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On the convergence of the differentiated trigonometric projection operators

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Let $C_{2\pi}$ be the set of 2π-periodic continuous functions and $\mathcal{T}_n$ the set of trigonometric polynomials of order at most n. We will consider projection operators $P_n \in C_{2\pi} \to \mathcal{T}_n$, i.e. linear operators $P_n(f, t)$ with the properties

(i) $P_n(f, t) \in \mathcal{T}_n$ if $f \in C_{2\pi}$

(ii) $P_n(f, t) \equiv f(t)$ if $f \in \mathcal{T}_n$.

Let r be a nonnegative integer, and consider the r times differentiated operator $P_n^{(r)}(f, t)$. One may ask: under what conditions will this operator uniformly converge to $f^{(r)}(t)$? To state a result in the positive direction, we need some definitions. Let

(1) $$\|P_n^{(r)}\| := \sup_{0 \not\equiv f \in C_{2\pi}} \frac{\|P_n^{(r)}(f, t)\|}{\|f\|}$$

be the norm of the r times differentiated operator ($\|\cdot\|$ denotes supremum norm over the real line), and let $E_n(g)$ be the best (uniform) trigonometric approximation of order n of $g \in C_{2\pi}$.

Theorem 1. *If $f^{(r)}(t)$ is continuous and $P_n \in C_{2\pi} \to \mathcal{T}_n$, then*

$$\|f^{(r)}(t) - P_n^{(r)}(f, t)\| = O(E_n(f^{(r)}) + E_n(f)\|P_n^{(r)}\|).$$

Here the O-sign refers to $n \to \infty$, while r is fixed. Hence a sufficient condition of the uniform convergence is

$$\lim_{n \to \infty} E_n(f)\|P_n^{(r)}\| = 0.$$

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Proof of Theorem 1. Let $T_n(t)$ be the best approximating polynomial of $f(t)$. Then according to a result of Czipszer and Freud [2] on simultaneous approximation

$$\|f^{(k)}(t)-T_n^{(k)}(t)\| \leqq c_0 E_n(f^{(k)}) \quad (k=0,1,\ldots,r).^{1)}$$

Using this result, as well as property (ii) of the projection operator P_n we get

$$\|f^{(r)}(t)-P_n^{(r)}(f,t)\| \leqq \|f^{(r)}(t)-T_n^{(r)}(t)\| + \|\{T_n(t)-P_n(T_n,t)\}^{(r)}\| +$$
$$+\|P_n^{(r)}(T_n-f,t)\| \leqq c_0 E_n(f^{(r)})+c_0 E_n(f)\|P_n^{(r)}\|.$$

Now we turn to the divergence phenomena of the operator $P_n^{(r)}(f,t)$. Let $\omega(t)$ be an arbitrary modulus of continuity, and define

$$C_r(\omega)=\left\{f(t)\,|\,f^{(r)}(t)\in C_{2\pi},\ \sup_{t>0}\frac{\omega(f^{(r)},t)}{\omega(t)}<\infty\right\}. \tag{2}$$

Theorem 2. *Given* $r\geqq 0$ *and a modulus of continuity* $\omega(t)$ *such that*

$$\lim_{t\to 0+}\frac{t}{\omega(t)}=0, \tag{3}$$

further a sequence of projection operators $P_n\in C_{2\pi}\to\mathcal{T}_n$, *there exists an* $f_r(t)\in C_r(\omega)$ *such that*

$$\limsup_{n\to\infty}\frac{\|f_r^{(r)}(t)-P_n^{(r)}(f,t)\|}{\omega\left(\frac{1}{n}\right)\log n}>0. \tag{4}$$

For the proof of Theorem 2 we need the following

Lemma. *Given r and n, there exists a function* $g_{nr}(t)\in C_{2\pi}$ *such that*

$$\|g_{nr}^{(j)}(t)\| \leqq c_1 n^j \quad (j=0,1,\ldots,r+1) \tag{5}$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g_{nr}^{(r)}(t)D_n(t)\,dt \geqq c_2 n^r \log n, \tag{6}$$

where

$$D_n(t)=\frac{\sin\frac{2n+1}{2}t}{2\sin\frac{t}{2}} \tag{7}$$

is the Dirichlet kernel.

[1] In what follows $c_0, c_1, \ldots$ will denote constants depending on r but independent of n.

Proof. We distinguish two cases.

Case 1. r is odd. Then let

$$g_{nr}(t) = (-1)^{(r+1)/2}(\operatorname{sgn} t)\cos\frac{2n+1}{2}t \quad \text{if} \quad \frac{2\pi}{2n+1} \leqq |t| \leqq \frac{2n\pi}{2n+1}. \tag{8}$$

To extend the definition of $g_{nr}(t)$ for $|t|<2\pi/(2n+1)$, let $h_{nr}(t)$ be that uniquely determined algebraic polynomial of degree at most $2r+3$ which satisfies the conditions

$$h_{nr}^{(j)}\left(-\frac{2\pi}{2n+1}\right) = g_{nr}^{(j)}\left(-\frac{2\pi}{2n+1}\right), \quad h_{nr}^{(j)}\left(\frac{2\pi}{2n+1}\right) = g_{nr}^{(j)}\left(\frac{2\pi}{2n+1}\right) \tag{9}$$

$$(j = 0, 1, \ldots, r+1).$$

Then let

$$g_{nr}(t) = h_{nr}(t) \quad \text{if} \quad |t| < \frac{2\pi}{2n+1}. \tag{10}$$

Assume

$$h_{nr}(t) = \sum_{k=0}^{2r+3} a_{kn}\left(\frac{2n+1}{2\pi}t\right)^k, \tag{11}$$

then by (9) and (8)

$$h_{nr}^{(j)}\left(-\frac{2\pi}{2n+1}\right) = \left(-\frac{2n+1}{2\pi}\right)^j \sum_{k=j}^{2r+3} a_{kn}k(k-1)\ldots(k-j+1)(-1)^k =$$

$$= g_{nr}^{(j)}\left(-\frac{2\pi}{2n+1}\right) = O(n^j)$$

$$(j = 0, 1, \ldots, r+1),$$

i.e.

$$\sum_{k=j}^{2r+3} (-1)^k k(k-1)\ldots(k-j+1)a_{kn} = O(1) \quad (j = 0, 1, \ldots, r+1). \tag{12}$$

Similarly, from the second group of conditions in (9),

$$\sum_{k=j}^{2r+3} k(k-1)\ldots(k-j+1)a_{kn} = O(1) \quad (j = 0, 1, \ldots, r+1). \tag{13}$$

(12) and (13) together can be considered as a system of linear equations for the unknowns a_{kn}. Since $h_{nr}(t)$ is uniquely determined, this system is uniquely solvable and

$$|a_{kn}| \leqq c_8 \quad (k = 0, 1, \ldots, 2r+3).$$

Thus by (10) and (11) we get for $j=0, 1, \ldots, r+1$

$$|g_{nr}^{(j)}(t)| = |h_{nr}^{(j)}(t)| \leq$$

$$\leq \left(\frac{2n+1}{2\pi}\right)^j \sum_{k=j}^{2r+3} k(k-1)\ldots(k-j+1)|a_{kn}| \leq c_1 n^j \quad \text{if} \quad |t| \leq \frac{2\pi}{2n+1}.$$

Now $g_{nr}(t)$ is defined on $|t| \leq 2\pi/(2n+1)$, and extending the definition by translations of length 2π, the only missing interval is $\left(\frac{2n\pi}{2n+1}, \frac{2(n+1)\pi}{2n+1}\right)$ (and its translates). In this interval the construction is similar: let $H_{nr}(t)$ be that uniquely determined algebraic polynomial of degree at most $2r+3$ for which

$$H_{nr}^{(j)}\left(\frac{2n\pi}{2n+1}\right) = g_{nr}^{(j)}\left(\frac{2n\pi}{2n+1}\right), \quad H_{nr}^{(j)}\left(\frac{2(n+1)\pi}{2n+1}\right) = g_{nr}^{(j)}\left(\frac{2(n+1)\pi}{2n+1}\right)$$

$$(j = 0, 1, \ldots, r+1),$$

and let

$$g_{nr}(t) = H_{nr}(t) \quad \text{if} \quad \frac{2n\pi}{2n+1} < t < \frac{2(n+1)\pi}{2n+1}.$$

Thus the definition of $g_{nr}(t)$ is complete. Property (5) on the interval $\left[\frac{2n\pi}{2n+1}, \frac{2(n+1)\pi}{2n+1}\right]$ can be easily established.

The only thing remained to prove is (6). Since by (7) $\|D_n(t)\| = n+1/2$, we get from (8) and (5)

$$\frac{1}{\pi}\int_{-\pi}^{\pi} g_{nr}^{(r)}(t)D_n(t)\,dt \geq \frac{2}{\pi}\left(\frac{2n+1}{2}\right)^r \int_{\frac{2\pi}{2n+1}}^{\frac{2n\pi}{2n+1}} \frac{\sin^2\frac{2n+1}{2}t}{2\sin\frac{t}{2}}\,dt -$$

$$-\frac{6}{2n+1}\|g_{nr}^{(r)}(t)D_n(t)\| \geq \frac{1}{\pi}\left(\frac{2n+1}{2}\right)^r \sum_{k=1}^{n-1} \int_{\frac{(4k+1)\pi}{2(2n+1)}}^{\frac{(4k+3)\pi}{2(2n+1)}} \frac{dt}{t} - 3c_1 n^r \geq$$

$$\geq \frac{1}{\pi}\left(\frac{2n+1}{2}\right)^r 2\sum_{k=1}^{n-1} \frac{1}{4k+3} - 3c_1 n^r \geq c_2 n^r \log n.$$

Case 2. r is even. Now the definition of $g_{nr}(t)$ starts with

$$g_{nr}(t) = (-1)^{r/2}(\operatorname{sgn} t)\sin\frac{2n+1}{2}t \quad \text{if} \quad \frac{2\pi}{2n+1} \leq |t| \leq \frac{2n\pi}{2n+1}$$

instead of (8). The rest of the proof is very similar to Case 1, and we omit the details.

Proof of Theorem 2. Since $P_n(f, t)$ is a projection operator, according to the Berman—Faber—Marcinkiewicz relation we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n(f(\cdot+u), x-u)\,du = S_n(f, x)$$

where

$$S_n(f, x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)D_n(t)\,dt$$

is the n^{th} partial sum of the Fourier series of $f(x)$ (see e.g. Lorentz [3], p. 97). Applying this for $f(x)=g_{nr}(x)$, differentiating r times and setting $x=0$ we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_n^{(r)}(g_{nr}(\cdot+u), -u)\,du = \frac{1}{\pi}\int_{-\pi}^{\pi} g_{nr}^{(r)}(t)D_n(t)\,dt.$$

Let u_n be a point where $|P_n^{(r)}(g_{nr}(\cdot+u), -u)|$ attains its maximum, then by (6) we get

(14) $$\|P_n^{(r)}(g_{nr}(\cdot+u_n), t)\| \geqq |P_n^{(r)}(g_{nr}(\cdot+u_n), -u_n)| \geqq c_2 n^r \log n.$$

Now define a sequence of integers $n_1<n_2<\dots$ with the following properties: let

(15) $$\omega\left(\frac{1}{n_1}\right) \leqq \frac{c_2}{8c_1}, \quad n_1 > e^{8/c_2}$$

and assume that $n_1, n_2, \dots, n_{j-1}$ has been already defined.

If there exists a k, $1\leqq k\leqq j-1$, such that for infinitely many n's we have

$$\|g_{n_k r}^{(r)}(t) - P_n^{(r)}(g_{n_k r}(\cdot+u_{n_k}), t)\| \geqq c_1\omega(1/n)\log n$$

then this $g_{n_k r}(t)$ will satisfy the requirements of the theorem. If this is not the case, then for sufficiently large n's

$$\|g_{n_k r}^{(r)}(t) - P_n^{(r)}(g_{n_k r}(\cdot+u_{n_k}), t)\| < c_1\omega(1/n)\log n \quad (k = 1, \dots, j-1).$$

Now choose n_j in this case such that

(16) $$\|g_{n_k r}^{(r)}(t) - P_{n_j}^{(r)}(g_{n_k r}(\cdot+u_{n_k}), t)\| < c_1\omega(1/n_j)\log n_j \quad (k = 1, \dots, j-1)$$

and

(17) $$\frac{2n_{j-1}}{n_j}\,\omega(1/n_{j-1}) \leqq \omega(1/n_j) \leqq \min\left(\frac{1}{2}\,\omega(1/n_{j-1}), \frac{c_2}{4c_1\|P_{n_{j-1}}^{(r)}\|}\right)$$

hold. (The left hand side inequality is possible because of (3).)

We may assume that we can construct an infinite sequence of indices this way. Define

$$f_r(t) = \sum_{k=1}^{\infty} \frac{g_{n_k r}(t+u_{n_k})}{n_k^r}\,\omega(1/n_k).$$

Here the right hand side series, even after differentiating r times, uniformly converges by (5) and (17). Moreover, if $0<\delta\leqq h$ then

$$|f_r^{(r)}(t+\delta)-f_r^{(r)}(t)|\leqq\sum_{k=1}^{\infty}\frac{|g_{n_kr}^{(r)}(t+\delta+u_{n_k})-g_{n_kr}^{(r)}(t+u_{n_k})|}{n_k^r}\,\omega(1/n_k).$$

Let $0<h<1/n_1$ and j be that index for which

$$1/n_{j+1}\leqq h<1/n_j.$$

Then by (5) and (17)

$$|f_r^{(r)}(t+\delta)-f_r^{(r)}(t)|\leqq\sum_{k=1}^{j}\frac{\delta\|g_{n_kr}^{(r)}(t)\|}{n_k^r}\,\omega(1/n_k)+\sum_{k=j+1}^{\infty}\frac{2\|g_{n_kr}^{(r)}(t)\|}{n_k^r}\,\omega(1/n_k)\leqq$$

$$\leqq c_1\delta\sum_{k=1}^{j}n_k\omega(1/n_k)+2c_1\sum_{k=j+1}^{\infty}\omega(1/n_{k+1})\leqq 2c_1hn_j\omega(1/n_j)+4c_1\omega(1/n_{j+1})\leqq 8c_1\omega(h),$$

i.e. $f_r(t)\in C_r(\omega)$ (cf. (2)).

Finally, to show (4) we obtain by (14), (16), (5), (17) and (15)

$$\|f_r^{(r)}(t)-P_{n_j}^{(r)}(f_r,t)\|=\left\|\sum_{k=1}^{\infty}\frac{g_{n_kr}^{(r)}(t+u_{n_k})-P_{n_j}^{(r)}(g_{n_kr}(\cdot+u_{n_k}),t)}{n_k^r}\,\omega(1/n_k)\right\|\geqq$$

$$\geqq\frac{\|P_{n_j}^{(r)}(g_{n_jr}(\cdot+u_{n_j}),t)\|}{n_j^r}\,\omega(1/n_j)-\sum_{k=1}^{j-1}\frac{\|g_{n_kr}^{(r)}(t+u_{n_k})-P_{n_j}^{(r)}(g_{n_kr}(\cdot+u_{n_k}),t)\|}{n_k^r}\,\omega(1/n_k)-$$

$$-\sum_{k=j}^{\infty}\frac{\|g_{n_kr}^{(r)}\|}{n_k^r}\,\omega(1/n_k)-\sum_{k=j+1}^{\infty}\frac{\|P_{n_j}^{(r)}(g_{n_kr}(\cdot+u_{n_k}),t)\|}{n_k^r}\,\omega(1/n_k)\geqq$$

$$\geqq c_2\omega(1/n_j)\log n_j-c_1\omega(1/n_j)\log n_j\sum_{k=1}^{\infty}\omega(1/n_k)-\sum_{k=j}^{\infty}\omega(1/n_k)-$$

$$-c_1\|P_{n_j}^{(r)}\|\sum_{k=j+1}^{\infty}\omega(1/n_k)\geqq c_2\omega(1/n_j)\log n_j-2c_1\omega(1/n_1)\omega(1/n_j)\log n_j-$$

$$-2\omega(1/n_j)-2c_1\|P_{n_j}^{(r)}\|\,\omega(1/n_{j+1})\geqq c_2\omega(1/n_j)\log n_j-\frac{c_2}{4}\,\omega(1/n_j)\log n_j-$$

$$-\frac{c_2}{4}\,\omega(1/n_j)\log n_j-\frac{c_2}{4}\,\omega(1/n_j)\log n_j=\frac{c_2}{4}\,\omega(1/n_j)\log n_j\quad(j=1,2,\ldots).$$

$\omega(t)=o(t)$ is excluded in Theorem 2, by condition (3). With a slight modification of the proof we can easily get the following statement in this case.

Theorem 3. *Given* $r \geqq 0$, *a sequence of projection operators* $P_n \in C_{2\pi} \to \mathcal{T}_n$, *and a sequence* $\varepsilon_1 \geqq \varepsilon_2 \geqq \dots$, $\lim_{n\to\infty} \varepsilon_n = 0$, *there exists an* $f_r(t) \in C_{2\pi}$ *such that* $f_r^{(r)}(t) \in$ $\in \mathrm{Lip}\, 1$ *and*

$$\limsup_{n\to\infty} \frac{\|f_r^{(r)}(t) - P_n^{(r)}(f, t)\|}{\varepsilon_n \log n/n} > 0. \tag{18}$$

We do not give the details of the proof of this theorem. We only mention that now

$$f_r(t) = \sum_{k=1}^{\infty} \frac{\varepsilon_{n_k}}{n_k^{r+1}} g_{n_k r}(t + u_{n_k})$$

will be the function satisfying (18), where $n_1 < n_2 < \dots$ is a properly chosen sequence of indices.

An obvious consequence of Theorem 1 is that if $f(t) \in C_r(\omega)$ then

$$\|f^{(r)}(t) - P_n^{(r)}(f, t)\| = O\big(n^{-r}\omega(1/n)\|P_n^{(r)}\|\big). \tag{19}$$

Since here $\|P_n^{(r)}\| \geqq c_3 n^r \log n$ for any projection operator P_n (cf. Berman [1]), the best estimate one can obtain from (19) is

$$\|f^{(r)}(t) - P_n^{(r)}(f, t)\| = O\big(\omega(1/n) \log n\big) \quad (f(t) \in C_r(\omega)).$$

This shows that the results of Theorems 2 and 3 are sharp.

In particular, our theorems can be applied to the differentiated partial sums of the Fourier series and to the differentiated interpolating polynomials based on arbitrary systems of nodes.

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