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Characterization of locally bounded functions with a finite number of negative squares

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1. Introduction

Throughout the paper G denotes a locally compact commutative group.

Let f be a complex-valued function on G. The function f is called Hermitian if $f(-x)=\overline{f(x)}$ holds for every $x\in G$. If k is a nonnegative integer the Hermitian function f is said to have k negative squares if the Hermitian matrix

$$(1) \qquad (f(x_i-x_j))_{i,j=1}^n$$

has at most k negative eigenvalues for any choice of n and $x_1, \dots, x_n \in G$, and for some choice of $x_1, \dots, x_n$ the matrix (1) has exactly k negative eigenvalues. This definition reduces to that of a positive definite function in the case $k=0$. We denote by $P_k(G)$ ($P_k^c(G)$) the set of all (continuous) functions on G which have k negative squares.

For a function $f\in P_k^c(G)$, where G is second countable, an integral representation was given in [10]. The bounded functions in $P_k^c(G)$ are exactly the Fourier transforms of such measures on the character group of G which assigne negative measure to k points and which are nonnegative outside of these points [9, 10]. A survey and bibliography about functions with k negative squares can be found in [1, 10, 12].

It is the aim of this note to characterize those functions $f\in P_k(G)$ which are locally bounded, i.e., bounded on every compact set $K\subset G$. As was shown in [11], every measurable function f with k negative squares on an arbitrary locally compact group is locally bounded. Moreover, f has the decomposition $f=f_c+p$, where f_c is a continuous function with k negative squares and p is a positive definite function vanishing almost everywhere on G [10].

If f is not measurable and $k>0$, then it may be unbounded on every open set. To see this let l be a nonmeasurable real-valued function on $\mathbf{R}$ satisfying the

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equation $l(x+y)=l(x)+l(y)$ $(x, y\in \mathbf{R})$. Then the function $f=il$ has one negative square and f is unbounded on every open set $V\subset \mathbf{R}$.

The main result of the present paper is the following

Theorem 1. *Every locally bounded function $f\in P_k(G)$ has the decomposition*

$$f=\gamma_1 f_1+\ldots+\gamma_n f_n+p \tag{2}$$

where

(i) *γ_j is a bounded (continuous or discontinuous) character of G $(j=1, \ldots, n)$;*
(ii) *f_j is a continuous function with k_j negative squares and $k_1+\ldots+k_n=k$;*
(iii) *p is a positive definite function.*

Recall that a complex-valued Hermitian function defined on G is said to be conditionally positive definite if

$$\sum_{i,j=1}^{n} f(x_i-x_j)c_i\bar{c}_j \geqq 0$$

holds for every choice of $x_1, \ldots, x_n\in G$ and for every choice of complex numbers $c_1, \ldots, c_n$ such that $c_1+\ldots+c_n=0$. It is easy to see that a conditionally positive definite function has at most one negative square. For a bibliography about conditionally positive definite functions we refer to [2, 4].

The above theorem has the following

Corollary 1. *Let f be a conditionally positive definite function on G which is bounded on a set of positive Haar measure. Then f has the decomposition*

$$f=f_c+p$$

where f_c is a continuous conditionally positive definite function and p is positive definite.

We remark that a conditionally positive definite function f is bounded if and only if $f=p+m$, where $m\in \mathbf{R}$ and p is a positive definite function [2]. The function $f=il$ introduced above is a conditionally positive definite function which is unbounded on every set $V\subset \mathbf{R}$ of positive Haar measure.

2. Notation and preliminaries

(2.1) Let k be a nonnegative integer. Throughout the paper the symbol Π_k denotes a π_k-space with rank of negativity k. We shall assume familiarity with basic information about π_k-spaces as found in [3, 5].

Let

$$\Pi_k=\Pi_+\oplus\Pi_- \tag{3}$$

be a fixed decomposition of Π_k where Π_+ is a positive subspace and Π_- is a negative k-dimensional subspace. Representing each vector $v \in \Pi_k$ in the form $v = v_+ + v_-$ ($v_+ \in \Pi_+$, $v_- \in \Pi_-$) we introduce a new scalar product $[\,,\,]$ in Π_k by

$$[v, w] = (v_+, w_+) - (v_-, w_-), \quad v, w \in \Pi_k. \tag{4}$$

This scalar product is positive definite and Π_k can be regarded as a Hilbert space with scalar product $[\,,\,]$ and with the norm

$$\|v\| = \sqrt{[v, v]}. \tag{5}$$

The scalar product (v, w) is continuous with respect to the norm (5) in both variables v and w.

Let $\{e_1, \ldots, e_k\}$ be a basis of Π_- such that $[e_i, e_j] = -(e_i, e_j) = \delta_{ij}$. Then we have for any $v \in \Pi_k$

$$\|v\|^2 = [v, v] = (v, v) + 2 \sum_{i=1}^{k} |(e_i, v)|^2. \tag{6}$$

Recall that a linear operator U in Π_k is called unitary if it maps Π_k onto Π_k and preserves the scalar product $(\,,\,)$ of Π_k, i.e.,

$$(Uv, Uw) = (v, w) \quad \text{for all} \quad v, w \in \Pi_k.$$

By a unitary representation of G in Π_k there is meant a mapping $x \to U_x$ of G satisfying the following conditions:

(i) $U_0 = I$ where I is the identity operator in Π_k;

(ii) $U_{x+y} = U_x U_y$ for any $x, y \in G$;

(iii) U_x is a unitary operator in Π_k for all $x \in G$.

We shall need the following correspondence between cyclic unitary representations of G in π_k-spaces and functions of the class $P_k(G)$ [10, Satz 9.2].

Theorem 2. *For an arbitrary function $f \in P_k(G)$ there exists a π_k-space $\Pi_k(f)$ with the following properties:*

(i) *the elements of $\Pi_k(f)$ are complex-valued functions on G, $f \in \Pi_k(f)$, and $\Pi_k(f)$ is invariant under translations;*

(ii) *the linear manifold $T(f)$ spanned by all translations of f is dense in $\Pi_k(f)$;*

(iii) *$x \to U_x$ is a cyclic unitary representation of G in $\Pi_k(f)$, where U_x is defined by*

$$(U_x g)(y) = g(y-x), \quad g \in \Pi_k(f), \quad x, y \in G;$$

(iv) *$g(x) = (g, U_x f)$, $g \in \Pi_k(f)$, $x \in G$;*

(v) *if f is locally bounded then every function $g \in \Pi_k(f)$ is locally bounded.*

We now prove a further assertion.

(vi) *If f is locally bounded then the function $x \to \|U_x\|$ is locally bounded.*[1]

[1] The operator norm is induced by the vector norm (5).

Proof of (vi). It follows from the proof of Satz 9.2 in [10] that

$$\Pi_k(f) = P \oplus N$$

where P is a positive subspace and N is a negative k-dimensional subspace such that every function $h \in N$ is a finite linear combination of translations of f, i.e., $h \in T(f)$. Let $\{e_1, \ldots, e_k\}$ be an orthonormal basis of N. By (6) we have

$$\|U_x g\|^2 = (g, g) + 2 \sum_{i=1}^{k} |(U_x g, e_i)|^2, \quad g \in \Pi_k(f).$$

From $e_i \in T(f)$ and (iv) it follows easily that the function $h_i(-x) = (U_{-x} g, e_i) = (g, U_x e_i)$ is a finite linear combination of translations of g $(i = 1, \ldots, n)$. By (v), g is locally bounded, so the function $x \to \|U_x g\|^2$ is locally bounded for every $g \in \Pi_n(f)$. The local boundedness of $x \to \|U_x\|$ follows now from the Banach—Steinhaus Theorem.

(2.2) Let $x \to U_x$ be a representation of G by invertible bounded linear operators on a Hilbert space $\mathfrak{H}$. We say that the representation $x \to U_x$ is locally bounded if the function $x \to \|U_x\|$ is locally bounded. Denote by $\mathfrak{H}_c$ the subspace of continuously translating elements of $\mathfrak{H}$, i.e., the set of all $h \in \mathfrak{H}$ for which $x \to U_x h$ is continuous from G into $\mathfrak{H}$ in its weak topology. Let $\mathscr{V}$ denote the set of all neighbourhoods V of the zero of G, $U_V = \{U_x \colon x \in V\}$, and $\mathscr{C}(U_V h)$ the closed convex hull of the "partial orbit" $U_V h = \{U_x h \colon x \in V\}$. The subspace $\mathfrak{H}_0$ of elements averaging to $0 \in \mathfrak{H}$ is the set of all $h \in \mathfrak{H}$ for which

$$0 \in \bigcap_{V \in \mathscr{V}} \mathscr{C}(U_V h).$$

K. DELEEUW and I. GLICKSBERG [6, Th. 2.7] proved the following

Theorem 3. *Let $x \to U_x$ be a locally bounded representation of G in a Hilbert space $\mathfrak{H}$. Then $\mathfrak{H}_c$ and $\mathfrak{H}_0$ are closed (U_x)-invariant subspaces and $\mathfrak{H}$ is the orthogonal direct sum of $\mathfrak{H}_c$ and $\mathfrak{H}_0$.*

Let now $f \in P_k(G)$ be a locally bounded function and consider the unitary representation $x \to U_x$ of G in $\Pi_k(f)$. By (vi) this representation is locally bounded with respect to the positive definite scalar product (4). (Note that local boundedness of $x \to U_x$ does not depend on the special decomposition (3).) It follows from the definition of $\mathfrak{H}_c$ and from (iv) that every $h \in \mathfrak{H}_c$ is a continuous function. When $h \in \mathfrak{H}_0$ then h has the following property: for $\varepsilon > 0$ and any $V \in \mathscr{V}$ for which $\sup_{x \in V} \|U_x\| < \infty$ there exist $x_1, \ldots, x_n \in V$ and positive numbers $p_1, \ldots, p_n$ summing to 1 such that

$$\left|\sum_{i=1}^{n} p_i h(x - x_i)\right| < \varepsilon \quad \text{for all} \quad x \in V. \tag{7}$$

Indeed, by the definition of $\mathfrak{H}_0$ there are $x_1, \ldots, x_n \in V$ and positive numbers $p_1, \ldots, p_n$ summing to 1 such that

$$\Big\|\sum_{i=1}^{n} p_i U_{x_i} h\Big\| < \varepsilon / \sup_{x\in V} \|U_x f\|.$$

By (iv) we have

$$\Big|\sum_{i=1}^{n} p_i h(x-x_i)\Big| = \Big|\Big(\sum_{i=1}^{n} p_i U_{x_i} h\Big)(x)\Big| = \Big|\Big(\Big(\sum_{i=1}^{n} p_i U_{x_i} h\Big), U_x f\Big)\Big| \leqq$$

$$\leqq \Big\|\sum_{i=1}^{n} p_i U_{x_i} h\Big\| \|U_x f\| < \varepsilon \quad \text{for} \quad x \in V.$$

(2.3) Let G^d be the discrete version of G. The character group of G^d is denoted by Γ^d. We introduce the notation Γ_u^d for the set of unbounded characters of G^d, i.e., the set of complex-valued unbounded functions γ on G^d for which $\gamma(0)=1$ and $\gamma(x+y)=\gamma(x)\gamma(y)$ hold. Let

$$\Gamma'^d = \Gamma^d \cup \Gamma_u^d.$$

In the proof of Theorem 1 we shall need the following result which is the discrete version of Folgerung 11.7 in [10] (see also Theorem 3.1 in [8]).

Theorem 4. *For every $f \in P_k(G)$ there exist positive integers k_i, functions $f_i \in P_{k_i}(G)$ and $\gamma_i \in \Gamma'^d$ $(i=1, \ldots, n)$ with the following properties:*

(a) $f = f_1 + \ldots + f_n$;

(b) $k = k_1 + \ldots + k_n$;

(c) $f_i \in \Pi_k(f)$ $(i=1, \ldots, n)$;

(d) *the only common nonpositive eigenvector of the translation operators U_x in $\Pi_{k_i}(f_i)$ are γ_i and $\overline{\gamma_i^{-1}}$.*[2)]

When f is locally bounded then by (c) and (v) in Theorem 2 the functions f_i are locally bounded as well.

3. Proof of Theorem 1 and Corollary 1

(3.1) Let $f \in P_k(G)$ be a locally bounded function and consider the locally bounded unitary representation $x \to U_x$ of G in $\Pi_k(f)$. By Theorem 4 we can restrict ourselves to the case where the only common nonpositive eigenvectors of the operators U_x are $\gamma, \overline{\gamma^{-1}} \in \Gamma'^d$. Since $\dfrac{|\gamma|}{\gamma}$ is a bounded character of G, the (locally bounded)

[2)] Note that $\gamma_i = \overline{\gamma_i^{-1}}$ if and only if $\gamma_i \in \Gamma^d$.

function $f' = \frac{|\gamma|}{\gamma} f$ has k negative squares. Moreover, the only common nonpositive eigenvectors of the translation operators U_x in $\Pi_k(f')$ are $|\gamma|$ and $|\gamma|^{-1}$ (see for this (11.5)(a) and (3.2)(c) in [10]). Thus, the proof of Theorem 1 will be complete if we verify the following.

Proposition 1. *Let $f \in P_k(G)$ be a locally bounded function. If the only common nonpositive eigenvectors of the operators U_x in $\Pi_k(f)$ are γ and γ^{-1}, and if they are positive then*

$$f = f_c + p,$$

where $f_c \in P_k^c(G)$ and $p \in P_0(G)$.

Proof. We consider the π_k-space $\Pi_k(f)$ as a Hilbert space with the scalar product $[\,,]$ in (4). By Theorem 3 $\Pi_k(f)$ is the $[\,,]$-orthogonal direct sum of the closed (U_x)-invariant subspaces X_c and X_0. Considering X_0 as subspace of the π_k-space $\Pi_k(f)$ there are three possibilities:

(i) X_0 is a π_l-space $(l \geqq 1)$;

(ii) X_0 is degenerate;

(iii) X_0 is a Hilbert space.

In the first case the commuting unitary operators U_x have a common nonpositive eigenvector in X_0 [7] which by our assumption must be γ or γ^{-1}. In the second case the isotropic part N of X_0 is (U_x)-invariant and finite dimensional. Hence the commuting operators U_x have a common eigenvector in N which must be again γ or γ^{-1}. Thus, in both cases we have $\gamma \in X_0$ or $\gamma^{-1} \in X_0$. Suppose for example $\gamma \in X_0$ and let V be an open symmetric neighbourhood of zero such that γ is bounded on V:

$$\gamma(x) < K \quad (x \in V).$$

As $\gamma(-x)\gamma(x) = 1$, we get

$$1/K < \gamma(x) < K \quad (x \in V).$$

Consequently, for any $x, x_i \in V$ $(i = 1, \ldots, n)$ and arbitrary positive numbers $p_1, \ldots, p_n$ summing to 1 we have:

$$\sum_{i=1}^{n} \gamma(x - x_i) p_i = \gamma(x) \sum_{i=1}^{n} \gamma(-x_i) p_i > \gamma(x)/K > 1/K^2,$$

in contradiction to (7). Hence (i) and (ii) are not possible and so X_0 is a Hilbert space.

Let X_c' denote the $(\,,)$-orthogonal complement of X_0. Then X_c' is a closed (U_x)-invariant π_k-space and

(8) $$\Pi_k(f) = X_c' \oplus X_0.$$

(the symbol $\oplus$ denotes (,)-orthogonal direct sum). On the other hand, X_c' is a Hilbert space with respect to the scalar product [,], and the restriction $x\to U_x'$ of $x\to U_x$ to X_c' is a locally bounded representation of G in X_c'. If $h'\in X_c'$ averages to zero with respect to the representation $x\to U_x'$ then it averages to zero with respect to $x\to U_x$ as well. Since X_0 consists of all $h\in\Pi_k(f)$ averaging to zero we necessarily have $h'=0$. Applying Theorem 3 to the representation $x\to U_x'$ in X_c' we see that every $h\in X_c'$ is continuously translating. Hence the function $x\to[g, U_x h]$ is continuous, from which the continuity of $x\to(g, U_x h)$ follows $(g, h\in X_c')$.

Let now $f=f_c+p$ $(f_c\in X_c',\ p\in X_0)$ be the decomposition of f corresponding to (8). We have

$$f(x)=(f, U_x f)=(f_c+p, U_x f_c+U_x p)=(f_c, U_x f_c)+(p, U_x p).$$

Moreover,

$$f_c(x)=(f_c, U_x f)=(f_c, U_x f_c)+(f_c, U_x p)=(f_c, U_x f_c) \tag{9}$$

and analogously

$$p(x)=(p, U_x p).$$

It follows from (9) that f_c is continuous. The function f is a cyclic vector for $x\to U_x$ and so f_c is cyclic for $x\to U_x'$. Thus, f_c has k negative squares [10, Satz 11.1]. Since X_0 is a Hilbert space (with respect to (,)) the function p is positive definite, completing the proof of Proposition 1.

(3.2) Let now f be a conditionally positive definite function which is bounded on a set $A\subset G$ of positive Haar measure. By a well known property of the Haar measure, $A-A$ contains an open set $V\neq\emptyset$. It follows from the inequality

$$\sqrt{|f(x-y)|}\leq\sqrt{|f(x)|}+\sqrt{|f(y)|},\quad x, y\in G, \tag{10}$$

that f is bounded on V. Moreover, (10) implies that f is bounded on $y+V$ for every $y\in G$. Since compact sets can be covered by finitely many sets V_i of the form $V_i=y_i+V$, f is locally bounded.

Let us consider the (locally bounded) unitary representation $x\to U_x$ in $\Pi_1(f)$ (we neglect the trivial case where f is positive definite). By [10, (11.5)] the only common nonpositive eigenvector of the operators U_x is $\gamma=1$. Therefore, we can apply Proposition 1 to obtain the decomposition

$$f=f_c+p,$$

where $f_c\in P_1^c(G)$ and $p\in P_0(G)$. All what remains to prove is that f_c is conditionally positive definite. Since $T(f)$ is dense in $\Pi_1(f)$, there is a sequence w_n of finitely supported complex measures on G such that

$$f_c=\lim_{n\to\infty} f*w_n$$

(the symbol $*$ denotes convolution). By (9) and (iv) in Theorem 2 we have

$$f_c(x) = (f_c, U_x f_c) = \lim_{n\to\infty} (f*w_n, U_x(f*w_n)) =$$
$$= \lim_{n\to\infty} (f*w_n*\tilde{w}_n, U_x f) = \lim_{n\to\infty} f*w_n*\tilde{w}_n(x),$$

where $\tilde{w}_n$ is defined by $\tilde{w}_n(\{-x\})=\overline{w_n(\{x\})}$. It follows immediately from the definition of conditional positive definiteness that the functions $f*w_n*\tilde{w}_n$ and so f_c are conditionally positive definite. The proof of Corollary 1 is complete.

Remark 1. As we have seen, boundedness on a set of positive Haar measure of a conditionally positive definite function implies local boundedness. It would be interesting to know whether a similar assertion holds for functions with a finite number of negative squares.

Remark 2. Corollary 1 probably holds even for noncommutative groups while the problem of characterization of locally bounded functions $f\in P_k(G)$ seems to be very difficult if G is not commutative.

Remark 3. Let G be an arbitrary commutative topological group. We say that a complex-valued function g on G is locally bounded if there exists an open set $V\subset G$ such that g is bounded on $y+V$ for every $y\in G$. Let now $f\in P_k(G)$ be a locally bounded function and consider the representation $x\to U_x$ in $\Pi_k(f)$. It follows by the same arguments as in the proof of property (vi) that the function $x\to\|U_x\|$ is locally bounded. Since Theorem 3 holds for an arbitrary commutative topological group G [6, Th. 2.7] we can repeat the proof of Theorem 1 to get the decomposition (2) of f.

References

[1] T. Ja. Azizov, I. S. Iohvidov, Linear operators in spaces with an indefinite metric and their applications, *Itogi Nauki i Techn.*, Ser. Mat. Anal., **17** (1979), 113—205 (Russian).

[2] Ch. Berg, G. Forst, *Potential Theory on Locally Compact Abelian Groups*, Springer-Verlag (Berlin—Heidelberg—New York, 1977).

[3] J. Bognár, *Indefinite Inner Product Spaces*, Springer-Verlag (Berlin—Heidelberg—New York, 1974).

[4] H. Heyer, *Probability Measures on Locally Compact Groups*, Springer-Verlag (Berlin—Heidelberg—New York, 1977).

[5] I. S. Iohvidov, M. G. Krein and H. Langer, *Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric*, Akademie-Verlag (Berlin, 1982).

[6] K. deLeeuw, I. Glicksberg, The decomposition of certain group representations, *J. Anal. Math.*, **15** (1965), 135—192.

[7] M. A. Naimark, On commuting unitary operators in spaces with an indefinite metric, *Acta Sci. Math.*, **24** (1963), 177—189.

[8] Z. Sasvári, Indefinite functions on commutative groups, *Monatshefte Math.*, **100** (1985), 223—238.

[9] Z. Sasvári, On bounded functions with a finite number of negative squares, *Monatshefte Math.*, **99** (1985), 223—234.

[10] Z. Sasvári, Definisierbare Funktionen auf Gruppen, *Dissertationes Math.*, **281** (1989), 1—83.

[11] Z. Sasvári, On measurable functions with a finite number of negative squares, *Acta Sci. Math.*, **50** (1986), 359—363.

[12] J. Stewart, Positive definite functions and generalizations, an historical survey, *Rocky Mountain J. Math.*, 6 (1976), 409—434.

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