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Injection of shifts into contractions

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The structure of unilateral shifts is well understood. Hence any relation between a contraction and a unilateral shift can be very useful. Here we only quote a recent result of H. BERCOVICI and K. TAKAHASHI (cf. [1]) claiming that a contraction T is reflexive whenever the set $\mathcal{I}(T, S)=\{A\colon AT=SA\}$ of intertwining operators contains a nonzero element, where S denotes the simple unilateral shift. In 1974 B. SZ.-NAGY and C. FOIAŞ proved the following (cf. [7, Corollary 2]):

Theorem 0. *If T is a contraction of class C_{10} with finite defect indices d_T and d_{T*}, then*

$$S^{(k)} \overset{\text{c.i.}}{\prec} T \prec S^{(k)}, \quad \textit{where} \quad k = d_{T*}-d_T.$$

Here $S^{(k)}$ stands for the unilateral shift of multiplicity k, i.e. for the orthogonal sum of k copies of the simple unilateral shift $S=S^{(1)}$. $T\prec S^{(k)}$ denotes that T is a quasiaffine transform of $S^{(k)}$, i.e. $\mathcal{I}(T, S^{(k)})$ contains a quasiaffinity (an operator with trivial kernel and dense range). The meaning of the notation $S^{(k)} \overset{\text{c.i.}}{\prec} T$ is that $S^{(k)}$ can be completely injected into T, i.e. $\mathcal{I}(S^{(k)}, T)$ contains a subsystem Φ consisting of injections such that $\vee\{\text{ran } A\colon A\in\Phi\}=\text{dom } T$. In connection with other notions concerning contractions readers are referred to the monograph [9].

We remark that, as it was illustrated by an example in [7], the relation $S^{(k)} \overset{\text{c.i.}}{\prec} T$ in Theorem 0 can not be generally replaced by $S^{(k)}\prec T$.

Definition. Let T be a completely non-unitary (c.n.u.) contraction. If the space of T is separable then the number

$$\mu_{*,T} = \operatorname*{ess\,sup}_{\zeta\in\partial\mathbf{D}} \text{rank}\, \Delta_{*,T}(\zeta)\in[0, \infty]$$

will be called the *$*$-multiplicity* of T. In the general case $\mu_{*,T}$ is defined as the least upper bound of the $*$-multiplicities of the restrictions of T to its separable reducing subspaces.

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Here $\Delta_{*,T}(\zeta)=[I-\Theta_T(\zeta)\Theta_T(\zeta)^*]^{1/2}$ is the defect function of the adjoint of the characteristic function Θ_T of T, and the essential upper bound is taken with respect to the normalized Lebesgue measure m on the boundary $\partial\mathbf{D}$ of the open unit disc $\mathbf{D}$.

The $*$-multiplicity $\mu_{*,T}$ of T coincides with the usual multiplicity of the unitary operator $R_{*,T}$ of multiplication by the identical function $\chi(\zeta)=\zeta$ on the Hilbert space $(\Delta_{*,T}L^2(\mathfrak{D}_{T^*}))^-$. (Cf. [3].) Furthermore, we can observe that if T is of class $C_{\cdot 0}$ with $d_T<\infty$, then rank $\Delta_{*,T}(\zeta)=d_{T^*}-d_T$ a.e., whence $\mu_{*,T}=d_{T^*}-d_T$. Now, it is natural to ask how the statement of Theorem 0 alters if $\mu_{*,T}<\infty$ is assumed instead of $d_{T^*}<\infty$.

First we note that by a result of Takahashi (cf. [10, Proposition 2]) $S^{(k)}\overset{\text{c. i.}}{\prec} T$ is already a consequence of the relation $T\prec S^{(k)}$. However $T\prec S^{(k)}$ does not hold in general. This is shown by the following.

Example. Let us consider a contraction T of class C_{10} such that rank $\Delta_{*,T}=\chi_\alpha$ a.e., where χ_α denotes the characteristic function of a Borel set $\alpha\subset\partial\mathbf{D}$ of measure $0<m(\alpha)<1$. (The existence of such a contraction was proved in [4].) Now the $*$-multiplicity of T is 1.

Let us assume that T is the quasi-affine transform of $S^{(k)}$, for some $1\leqq k\leqq\infty$, and let $C\in\mathscr{I}(T,S^{(k)})$ be a quasi-affinity. Let $U^{(k)}$ denote the minimal unitary extension of $S^{(k)}$. The operator C can be considered as an element of $\mathscr{I}(T,U^k)$. In view of [5, Proposition 4] there exists an operator $D\in\mathscr{I}(R_{*,T},U^{(k)})$ such that $C=DX$, where $X\in\mathscr{I}(T,R_{*,T})$ is a canonical intertwining operator. Since $R^*_{*,T}|(\operatorname{ran}X)^\perp$ is always of class C_{10} (cf. [,5 Proposition 4]) and since $R_{*,T}$ is now reductive, it follows that X has dense range. We infer that $(\operatorname{ran}D)^-=(\operatorname{ran}C)^-=$ $=\operatorname{dom}S^{(k)}$, so D can be considered as a quasi-surjective operator from $\mathscr{I}(R_{*,T},S^{(k)})$, whence $D^*\in\mathscr{I}(S^{*(k)},R^*_{*,T})$ is an injection. This yields that $\{0\}=\ker R^*_{*,T}\supset$ $\supset D^*\ker S^{*(k)}\neq\{0\}$, what is a contradiction.

Therefore $T\prec S^{(k)}$ is not true, for any $1\leqq k\leqq\infty$.

In [10] K. Takahashi characterized, in terms of the characteristic function, contractions which are quasi-affine transforms of unilateral shifts of finite multiplicity. While in [11] P. Y. Wu gave a characterization for contractions which are quasi-similar to unilateral shifts of finite multiplicity.

Though, as we have seen, $T\prec S^{(k)}$ $(k=\mu_{*,T})$ loses validity in Theorem 0 if $d_{T^*}=\infty$, we shall prove that the relation $S^{(k)}\overset{\text{c.i.}}{\prec} T$ $(k=\mu_{*,T})$ does remain true in a very general setting. This is expressed in the following theorem, the main result of our paper.

Theorem. *If T is a c.n.u. contraction with $*$-multiplicity $1 \leqq \mu_{*,T} < \infty$, then*

$$S^{(k)} \overset{\text{c.i.}}{\prec} T, \quad \text{where} \quad k = \mu_{*,T}.$$

We remark that injection of shifts into strict contractions was investigated in [8] and [12]. A contraction T is called strict if $\|T\| < 1$, in which case $\mu_{*,T} = 0$.

In proving our theorem we can assume that T acts on a separable Hilbert space $\mathfrak{H}$. In fact, in the opposite case $\mathfrak{H}$ can be decomposed into the orthogonal sum of separable subspaces reducing for T, and then the characteristic function of T will be the orthogonal sum of the characteristic functions of the restrictions of T. Hence in the sequel every Hilbert space will be supposed to be separable.

Since T is c.n.u. it can be given as a model operator (cf. [9, Chapter VI]). So let $\{\Theta, \mathfrak{E}, \mathfrak{E}_*\}$ be a purely contractive analytic function, its defect function is $\Delta = [I - \Theta^* \Theta]^{1/2}$. Let U_+ denote the operator of multiplication by the identical function $\chi(\zeta) = \zeta$ on the Hilbert space $\mathfrak{K}_+ = H^2(\mathfrak{E}_*) \oplus (\Delta L^2(\mathfrak{E}))^-$. The c.n.u. contraction T is defined on the Hilbert space $\mathfrak{H} = \mathfrak{K}_+ \ominus \{\Theta w \oplus \Delta w \colon w \in H^2(\mathfrak{E})\}$ as $T = PU_+|\mathfrak{H}$, where P denotes the orthogonal projection onto $\mathfrak{H}$ in $\mathfrak{K}_+$. The $*$-multiplicity of T is $\mu_{*,T} = \operatorname{ess\,sup}_\zeta \operatorname{rank} \Delta_*(\zeta)$, where $\Delta_* = [I - \Theta\Theta^*]^{1/2}$.

The proof of the Theorem is based on the following.

Lemma. *Let h be a function in $L^2(\mathfrak{E}_*)$ such that $\|h(\zeta)\|_{\mathfrak{E}_*} = 1$ a.e. Then for any non-zero function $f \in H^2(\mathfrak{E}_*)$ and for any number $0 < c < 1$, there exists an analytic function $u \in H^2(\mathfrak{E}_*)$ such that*

(1) $$\|u(\zeta)\|_{\mathfrak{E}_*} \leqq 1 \quad \text{a.e.},$$

(2) $$|\langle u(\zeta), h(\zeta)\rangle_{\mathfrak{E}_*}| \geqq c \quad \text{a.e., and}$$

(3) $$\langle u, f\rangle_{H^2(\mathfrak{E}_*)} \neq 0.$$

Proof. First we show that a function $u \in H^2(\mathfrak{E}_*)$ can be found with the properties (1) and (2). The proof of this is essentially the same as the proof of the Lemma in [6]. For the sake of easy reference we give the details.

Let $\{x_j\}_{j=1}^\infty$ be a dense sequence on the unit sphere of $\mathfrak{E}_*$, and for every j let us consider the function $h_j(\zeta) = \langle x_j, h(\zeta)\rangle_{\mathfrak{E}_*}$ $(\zeta \in \partial\mathbf{D})$, $h_j \in L^2$. Then we have

(4) $$1 = \|h(\zeta)\|_{\mathfrak{E}_*} = \sup_j |h_j(\zeta)|, \quad \text{for a.e.} \quad \zeta \in \partial\mathbf{D}.$$

Let $0 < \nu < 1$ be arbitrary, and define $\{\alpha_j\}_{j=1}^\infty$ as $\alpha_1 = \alpha_1^{(0)}$, $\alpha_j = \alpha_j^{(0)} \setminus \left(\bigcup_{i=1}^{j-1} \alpha_i\right)$ $(j \geqq 2)$, where $\alpha_j^{(0)} = \{\zeta \in \partial\mathbf{D} \colon |h_j(\zeta)| > \nu\}$. The sequence $\{\alpha_j\}_{j=1}^\infty$ consists of pairwise disjoint Borel sets, and by (4) we have

(5) $$m\left(\partial\mathbf{D} \setminus \left(\bigcup_j \alpha_j\right)\right) = 0.$$

Let $\{\mu_j\}_{j=1}^{\infty}$ be a sequence of positive numbers such that $0<\mu=\sum_{j=1}^{\infty}\mu_j<1$. For every j, let us consider an outer function $\hat{u}_j\in H^\infty$ with absolute value $|\hat{u}_j|=(1-\mu)\chi_{\alpha_j}+\mu_j\chi_{\partial\mathbf{D}\setminus\alpha_j}$ a.e. on $\partial\mathbf{D}$, and let us define $u_j=\hat{u}_j x_j\in H^2(\mathfrak{E}_*)$.

For every j and for a.e. $\zeta\in\alpha_j$ we can write $\sum_{i=1}^{\infty}\|u_i(\zeta)\|_{\mathfrak{E}_*}=1-\mu+\sum_{\substack{i=1\\ i\neq j}}^{\infty}\mu_i\leqq 1-\mu+\mu=1$. Hence in view of (5) $\sum_{j=1}^{\infty}\|u_j(\zeta)\|_{\mathfrak{E}_*}\leqq 1$ and so $\sum_{j=1}^{\infty}u_j(\zeta)$ strongly converges in $\mathfrak{E}_*$ a.e. on $\partial\mathbf{D}$. The limit function $u(\zeta)=\sum_{j=1}^{\infty}u_j(\zeta)$ satisfies (1), therefore, $u\in L^2(\mathfrak{E}_*)$. Furthermore, Lebesgue's dominated theorem ensures that $\lim_{n\to\infty}\left\|\sum_{j=1}^{n}u_j-u\right\|_{L^2(\mathfrak{E}_*)}=0$, whence $u\in H^2(\mathfrak{E}_*)$.

For every j and for a.e. $\zeta\in\alpha_j$ we have

$$
(6)\qquad |\langle u(\zeta),h(\zeta)\rangle_{\mathfrak{E}_*}|=\left|\sum_{i=1}^{\infty}\langle u_i(\zeta),h(\zeta)\rangle_{\mathfrak{E}_*}\right|=\left|\sum_{i=1}^{\infty}\hat{u}_i(\zeta)h_i(\zeta)\right|\geqq
$$

$$
\geqq|\hat{u}_j(\zeta)||h_j(\zeta)|-\sum_{\substack{i=1\\ i\neq j}}^{\infty}|\hat{u}_i(\zeta)||h_i(\zeta)|\geqq(1-\mu)\nu-\sum_{i=1}^{\infty}\mu_i=(1-\mu)\nu-\mu.
$$

If μ and ν are chosen sufficiently close to 0 and 1, respectively, then $(1-\mu)\nu-\mu\geqq c$, and so (2) is implied by (5) and (6).

Now, let us take real numbers c_1 and c_2 satisfying $c<c_1<c_2<1$. By the previous part of the proof we can find a function $u_1\in H^2(\mathfrak{E}_*)$ such that (1) and (2) hold with c_1/c_2 in place of c. Then for the function $u_2=c_2u_1\in H^2(\mathfrak{E}_*)$ we have

$$
(7)\qquad \|u_2(\zeta)\|_{\mathfrak{E}_*}\leqq c_2\quad\text{a.e., and}
$$

$$
(8)\qquad |\langle u_2(\zeta),h(\zeta)\rangle_{\mathfrak{E}_*}|\geqq c_1\quad\text{a.e..}
$$

Let δ denote the positive number $\delta=\min\{c_1-c,1-c_2\}$, and for any integer $n\geqq 0$ and for any vector $a\in\mathfrak{E}_*$, $\|a\|\leqq\delta$ let us define the function $u_{n,a}\in H^2(\mathfrak{E}_*)$ as $u_{n,a}=u_2+\chi^n a$. By (7) and (8) it easily follows that $u_{n,a}$ has the properties (1) and (2). Let us assume that (3) is not true, for any choice of n and a. Then taking $a=0$ we obtain $\langle u_2,f\rangle_{H^2(\mathfrak{E}_*)}=0$, whence $\langle\chi^n a,f\rangle=\langle u_{n,a},f\rangle=0$ for every $n\geqq 0$ and $a\in\mathfrak{E}_*$, $\|a\|\leqq\delta$. But the set $\{\chi^n a\colon n\geqq 0, a\in\mathfrak{E}_*, \|a\|\leqq\delta\}$ is total in $H^2(\mathfrak{E}_*)$ and $f\in H^2(\mathfrak{E}_*)$, so f must be zero, which is a contradiction.

Therefore, the function $u=u_{n,a}\in H^2(\mathfrak{E}_*)$ possesses the properties (1)—(3) for an appropriate choice of $n\geqq 0$ and $a\in\mathfrak{E}_*$, $\|a\|\leqq\delta$.

Now we turn to the

Proof of the Theorem. Let k denote the $*$-multiplicity of T: $1\leqq k=\mu_{*,T}<\infty$.

1) First we show that there exists an injection A in $\mathscr{I}(S^{(k)}, T)$.

The operator $X_+\colon \mathfrak{K}_+\to(\Delta_*(L^2(\mathfrak{E}_*))^-$, $X_+(u\oplus v)=(-\Delta_* u+\Theta v)$ intertwines U_+ with the operator R_* of multiplication by χ on the space $(\Delta_* L^2(\mathfrak{E}_*))^-$, $X_+\in\mathscr{I}(U_+, R_*)$. In view of the commuting relation $\Delta_*\Theta=\Theta\Delta$ it is immediate that $X_+(\mathfrak{K}_+\ominus\mathfrak{H})=\{0\}$, and so the operator $X=X_+|\mathfrak{H}$ belongs to $\mathscr{I}(T, R_*)$ and the relation

$$X_+ = XP \tag{9}$$

holds. (A detailed study of the operator X can be found in [5].)

Since $\Delta_*(\zeta)$ is a positive operator of finite rank a.e. and $\operatorname{ess\,sup}_\zeta \operatorname{rank} \Delta_*(\zeta)=k$, we conclude that $\Delta_*(\zeta)$ is of the form

$$\Delta_*(\zeta) = \sum_{j=1}^{k} \delta_j(\zeta) h_j(\zeta)\otimes h_j(\zeta), \tag{10}$$

where

$$h_j\in L^2(\mathfrak{E}_*) \quad \text{for every} \quad 1\leqq j\leqq k,$$

$$\{h_j(\zeta)\}_{j=1}^k \text{ is an orthonormal system in } \mathfrak{E}_* \text{ a. e. on } \partial\mathbf{D},$$

$$0\leqq\delta_j\in L^\infty \quad \text{for every} \quad 1\leqq j\leqq k, \tag{11}$$

$$1\geqq\delta_1(\zeta)\geqq\delta_2(\zeta)\geqq\ldots\geqq\delta_k(\zeta) \quad \text{a.e.} \quad \text{on} \quad \partial\mathbf{D}, \quad \text{and}$$

$$m(\alpha_k)>0, \quad \text{where} \quad \alpha_k=\{\zeta\in\partial\mathbf{D}\colon \delta_k(\zeta)\neq 0\}.$$

(Indeed, the function $\delta_1(\zeta)=\|\Delta_*(\zeta)\|_{\mathfrak{E}_*}$ is measurable, and an easy application of [2, Lemma II.1.1] guarantees the existence of a function $h_1\in L^2(\mathfrak{E}_*)$ such that $\|h_1(\zeta)\|_{\mathfrak{E}_*}=1$ a.e. and $h_1(\zeta)\in\ker(\Delta_*(\zeta)-\delta_1(\zeta)I)$, whenever $\ker(\Delta_*(\zeta)-\delta_1(\zeta)I)\neq\{0\}$. The functions $\delta_2\in L^\infty$ and $h_2\in L^2(\mathfrak{E}_*)$ can be obtained from $\Delta_*-\delta_1 h_1$ in place of Δ_* in an analogous way; and so on.)

Let $0<c<1$ be arbitrary. In virtue of our Lemma, for every $1\leqq j\leqq k$, we can find a function $u_j\in H^2(\mathfrak{E}_*)$ such that

$$\|u_j(\zeta)\|_{\mathfrak{E}_*}\leqq 1 \quad \text{a.e., and} \tag{12}$$

$$|\langle u_j(\zeta), h_j(\zeta)\rangle_{\mathfrak{E}_*}|\geqq c \quad \text{a.e..} \tag{13}$$

Let $\{e_j\}_{j=1}^k$ be an orthonormal basis on a Hilbert space $\mathfrak{G}$. The operator of multiplication by χ on the space $H^2(\mathfrak{G})$ is a unilateral shift of multiplicity k, which will be denoted by $S^{(k)}$. Since on account of (12), for any sequence $\{\xi_j\}_{j=1}^k\subset H^2$, we have

$$\Big\|\sum_{j=1}^k \xi_j u_j\Big\|_{H^2(\mathfrak{E}_*)}\leqq\sum_{j=1}^k\|\xi_j u_j\|_{H^2(\mathfrak{E}_*)}=\sum_{j=1}^k\Big(\int_{\partial\mathbf{D}}|\xi_j|^2\|u_j\|^2_{\mathfrak{E}_*}\,dm\Big)^{1/2}\leqq$$

$$\leqq\sum_{j=1}^k\|\xi_j\|_{H^2}\leqq k^{1/2}\Big\|\sum_{j=1}^k\xi_j e_j\Big\|_{H^2(\mathfrak{G})},$$

it follows that by the definition

$$W(\sum_{j=1}^{k}\xi_j e_j) = \sum_{j=1}^{k}\xi_j u_j, \quad \{\xi_j\}_{j=1}^{k}\subset H^2,$$

we obtain a bounded, linear operator, belonging to $\mathscr{I}(S^{(k)}, U_+)$. Now, in virtue of [9, Theorem I.4.1] the operator

$$A = PW \tag{14}$$

belongs to $\mathscr{I}(S^{(k)}, T)$.

We are going to prove that A is *injective* if c is sufficiently close to 1. First of all we observe that by (9) and (14)

$$XA = X_+W \tag{15}$$

holds, hence the injectivity of A is a consequence of the injectivity of X_+W.

Let us assume that $X_+W(\sum_{j=1}^{k}\xi_j e_j)=0$, for a sequence $\{\xi_j\}_{j=1}^{k}\subset H^2$. On account of (10) this means that for a.e. $\zeta\in\partial\mathbf{D}$ we have

$$0 = (X_+W\sum_{j=1}^{k}\xi_j e_j)(\zeta) = -\Delta_*(\zeta)\sum_{j=1}^{k}\xi_j(\zeta)u_j(\zeta) =$$

$$= -[\sum_{i=1}^{k}\delta_i(\zeta)h_i(\zeta)\otimes h_i(\zeta)]\sum_{j=1}^{k}\xi_j(\zeta)u_j(\zeta) = -\sum_{i=1}^{k}\delta_i(\zeta)(\sum_{j=1}^{k}\xi_j(\zeta)\langle u_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*})h_i(\zeta).$$

Making use of (11) we obtain that

$$\sum_{j=1}^{k}\xi_j(\zeta)\langle u_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*} = 0, \quad 1\leqq i\leqq k, \tag{16}$$

for a.e. $\zeta\in\alpha_k$.

Let us introduce the operators $B(\zeta)$, $C(\zeta)$, $D(\zeta)$ ($\zeta\in\partial\mathbf{D}$) acting on $\mathfrak{G}$ such that their matrices $[b_{ij}(\zeta)]_{i,j=1}^{k}$, $[c_{ij}(\zeta)]_{i,j=1}^{k}$, $[d_{ij}(\zeta)]_{i,j=1}^{k}$, respectively, in the basis $\{e_j\}_{j=1}^{k}$ are of the following form:

$$b_{ij}(\zeta) = \langle u_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*}, \quad 1\leqq i, j\leqq k,$$

$$c_{ij}(\zeta) = \begin{cases} b_{ij}(\zeta) & \text{if } i=j \\ 0 & \text{otherwise,}\end{cases}$$

$$d_{ij}(\zeta) = \begin{cases} 0 & \text{if } i=j \\ -b_{ij}(\zeta) & \text{otherwise.}\end{cases}$$

By (13) we see that $|c_{jj}(\zeta)|=|b_{jj}(\zeta)|=|\langle u_j(\zeta), h_j(\zeta)\rangle_{\mathfrak{E}_*}|\geqq c$ a.e. $(1\leqq j\leqq k)$, hence $C(\zeta)$ is invertible and

$$\|C(\zeta)^{-1}\|\leqq c^{-1} \quad \text{a.e..} \tag{17}$$

On the other hand, if $i \neq j$ then by (12) and (13)

$$|d_{ij}(\zeta)| = |b_{ij}(\zeta)| = |\langle u_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*}| = \left|\langle u_j(\zeta) - \langle u_j(\zeta), h_j(\zeta)\rangle_{\mathfrak{E}_*} h_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*}\right| \leqq$$
$$\leqq \|u_j(\zeta) - \langle u_j(\zeta), h_j(\zeta)\rangle_{\mathfrak{E}_*} h_j(\zeta)\|_{\mathfrak{E}_*} = [\|u_j(\zeta)\|^2_{\mathfrak{E}_*} - |\langle u_j(\zeta), h_j(\zeta)\rangle_{\mathfrak{E}_*}|^2]^{1/2} \leqq (1-c^2)^{1/2},$$

and so

$$\|D(\zeta)\| \leqq \sum_{i=1}^{k} \Big(\sum_{j=1}^{k} |d_{ij}(\zeta)|^2\Big)^{1/2} \leqq (1-c^2)^{1/2} k^{3/2} \quad \text{a.e..} \tag{18}$$

Consequently, if c satisfies

$$1 > c > k^{3/2}(k^3+1)^{-1/2}, \tag{19}$$

then $k^{3/2}(1-c^2)^{1/2} < c$, and by the inequalities (17), (18) we infer $\|D(\zeta)\| < \|C(\zeta)^{-1}\|^{-1}$. Then the operator $B(\zeta) = C(\zeta) - D(\zeta) = C(\zeta)[I - C(\zeta)^{-1} D(\zeta)]$ will be invertible and

$$\|B(\zeta)^{-1}\| \leqq \|C(\zeta)^{-1}\| (1 - \|C(\zeta)^{-1}\| \|D(\zeta)\|)^{-1} \leqq \tag{20}$$
$$\leqq c^{-1}\big(1 - k^{3/2}(1-c^2)^{1/2} c^{-1}\big)^{-1} = \big(c - k^{3/2}(1-c^2)^{1/2}\big)^{-1} \quad \text{a.e..}$$

Since the matrix of $B(\zeta)$ coincides with the matrix of the system of equations (16), it follows that $\xi_j(\zeta) = 0$ for every $1 \leqq j \leqq k$ and for a.e. $\zeta \in \alpha_k$. But α_k is of positive measure and the functions ξ_j are from the Hardy class H^2, so we conclude that $\xi_j = 0$, for every $1 \leqq j \leqq k$.

Therefore, taking into consideration (15) we obtain that under the assumption (19) the operator $A \in \mathscr{I}(S^{(k)}, T)$ defined before is injective.

2) To prove that $S^{(k)}$ can be completely injected into T it is enough to show that for any non-zero vector h in $\mathfrak{H}$ the injection $A \in \mathscr{I}(S^{(k)}, T)$ can be chosen in such a way that h is not orthogonal to the range of A.

Let us be given first $0 \neq f \in H^2(\mathfrak{E}_*)$ and $g \in (\Delta L^2(\mathfrak{E}))^-$ such that $f \oplus g \in \mathfrak{H}$. Our Lemma ensures the existence of a function $u_1 \in H^2(\mathfrak{E}_*)$ for which beyond (12) and (13) even the relation $\langle u_1, f\rangle_{H^2(\mathfrak{E}_*)} \neq 0$ holds. In this case $\langle Ae_1, f \oplus g\rangle_{\mathfrak{H}} = \langle P(u_1 \oplus 0), f \oplus g\rangle_{\mathfrak{K}_+} = \langle u_1 \oplus 0, P(f \oplus g)\rangle_{\mathfrak{K}_+} = \langle u_1 \oplus 0, f \oplus g\rangle_{\mathfrak{K}_+} = \langle u_1, f\rangle_{H^2(\mathfrak{E}_*)} \neq 0$, i.e. $f \oplus g$ is not orthogonal onto ran A.

Let us assume now that $0 \neq g \in \mathfrak{H} \cap (\Delta L^2(\mathfrak{E}))^-$. Let $\lambda > 1$ be a real number such that the set $\alpha = \{\zeta \in \partial\mathbf{D} \colon \lambda^{-1} < \|g(\zeta)\|_{\mathfrak{E}} < \lambda\}$ is of positive measure. Let $\varrho > 0$ be arbitrary and let us consider the functions $\{u_j\}_{j=1}^k \subset H^2(\mathfrak{E}_*)$ occuring in the first part of the proof. Since for any $\xi_1 \in H^2$ we have

$$\|\xi_1(u_1 \oplus \varrho\chi_\alpha g)\|_{\mathfrak{K}_+} = \Big(\int_{\partial\mathbf{D}} |\zeta_1|^2 (\|u_1\|^2_{\mathfrak{E}_*} + \varrho^2 \chi_\alpha \|g\|^2_{\mathfrak{E}})\, dm\Big)^{1/2} \leqq (1 + \varrho^2\lambda^2)^{1/2} \|\xi_1\|_{H^2},$$

it follows that the definition

$$W_\varrho\Big(\sum_{j=1}^{k} \xi_j e_j\Big) = \xi_1(u_1 \oplus \varrho\chi_\alpha g) + \sum_{j=2}^{k} \xi_j u_j \quad (\{\xi_j\}_{j=1}^k \subset H^2)$$

gives a bounded linear operator ($\|W_\varrho\| \leq k^{1/2}(1+\varrho^2\lambda^2)^{1/2}$) belonging to $\mathcal{I}(S^{(k)}, U_+)$. We define $A_\varrho \in \mathcal{I}(S^{(k)}, T)$ by $A_\varrho = PW_\varrho$. Since $XA_\varrho = X_+ W_\varrho$, the injectivity of A_ϱ is again implied by the injectivity of $X_+ W_\varrho$.

For any $\{\xi_j\}_{j=1}^k \subset H^2$ we have

$$X_+ W_\varrho\big(\sum_{j=1}^k \xi_j e_j\big)(\zeta) = X_+\big(\sum_{j=1}^k \xi_j u_j \oplus \xi_1 \varrho \chi_\alpha g\big)(\zeta) =$$

$$= -\Delta_*(\zeta) \sum_{j=1}^k \xi_j(\zeta) u_j(\zeta) + \Theta(\zeta)\xi_1(\zeta)\varrho\chi_\alpha(\zeta) g(\zeta) =$$

$$= \sum_{i=1}^k \big[-\delta_i(\zeta) \sum_{j=1}^k \xi_j(\zeta)\langle u_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*} + \xi_1(\zeta)\varrho\langle \Theta(\zeta)\chi_\alpha(\zeta)g(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*}\big] h_i(\zeta) \quad \text{a.e.}.$$

Hence $X_+ W_\varrho\big(\sum_{j=1}^k \xi_j e_j\big) = 0$ yields that

(21)

$$\delta_i(\zeta) \sum_{j=1}^k \xi_j(\zeta)\langle u_j(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*} - \xi_1(\zeta)\varrho\langle \Theta(\zeta)\chi_\alpha(\zeta)g(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*} = 0, \quad 1 \leq i \leq k,$$

holds for a.e. $\zeta \in \alpha_k$.

Let $E_\varrho(\zeta)$ ($\zeta \in \alpha_k$) stand for the operator acting on $\mathfrak{G}$ with matrix in the basis $\{e_j\}_{j=1}^k$ of the form

$$e_{ij}^{(\varrho)}(\zeta) = \begin{cases} \varrho\delta_i(\zeta)^{-1}\langle \Theta(\zeta)\chi_\alpha(\zeta)g(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*} & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

By (11) we infer that

$$|e_{i1}^{(\varrho)}(\zeta)| = \varrho|\delta_i(\zeta)|^{-1}|\langle \Theta(\zeta)\chi_\alpha(\zeta)g(\zeta), h_i(\zeta)\rangle_{\mathfrak{E}_*}| \leq$$
$$\leq \varrho|\delta_i(\zeta)|^{-1}\chi_\alpha(\zeta)\|g(\zeta)\|_{\mathfrak{E}} \leq \varrho\lambda|\delta_k(\zeta)|^{-1}$$

is true for every $1 \leq i \leq k$ and a.e. $\zeta \in \alpha_k$, whence

(22) $$\|E_\varrho(\zeta)\| = \big(\sum_{i=1}^k |e_{i1}^{(\varrho)}(\zeta)|^2\big)^{1/2} \leq k^{1/2}\varrho\lambda|\delta_k(\zeta)|^{-1} \quad \text{a.e. on } \alpha_k.$$

Let us consider a Borel set $\beta \subset \alpha_k$ of positive measure and a positive number $\lambda' > 0$ such that $|\delta_k(\zeta)|^{-1} \leq \lambda'$ for a.e. $\zeta \in \beta$. Let us assume that the functions $\{u_j\}_{j=1}^k$ correspond to a number c satisfying (19). Now, if $\varrho > 0$ fulfils the inequality

(23) $$\varrho k^{1/2}\lambda\lambda' < c - k^{3/2}(1-c^2)^{1/2},$$

then by (20) and (22) we obtain that $\|E_\varrho(\zeta)\| < \|B(\zeta)^{-1}\|^{-1}$ and so $B_1(\zeta) = B(\zeta) - E(\zeta)$ is invertible a.e. on β. In view of (21) we infer that $\xi_j(\zeta) = 0$ ($1 \leq j \leq k$) a.e. on β, and since $m(\beta) > 0$ that $\xi_j = 0$ ($1 \leq j \leq k$).

Therefore, the operator $A_\varrho \in \mathscr{I}(S^{(k)}; T)$ defined before will be an injection whenever c and $\varrho > 0$ satisfy the inequalities (19) and (23), respectively. At the same time we have

$$\langle A_\varrho e_1, 0 \oplus g \rangle_{\mathfrak{H}} = \langle P(u_1 \oplus \varrho \chi_\alpha g), 0 \oplus g \rangle_{\mathfrak{K}_+} = \langle u_1 \oplus \varrho \chi_\alpha g, P(0 \oplus g) \rangle_{\mathfrak{K}_+} =$$

$$= \langle u_1 \oplus \varrho \chi_\alpha g, 0 \oplus g \rangle_{\mathfrak{K}_+} = \langle \varrho \chi_\alpha g, g \rangle_{L^2(\mathfrak{E})} = \varrho \Big(\int_\alpha \|g\|_{\mathfrak{E}}^2 \, dm \Big)^{1/2} \geq \varrho \lambda^{-1} m(\alpha)^{1/2} > 0,$$

i.e. g is not orthogonal to ran A_ϱ.

According to [7, Theorem 5], if T is a contraction of class $C_{\cdot 0}$ with finite defect indices d_T, d_{T^*} and if $S^{(k)} \overset{i}{\prec} T$, then $k \leq d_{T^*} - d_T = \mu_{*,T}$. Hence, under the assumptions of Theorem 0, $\mu_{*,T}$ is the maximum of the multiplicities of those unilateral shifts which can be completely injected into T. The following example shows that this statement fails if $d_{T^*} = \infty$.

Example. Let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of pairwise disjoint Borel subsets of $\partial \mathbf{D}$ of positive measure. For every n, let T_n be a contraction of class C_{10} such that rank $\Delta_{*,T_n} = \chi_{\alpha_n}$ a.e. (cf. [4]). Then the orthogonal sum $T = \bigoplus_{n=1}^\infty T_n$ is also of class C_{10} with rank $\Delta_{*,T} = \text{rank} \bigoplus_{n=1}^\infty \Delta_{*,T_n} = \chi_{\bigcup_{n=1}^\infty \alpha_n}$ a.e., whence $\mu_{*,T} = 1$. By our Theorem $S \overset{\text{c.i.}}{\prec} T_n$ for every n, which results in that $S^{(\infty)} \overset{\text{c.i.}}{\prec} T$.

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