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On commutativity and spectral radius property of real generalized *-algebras

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Introduction. Let A denote a Banach algebra over the real field throughout this paper. Of course, a complex algebra is a real algebra as well, although the spectra will change (cf. [1], p. 70). Assume we have a linear operation $a \to a^*$ on A with the properties

(i) $a^{**}=a$,
(ii) $(ab)^*=b^*a^*$.

Then A is called a *-algebra. If we replace (ii) by

(ii') $(ab)^*=a^*b^*$

then we call A an auto-*-algebra. We say A is a generalized *-algebra if A is either a *-algebra or an auto-*-algebra (cf. [4], [6]). In such an algebra let

$$A_H = \{a\in A;\ a = a^*\},\quad A_J = \{a\in A;\ a = -a^*\},\quad A_N = \{a\in A;\ aa^* = a^*a\}.$$

We call the elements of A_H, A_J and A_N self-adjoint, skew-adjoint and normal, respectively.

In [6] A is called Hermitian if each self-adjoint element has purely real spectrum and A is called skew Hermitian if the spectra of the skew-adjoint elements do not contain any non-zero real number (the spectrum is defined as follows: a complex number z belongs to $\mathrm{Sp}(x)$ if and only if $z\cdot 1-x$ is not invertible in A_1, where A_1 is the complexification, and unitization if necessary, of A, see [1], p. 70). None of these properties implies the other one as simple examples show. This is a marked difference from the complex case.

We shall retain the above definition of skew Hermitianness but *we shall call A Hermitian if both properties are satisfied.*

Our main results then:

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Theorem 1. *If A is an Hermitian Banach auto-*-algebra then $A/\text{rad } A$ is commutative.*

Theorem 2. *If A is an Hermitian Banach generalized *-algebra then $r(a^*a) \geqq r(a)^2$ for any $a \in A$, where r denotes the spectral radius.*

Remarks. If the *-operation is the identical mapping then Theorem 1 reduces to a theorem of I. KAPLANSKY (see Thm. 8 in [2]) and, indeed, that result is the starting point of our proof. We should emphasize that the significance of Kaplansky's theorem for Hermitian auto-*-algebras was first pointed out by T. W. PALMER in [4], though [4] contains a wrong proof assuming the unitary elements form a group, which is not true in an auto-*-algebra. On the other hand, the authors of [6] simply overlooked that the proof of their key Gelfand—Naimark type theorem (Theorem 2.3 in [6]) does not work for auto-*-algebras. Now our Theorem 1 implies that all results of [4] and [6] are true.

Finally we shall include a version of Theorem 2 which answers a question in [6] (see Proposition 3 below).

To prove our theorems we shall need the following simple lemmas.

Lemma 1. *If A is skew Hermitian then every skew-adjoint element has purely imaginary spectrum.*

Proof. Suppose to the contrary that $a \in A_J$, $z \in \text{Sp}(a)$ and z is not imaginary. Then z can not be real, since A is skew Hermitian, and hence z^2 is not real. Thus z and z^3 are linearly independent over $\mathbf{R}$, and hence there are $s, t \in \mathbf{R}$ such that $sz + tz^3 = 1$. Then $\text{Sp}(sa+ta^3) \ni 1$, while $sa+ta^3$ is skew-adjoint; this is a contradiction.

Lemma 2. *If A is an auto-*-algebra then*

$$\text{rad } A_H = A_H \cap \text{rad } A$$

(see [1] for the concept of the Jacobson-radical).

Proof. The containment "$\supset$" follows at once from the "quasi-inverse-characterization" of the radical (see [1], p. 125).

Prove "$\subset$". Consider an element $a \in \text{rad } A_H$, and an irreducible representation p of A over the real vector space X. Then we have to show $p(a)=0$ (if this is true for all p then $a \in \text{rad } A$). If p is irreducible for A_H too, then we are done. If p is not irreducible then for any non-trivial A_H-invariant subspace M set

$$M' := \text{the linear span of } p(A_J)M.$$

Then the relations $A_H A_J \subset A_J$, $A_J A_J \subset A_H$ imply $p(A_H)M' \subset M'$ and $p(A_J)M' \subset M$. Hence $M+M'$ and $M \cap M'$ are invariant for $A_H + A_J = A$, and therefore $X = M \oplus$

$\oplus M'$. Now if M were not irreducible then one could find a non-trivial A_H-invariant subspace L in M, on the other hand, $X=L\oplus L'$ and clearly $L'\subset M'$, which is a contradiction. The same is true for M' since it is another invariant subspace. So we see $p|_{A_H}$ is a direct sum of two irreducible representations and hence $p(a)=0$.

Lemma 3. *Let A be an auto-*-algebra. Then*

$$\mathrm{Sp}_{A_H}(h) = \mathrm{Sp}_A(h) \quad \textit{for any} \quad h\in A_H.$$

Proof. If a self-adjoint element has a quasi-inverse (or inverse) in A, then this quasi-inverse (or inverse) is self-adjoint, too. Thus we get our statement using the well-known characterization of the spectrum (see [1], p. 70).

Lemma. 4. *Factorization by the radical does not effect the spectra except possibly for the number 0 in them.*

Proof. Use the "quasi-inverse-characterization" of the radical (see [1], p. 125) and the fact if x has a left- and a right-quasi-inverse then x is quasi-invertible.

Proof of Theorem 1. First observe that the "*" preserves the radical (use the characterizations of the radical from [1]). Hence $A/\mathrm{rad}\, A$ is a Banach auto-*-algebra and it is Hermitian by Lemma 4. Thus we can assume A is semi-simple. In this case A_H is semi-simple, too, by Lemma 2. If $\|a\|':=\|a^*\|$ then $\|\cdot\|'$ is another Banach algebra norm, hence by Johnson's theorem the two norms are equivalent (see [1], p. 130 for the proof of Johnson's theorem). Thus A_H is closed. Using Lemma 4 we see A_H is a semi-simple Banach algebra in which every element has purely real spectrum. This implies, by Theorem 8 of [2], that

(1) $$A_H \text{ is commutative.}$$

Let $h\to\hat{h}$ be the Gelfand transform on A_H. It is injective, because A_H is semi-simple. Next we will show

(2) $$\text{if } j\in A_J \text{ and } j^2=0 \text{ then } j=0.$$

Consider a fixed $j\in A_J$ for which $j^2=0$. Let $k\in A_J$ be arbitrary and $r\in\mathbf{R}$. Since A is skew Hermitian, thus $\mathrm{Sp}\,(rj+k)$ is imaginary, and hence, using Lemma 3, we have $0\geqq\widehat{(rj+k)^2}=r\widehat{(jk+kj)}+\widehat{k^2}$, for $j^2=0$. This is true for any r, therefore $\widehat{jk+kj}=0$, $jk+kj=0$. Thus $(jk)^2=j(-jk)k=0$, which implies $\widehat{(jk)}=0$, $jk=0$. Since $jk+kj=0$, we have $jk=kj=0$ for any $k\in A_J$. Now let $a\in A$ be arbitrary, and $h=(1/2)(a+a^*)$, $k=(1/2)(a-a^*)$. Then $aj=(h+k)j=hj\in A_J$, and therefore $jaj=0$, $(aj)^2=0$. We get from this $\mathrm{Sp}\,(aj)=\{0\}$ for each $a\in A$, and hence $j=0$ for A is semi-simple.

Next we want to show that

(3) $$khk=k^2h \quad \text{for any} \quad h\in A_H,\ k\in A_J.$$

Let $g=hk-kh$. Since $k^2\in A_H$, thus $k^2h=hk^2$, and hence $gk=-kg$. Therefore $(kg)^2=k\cdot(-kg)\cdot g$ and hence $\widehat{(kg)}^2=-\widehat{k^2}\widehat{g^2}$. Since $k, g\in A_J$ and $kg\in A_H$, thus k^2, g^2 have non-positive real spectra, while $(kg)^2$ has non-negative spectrum. Thus we can infer $\widehat{kg}=0$, $kg=0$, which is exactly (3).

Now we will prove that

$$(4) \qquad kh = hk \quad \text{for any} \quad h\in A_H, \quad k\in A_J.$$

Let $g=hk-kh$. Then $g^2=hkhk-hk^2h+khkh-kh^2k=0$ (use (3) for h, k in the 1st and 3rd term, and for h^2, k in the 4th term). Thus, by (2), we get $g=0$.

Finally, we will show that

$$(5) \qquad jk = kj \quad \text{for any} \quad j, k\in A_J.$$

Since $jk, kj\in A_H$, thus, by (4), $jkj=j^2k$ and $kjk=k^2j$; therefore $0=kjkj-kj^2k+ +jkjk-jk^2j$, in other words, $m^2=0$ where $m=kj-jk\in A_H$. Thus $\hat{m}=0$ and (5) is proved.

The theorem is proved by uniting (2), (4) and (5).

Remark. Since the complex radical of a complex algebra is the same as the real radical (cf. [1]), therefore Theorem 1 is valid for complex algebras, too. Of course, one should check that a complex Hermitian algebra is Hermitian in our sense as a real algebra. This follows from the fact if S is the complex spectrum of an element then the "real spectrum" is the set $S\cup\bar{S}$.

Proof of Theorem 2. By Lemma 4 we may again assume A is semi-simple. But then, by Theorem 1, A is a *-algebra anyway. So let A be an Hermitian Banach *-algebra. Let $p(x):=r(x^*x)^{1/2}$ for all $x\in A$. Now A satisfies the conditions of Lemma 3.1 from [6], therefore we can infer

$$(6) \qquad p \text{ is an algebra-seminorm on } A.$$

The proof of Lemma 41.2 in [1] (see p. 225) yields in the real case that

$$(7) \qquad \text{if} \quad 1\in \mathrm{Sp}(a) \quad \text{then} \quad p(a)\geqq 1.$$

We assert that

$$(8) \qquad 3p(a)\geqq r(a) \quad \text{for all} \quad a\in A.$$

If $r(a)=0$ then this is clear. If $r(a)>0$ then let $b=r(a)^{-1}a$. We can choose a $z\in\mathrm{Sp}(b)$ such that $|z|=1$. Let $c=(z+\bar{z})b-b^2$. Then $1=(z+\bar{z})z-z^2\in\mathrm{Sp}(c)$, and hence, by (7) and (6), we have

$$1\leqq p(c)\leqq |z+\bar{z}|p(b)+p(b)^2\leqq (2+p(b))\cdot p(b), \quad \text{thus} \quad p(b)\geqq 1/3$$

and (8) is proved.

Applying (8) to a^n we get $r(a)^n = r(a^n) \leqq 3p(a^n)$. Now use the submultiplicativity of p and tend with n to infinity. The theorem is proved.

Remark. Differently from the complex case (cf. [5]), $r(a^*a) \geqq r(a)^2$ does not imply A is Hermitian; e.g., if $A = \mathbf{C}$ (considered as a real algebra) and the * is the identical mapping then $r(a^*a) = r(a)^2$ for all a but A is not Hermitian.

Proposition 3. *Let A be a skew Hermitian Banach generalized *-algebra. Then $r(a^*a) = r(a)^2$ for any normal element a.*

Proof. Let $a \in A_N$ be fixed. Let B be the second commutant of the set $\{a, a^*\}$. Then B is a Banach algebra, closed under the involution and $\mathrm{Sp}_B(b) = \mathrm{Sp}_A(b)$ for any $b \in B$. Further, B is commutative for a is normal. Let f be a multiplicative linear functional on B. Let $f(a) = u$, $f(a^*) = v$. Since A is skew Hermitian, thus, by Lemma 1, $a - a^*$ and $a^2 - (a^*)^2$ both have imaginary spectrum, and hence $u - v$ and $u^2 - v^2$ are imaginary numbers. Thus if $u \neq v$ then $u + v$ is real and $v = \bar{u}$. In any case $|v| = |u|$, and hence $|f(a^*a)| = |f(a)|^2$. This is true for any multiplicative linear functional f on B, therefore $r(a^*a) = r(a)^2$.

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