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Notes on approximation by Riesz-means

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To my dear colleague L. Pintér on his 60th birthday

1. Let $f=f(x)$ be a continuous 2π-periodic function i.e. $f\in C_{2\pi}$, and let

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \tag{1}$$

be its Fourier series. Denote $s_n=s_n(x)=s_n(f;x)$ and $\sigma_n^\alpha=\sigma_n^\alpha(x)=\sigma_n^\alpha(f;x)$ the n-th partial sum and the n-th (C,α)-mean of (1), respectively, i.e.

$$\sigma_n^\alpha(x)=\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}s_\nu(x),\quad A_n^\alpha=\binom{n+\alpha}{n};$$

furthermore $\tilde{f}$ denotes the conjugate function of f, and $f^{(r)}$ is the r-th derivative of f.

Let $E_n(f)$ denote the best approximation of f by trigonometric polynomials of order at most n in the space $C_{2\pi}$, and let $\|\cdot\|$ denote the usual supremum norm.

We define two important strong means:

$$h_n(f,\beta,p;\ x):=\left\{\frac{1}{(n+1)^\beta}\sum_{k=0}^{n}(k+1)^{\beta-1}|s_k(x)-f(x)|^p\right\}^{1/p}\quad(\beta,p>0),$$

$$\sigma_n^\gamma|f,p;\ x|:=\left\{\frac{1}{A_n^\gamma}\sum_{k=0}^{n}A_{n-k}^{\gamma-1}|s_k(x)-f(x)|^p\right\}^{1/p}\quad(\gamma,p>0).$$

The first result on strong approximation by Fourier series has been connected with the following classical theorem of S. N. BERNSTEIN [3]:

If $f\in\operatorname{Lip}\alpha$ *then*

$$\|\sigma_n^1-f\|=O(n^{-\alpha})\quad\textit{for}\quad 0<\alpha<1 \tag{2}$$

and

$$\|\sigma_n^1-f\|=O(n^{-1}\log n)\quad\textit{for}\quad\alpha=1. \tag{3}$$

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Namely G. ALEXITS and D. KRÁLIK [2] sharpened this theorem by proving that the order of approximation given in estimates (2) and (3) can be achieved for the strong means $h_n(f, 1, 1; x)$, too; i.e.

$f \in \text{Lip}\,\alpha$ implies that

$$\|h_n(f, 1, 1;\ x)\| = \begin{cases} O(n^{-\alpha}) & \text{if } 0 < \alpha < 1, \\ O(n^{-1}\log n) & \text{if } \alpha = 1. \end{cases}$$

Improving further the result of Alexits and Králik we ([4], [5]) proved, among others, the following theorems:

Theorem A. *If $f^{(r)} \in \text{Lip}\,\alpha$, $0<\alpha\leqq 1$, $p>0$ and $\beta>(r+\alpha)p$ then*

$$h_n(f, \beta, p) := \|h_n(f, \beta, p;\ x)\| = O(n^{-r-\alpha}).$$

Theorem B. *If $f^{(r)} \in \text{Lip}\,\alpha$, $0<\alpha\leqq 1$, $p>0$ and $(r+\alpha)p<1$ then for arbitrary $\gamma>0$*

$$\sigma_n^\gamma|f, p| := \|\sigma_n^\gamma|f, p;\ x|\| = O(n^{-r-\alpha}).$$

It is clear that these estimations are best possible, namely, by the well-known result of Jackson $f^{(r)} \in \text{Lip}\,\alpha$ implies that $E_n(f) = O(n^{-r-\alpha})$.

The following theorems show that the conditions $\beta>(r+\alpha)p$ and $(r+\alpha)p<1$ are very essential with respect to the order of approximation. If they are not fulfilled then the strong means do not approximate in the order of best approximation.

Theorem C. *If $f^{(r)} \in \text{Lip}\,\alpha$, $0<\alpha\leqq 1$, $p>0$ and $\beta=(r+\alpha)p$ then we have only*

$$h_n(f, \beta, p) = O\big(n^{-r-\alpha}(\log n)^{1/p}\big).$$

Furthermore there exists a function f_1 such that $f_1^{(r)} \in \text{Lip}\,\alpha$, $0<\alpha\leqq 1$, but

$$h_n(f_1, \beta, p;\ 0) \geqq cn^{-r-\alpha}(\log n)^{1/p} \quad (c>0),$$

holds if n is large enough.

Theorem D. *If $f^{(r)} \in \text{Lip}\,\alpha$, $0<\alpha\leqq 1$, $p>0$, $\gamma>0$ and $(r+\alpha)p=1$ then we have only*

$$\sigma_n^\gamma|f, p| = O\big(n^{-r-\alpha}(\log n)^{1/p}\big).$$

Moreover, there exists a function f_2 such that $f_2^{(r)} \in \text{Lip}\,\alpha$, $0<\alpha\leqq 1$, and

$$\sigma_n^\gamma|f_2, p;\ 0| \geqq dn^{-r-\alpha}(\log n)^{1/p} \quad (d>0)$$

holds for sufficiently large n.

Analogous estimations for the conjugate functions have been proved, but now we do not treat them.

Analysing these results we can see that the strong means $\sigma_n^\gamma|f, p; x|$ behave like $\sigma_n^1|f, p; x|=h_n(f, 1, p; x)$, i.e. the strong means $h_n(f, \beta, p; x)$ are more sensible of parameter β regarding the order of approximation.

This phenomenon raises the following problems: If we consider the following regular ordinary Riesz-means

$$R_n(f, \beta, x) := \frac{\beta_n}{(n+1)^\beta} \sum_{k=0}^{n} (k+1)^{\beta-1} s_k(x) \quad \left(\beta_n^{-1} := (n+1)^{-\beta} \sum_{k=0}^{n} (k+1)^{\beta-1}\right)$$

and take the difference

$$\|R_n(f, \beta; x)-f(x)\|$$

i.e. if we consider the ordinary approximation instead of strong one for the Riesz-means, then at which value of the parameter β will a jump in the order of approximation appear, also at the parameter $\beta=r+\alpha$ $(p=1)$ as in the strong case? If $r=0$, then will the jump be at $\beta=\alpha$ independently of the value of α, regardless whether $\alpha<1$ or $\alpha=1$? The answer is affermative if $r=0$, and this shows that the analogue of Bernstein's theorem holds for the Riesz-means, but the jump of the order of approximation can appear at any value $\beta\leqq 1$ if the Lipschitz class has the same parameter. But if $r\neq 0$ then a curious phenomenon appears, namely if r is odd then the case $\alpha=1$ will be exceptional. The reason of this exception has its roots in the following classical result of M. Zamansky [10]: $f^{(r)}\in \mathrm{Lip}\, 1$ *if and only if*

$$\|f-R_n(f, r+1)\| = O(n^{-r-1}) \quad \textit{for an odd} \quad r, \quad \textit{and}$$
$$\|\tilde{f}-R_n(\tilde{f}, r+1)\| = O(n^{-r-1}) \quad \textit{for an even} \quad r.$$

We mention that the case $r=0$ of this theorem was proved by G. Alexits [1].

Now we formulate the statements mentioned above precisely, and refer to our paper [6] where the statements of Theorem E appear implicitly.

Theorem E. *Let* $f^{(r)}\in \mathrm{Lip}\,\alpha$, $0<\alpha\leqq 1$. *Then*

(i) *if r is even*

$$\|R_n(f, \beta; x)-f(x)\| = \begin{cases} O(n^{-r-\alpha}), & \textit{if} \quad r+\alpha<\beta, \\ O(n^{-r-\alpha}\log n), & \textit{if} \quad r+\alpha=\beta; \end{cases}$$

(ii) *if r is odd*

$$\|R_n(f, \beta; x)-f(x)\| = \begin{cases} O(n^{-r-\alpha}) & \textit{if} \quad r+\alpha<\beta \\ O(n^{-r-1}) & \textit{if} \quad r+1=\beta \quad (\alpha=1) \\ O(n^{-r-\alpha}\log n) & \textit{if} \quad r+\alpha=\beta \quad \textit{and} \quad \alpha<1, \end{cases}$$

hold true.

Furthermore, if whether r is even or $\alpha<1$, *then there exists a function* f_0 *such that* $f_0^{(r)}\in \mathrm{Lip}\,\alpha$, $0<\alpha\leqq 1$ *and*

(4) $$|R_n(f_0, r+\alpha; 0)-f_0(0)| \geqq c n^{-r-\alpha}\log n$$

holds with a positive $c=c(r, \alpha)$ *if n is large enough.*

We mention that analogous results for the conjugate functions also hold, and that the special case $\alpha=1$ of (4) is not proved in [6], but it is true, and our theorem to be proved includes this special case, too.

The results of Theorem A and C (B and D also) were generalized by V. Totik [9] as follows:

Theorem F. *If* $f\in W^r H^\omega$ *then for any* $\beta>0$ *and* $p>0$

$$h_n(f,\beta,p)=O(H^{\beta,p}_{r,\omega,n}) \tag{5}$$

holds, where

$$H^{\beta,p}_{r,\omega,n}:=\left\{\frac{1}{(n+1)^\beta}\sum_{k=1}^n k^{\beta-1}\left(\frac{1}{k^r}\,\omega\left(\frac{1}{k}\right)\right)^p\right\}^{1/p}.$$

Furthermore there exists a function f_r *such that* $f_r\in W^r H^\omega$, *but*

$$h_n(f_r,\beta,p;\,0)\geqq cH^{\beta,p}_{r,\omega,n}\quad (c>0).$$

The aim of our note is to show that Theorem E can be generalized for the class $W^r H^\omega$, i.e. to prove that the ordinary Riesz-means do not approximate better than the strong Riesz-means on the whole class $W^r H^\omega$ if r is even or if r is odd but $\sum_{k=1}^n \omega(1/k)=O(n\omega(1/n))$.

Our theorem reads:

Theorem. *If* $f\in W^r H^\omega$ *then for any* $\beta>0$

$$\|R_n(f,\beta;\,x)-f(x)\|=O(H^{\beta,1}_{r,\omega,n}) \tag{6}$$

holds.

Furthermore, if whether r is even or r is odd but $\sum_{k=1}^n \omega(1/k)=O(n\omega(1/n))$ *is fulfilled, then there exists a function* f_0 *such that* $f_0\in W^r H^\omega$ *and*

$$|R_n(f_0,\beta;\,0)-f_0(0)|\geqq cH^{\beta,1}_{r,\omega,n} \tag{7}$$

hold with a positive $c=c(\beta,r)$.

It is easy to verify that if r is even, $\beta=r+1$ and $\omega(\delta)=\delta$ $(\alpha=1)$ then (7) reduces to (4) as we stated above.

2. To prove our theorem we require the following lemmas.

We may assume, without restriction of generality, that the modulus of continuity ω is always concave. (See [8, p. 45].)

Lemma 1. *If* ω *is a modulus of continuity, then the function*

$$f^*(x):=\sum_{n=1}^\infty\big(\omega(1/n)-\omega(1/(n+1))\big)\cos nx$$

belongs to H^ω.

See Lemma 2.18 of [7] or V. Totik [9].

Lemma 2. *If the modulus of continuity ω satisfies the condition*

$$\sum_{k=1}^{n} \omega(1/k) = O(n\omega(1/n)), \tag{8}$$

then

$$g^*(x) := \sum_{k=1}^{\infty} (\omega(1/k) - \omega(1/(k+1))) \sin kx$$

belongs to H^ω.

Proof. Since

$$E_n(g^*) \leqq \|g^* - s_n(g^*)\| \leqq \omega(1/(n+1))$$

and

$$\omega\left(g^*, \frac{1}{n}\right) \leqq K\frac{1}{n} \sum_{k=0}^{n} E_k(g^*) \leqq K\frac{1}{n} \sum_{k=1}^{n+1} \omega(1/k),$$

so, by (8), $g^* \in H^\omega$.

Now we can start the proof of Theorem.

3. Proof of Theorem. The estimation (6) follows from (5) obviously. To prove the lower estimation (7) we define f_0 as follows:

$$f_0(x) := \sum_{n=1}^{\infty} n^{-r}(\omega(1/n) - \omega(1/(n+1))) \cos nx.$$

Since, by Lemmas 1 and 2, the functions f^* and g^* belong to H^ω and

$$f_0^{(r)}(x) = \begin{cases} \pm f^*(x) & \text{if } r \text{ is even,} \\ \pm g^*(x) & \text{if } r \text{ is odd,} \end{cases}$$

so $f_0 \in W^r H^\omega$.

A standard calculation gives that

$$R_n(f_0, \beta;\ 0) - f_0(0) = \frac{\beta_n}{(n+1)^\beta} \sum_{k=0}^{n} (k+1)^{\beta-1} \sum_{\nu=k+1}^{\infty} \nu^{-r}(\omega(1/\nu) - \omega(1/(\nu+1))) \geqq$$

$$\geqq \frac{d(\beta)}{n^\beta} \sum_{k=1}^{n} k^{\beta-1} \sum_{\nu=k}^{n} \nu^{-r}(\omega(1/\nu) - \omega(1/(\nu+1))) \geqq$$

$$\geqq \frac{d(\beta)}{n^\beta} \sum_{\nu=1}^{n} \nu^{-r}(\omega(1/\nu) - \omega(1/(\nu+1))) \sum_{k=1}^{\nu} k^{\beta-1} \geqq$$

$$\geqq \frac{d_1(\beta)}{n^\beta} \sum_{\nu=1}^{n} \nu^{\beta-r}(\omega(1/\nu) - \omega(1/(\nu+1))) \geqq$$

$$\geqq d(\beta, r) n^{-\beta} \sum_{\nu=1}^{n} \omega(1/\nu) \nu^{\beta-r-1} \geqq c(\beta, r) H_{r,\omega,n}^{\beta,1},$$

what proves (7).

Finally, we mention that a comparison of the statements of Theorem F and those of Theorem shows that if r is odd and

$$\sum_{k=1}^{n} \omega(1/k) \neq O(n\omega(1/n))$$

then the ordinary Riesz-means can approximate better than the strong ones, e.g. if $\omega(\delta)=\delta$.

Theorems C and E, in the special case $\alpha=1$, and $\beta=r+1$, also show this phenomenon clearly.

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