Proposition 2 provides us with one of the few criteria to establish if an infinite semigroup is repetitive.

Let us show that the semigroup $S=A^+\cup\{0\}/\sim$ (where A is a finite alphabet with at least three elements, 0 is a zero and $\sim$ is the congruence generated by the relation R on A^+ defined by $ww\ R\ 0$ for each $w\in A^+$) is non-repetitive.

In fact, the semigroup S is infinite (this is a consequence of the Thue construction of infinite square-free words over each alphabet with at least three elements, see [1]), evidently periodic and its elements have at most order 2. So, by Proposition 2, S cannot be repetitive.

On the contrary, the semigroup $S'=A'/\approx$ (where A is a finite alphabet, $\approx$ is the congruence generated by the relation R' defined by $ww\ R'\ w$ for each $w\in A^+$) is finite (see again [1]) and therefore repetitive (see [5]).

Acknowledgements. The author would like to thank J. Justin for his helpful comments in the preparation of this work.

References

[1] J. Berstel, Chapter 2 in: M. Lothaire, *Combinatorics on word,* Addison—Wesley (Reading, Mass. 1983).

[2] A. de Luca, On the Burnside problem for semigroups and languages, Atti Sem. Mat. Fis. Univ. Milano, **53** (1983).

[3] A. de Luca and A. Restivo, A finiteness condition for finitely generated semigroups, *Semigroup Forum,* **28** (1984), 123—134.

[4] J. Justin, Groupes et semigroupes à croissance lineaire, *C. R. Acad. Sci. Paris,* **273** (1971), 212—214.

[5] J. Justin, Generalization du Théorème de van der Waerden sur les semigroupes répétitifs, *J. Combin. Theory Ser. A,* **12** (1972), 357—367.

[6] A. Restivo and C. Reutenauer, On the Burnside problem, *J. Algebra,* **89** (1984), 102—104.

[7] M. P. Schützenberger, *Quelques problèmes combinatoires de la théorie des automates* (rédigé par J.-F. Perrot) Cours professé à l'Institut de Programmation, Fac. Sciences de Paris (1966—67).

UNIVERSITA DEGLI STUDI
INSTITUTO MATEMATICO „ULISSE DINI"
VIALE MORGAGNI 87/A
50134 FIREZNE, ITALY