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A characterization of (real or complex) Hermitian algebras and equivalent C^*-algebras

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0. Introduction

We use the symbol $\mathbf{F}$ to denote a field that is either the real field $\mathbf{R}$ or the complex field $\mathbf{C}$. We call an algebra A over $\mathbf{F}$ a *-algebra if there is a conjugate linear mapping "*" from A into A satisfying

(i) $(ab)^* = b^*a^*$ for all $a, b\in A$,

(ii) $(a^*)^* = a$ for all $a\in A$.

We call A an auto-*-algebra if we replace the axiom (i) by the axiom

(i′) $(ab)^* = a^*b^*$ for all $a, b\in A$.

We call A a generalized *-algebra if A is a *-algebra or an auto-*-algebra. An element $a\in A$ is called self-adjoint, if $a=a^*$, skew-adjoint, if $a=-a^*$; and normal, if $aa^*=a^*a$. Denote by A_H, A_J and A_N the sets of all self-adjoint, skew-adjoint and normal elements, respectively.

We will treat Banach generalized *-algebras, that are generalized *-algebras with complete algebra norm. We define the spectrum of an element with respect to an algebra containing it as in [1] (see pp. 19—20 and 70). Then it is known that

$$\max\{|z|;\ z\in \mathrm{Sp}(A, a)\} = \inf_n \|a^n\|^{1/n} = \lim_{n\to\infty} \|a^n\|^{1/n}$$

if $\|\,.\,\|$ is a complete algebra norm on A. We write in this case

$$r(a) := \inf_n \|a^n\|^{1/n}.$$

Let A be a Banach generalized *-algebra. A is called Hermitian if $\mathrm{Sp}(A, a)\subset\mathbf{R}$ for all $a\in A_H$, and skew-Hermitian if $\mathrm{Sp}(A, a)\subset i\cdot\mathbf{R}$ for all $a\in A_J$. Every Hermitian

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algebra over $\mathbf{C}$ is automatically skew-Hermitian, of course. But this assertion is not true for real algebras. We will prove that a real Banach generalized *-algebra A is Hermitian *and* skew-Hermitian if and only if its complexification $A_{\mathbf{C}}$ (see [1] pp. 68—69) is Hermitian (see Theorem 3 below).

We remark that there is an equivalent, but formally weaker, definition of the skew-Hermitian property demanding only $1 \notin \mathrm{Sp}(A, a)$ for all $a \in A_J$. It is not very hard to see that if $\mathrm{Sp}(A, a) \not\subset i \cdot \mathbf{R}$ for some $a \in A_J$ then there are $s, t \in \mathbf{R}$ such that $\mathrm{Sp}(A, sa+ta^3) \ni 1$, and $sa+ta^3 \in A_J$.

A is called a C^*-algebra, if it is isometrically *-isomorphic to a norm-closed *-subalgebra of the Banach *-algebra $B(\mathfrak{H})$ of all bounded $\mathbf{F}$-linear operators on some Hilbert space $\mathfrak{H}$ over $\mathbf{F}$. A is called an equivalent C^*-algebra, if it is homeomorphically *-isomorphic to some C^*-algebra. We will give a characterization of equivalent C^*-algebras in Theorem 1 below, which is a generalization of a result of PTÁK (see [4]).

We will prove the following characterization of Hermitian and skew-Hermitian algebras: A is Hermitian and skew Hermitian if and only if there is such a *-homomorphism π of A into some $B(\mathfrak{H})$ which preserves the spectral radius (see Theorem 2). In contrast to a lot of characterizations of complex Hermitian algebras, this is valid for real algebras, too.

Our results are based on the following lemma:

Lemma 0.1. *Let A be a Hermitian and skew-Hermitian Banach generalized *-algebra over $\mathbf{F}$. Then there is a Hilbert space $\mathfrak{H}$ over $\mathbf{F}$ and a *-homomorphism π: $A \mapsto B(\mathfrak{H})$ such that $\|\pi(a)\| = r(a^*a)^{1/2}$ for all $a \in A$. Moreover, $r(a) \leqq \|\pi(a)\|$ for all $a \in A$, and* $\mathrm{rad}(A) = \pi^{-1}(\{0\})$. *If A has a unit then π can be chosen so that $\pi(1) = 1$.*

Proof. First we suppose that A is a *-algebra. Let

$$A_p = \{a \in A_H;\ \mathrm{Sp}(A, a) \subset \mathbf{R}_+\}.$$

Then it is known that A_p is a cone and $a^*a \in A_p$ for all $a \in A$ (see [5]). This is also true for the unitization $A+\mathbf{F}$ of A, since $A+\mathbf{F}$ is Hermitian and skew-Hermitian as well. Thus it is not hard to see that we can find for any fixed $a \in A$ a self-adjoint positive functional such that $f(1)=1$ and $f(a^*a)=r(a^*a)$ so that the customary GNS-construction gives us a Hilbert space $\mathfrak{H}$ and a *-homomorphism π of A satisfying $\|\pi(a)\| = r(a^*a)^{1/2}$ for all $a \in A$. (For more detailed description see [2], Lemma 3.1 and [1] § 37. See also [4] for another proof in case $\mathbf{F}=\mathbf{C}$.)

Since $\mathrm{rad}(A) = \{a \in A;\ r(qa)=0$ for every $q \in A\}$ (see [1] p. 126), it is clear that $\mathrm{rad}(A) \subset N$, where $N := \pi^{-1}(\{0\})$. On the other hand, the author has proved in [3], that $r(a) \leqq r(a^*a)^{1/2}$ in a Hermitian and skew-Hermitian Banach *-algebra. Thus N is an ideal consisting of elements of spectrum $\{0\}$ whence $N \subset \mathrm{rad}(A)$. Moreover, we see that $r(a) \leqq \|\pi(a)\|$ for all $a \in A$.

Now we suppose A is an auto-*-algebra. Being a conjugate linear automorphism the "*" maps rad (A) onto itself. Let $B=A/\mathrm{rad}\,(A)$ and p be the canonical mapping $A\mapsto B$. Then it is known that

(1) $$\mathrm{Sp}\,(A, a)\setminus\{0\} = \mathrm{Sp}\,(B, p(a))\setminus\{0\} \quad \text{for all} \quad a\in A.$$

(It is not hard to deduce this fact from Proposition 24.16. (i), p. 125 in [1].)

Therefore B is a Hermitian and skew-Hermitian Banach auto-*-algebra. Moreover, B is semisimple (see [1] p. 126). Thus, by a result of the author (see [3]), B is commutative, and hence B is a *-algebra. Therefore we have a representation π_1 of B satisfying the statements of our lemma, and so by (1) $\pi:=\pi_1\circ p$ is a representation we asked.

1. A characterization of equivalent C*-algebras

Lemma 1.1. *Let A be a Banach-algebra over* $\mathbf{F}$, *and let g be an entire function on* $\mathbf{C}$, *satisfying $g'(0)\neq 0$. Further in case $\mathbf{F}=\mathbf{R}$ we assume that the Taylor-series of g at zero has only real coefficients. Then there is a function $f\colon \mathbf{R}_+\mapsto\mathbf{R}_+$ so that $\|x\|^2\leqq f(c)\cdot\|x^2\|$ whenever x is such that $\|g(tx)\|\leqq c$ for all $t\in\mathbf{R}_+$. ($g(a)$ may be in the unitization $A+\mathbf{F}$ of A, if A does not have a unit. We fix a norm on $A+\mathbf{F}$ in that case.)*

Proof. Let $g(z)=\sum_{n=0}^{\infty}\alpha_n z^n$. If $h(z)=\sum_{n=2}^{\infty}|\alpha_n|\cdot z^n$ then h is an entire function, too. Suppose that $\|g(tx)\|\leqq c$ for all $t\in\mathbf{R}_+$ for some $x\in A$ and $c\in\mathbf{R}_+$. We can assume that $\|x\|=1$ because both sides of the inequality $\|x\|^2\leqq f(c)\cdot\|x^2\|$ are multiplied by $|\lambda|^2$ when we replace x by λx, and the case $x=0$ is trivial. Then let $p=\|x^2\|^{1/3}$, thus we see that $p\leqq 1$ and $\|x^n\|\leqq(p^3)^{[n/2]}\leqq p^n$ for all $n\geqq 2$. Hence we have for all $t\in\mathbf{R}_+$

$$t = \|tx\| = |\alpha_1|^{-1}\cdot\Big\|g(tx)-\alpha_0\cdot 1-\sum_{n=2}^{\infty}\alpha_n t^n x^n\Big\| \leqq |\alpha_1|^{-1}\cdot\big(c+|\alpha_0|\cdot\|1\|+h(tp)\big).$$

Hence $p\neq 0$, and replace $t=p^{-1}$, we see that $p^{-1}\leqq\varphi(c)$, where $\varphi(c)=|\alpha_1|^{-1}\big(c+|\alpha_0|\cdot\|1\|+h(1)\big)$. Thus $\|x^2\|=p^3\geqq\varphi(c)^{-3}$, and so $f(c)=\varphi(c)^3$ satisfies our condition.

Lemma 1.2. *Let A and g be as in Lemma 1.1, and let $\langle x\rangle$ denote the real algebra generated by an element $x\in A$. Then the function f of Lemma 1.1 also satisfies $\|x\|\leqq f(c)\cdot r(x)$ whenever x is such that $\|g(a)\|\leqq c$ for all $a\in\langle x\rangle$.*

Proof. Assume that $\|g(a)\|\leqq c$ for all $a\in\langle x\rangle$ for some $x\in A$ and $c\in\mathbf{R}_+$. Then by Lemma 1.1 we have

$$\|a\|^2 \leqq f(c)\cdot\|a^2\| \quad \text{for all} \quad a\in\langle x\rangle.$$

Writing $a=x^{2^n}$, we can infer by induction that

$$\|x\|^{2^n} \leqq f(c)^{2^n-1} \cdot \|x^{2^n}\|$$

and hence, tending with n to infinity we get $\|x\| \leqq f(c) \cdot r(x)$.

Theorem 1. *Let A be a Banach generalized *-algebra over* $\mathbf{F}$. *Then A is an equivalent C^*-algebra if and only if there is a constant C such that*

(i) $\|\sin(h)\| \leqq C$ *for all* $h \in A_H$ *and,*
(ii) $\|\sinh(k)\| \leqq C$ *for all* $k \in A_J$.

Remark. Of course, in case $\mathbf{F}=\mathbf{C}$ (i) is equivalent to (ii).

Proof. First we assume that A is an equivalent C^*-algebra. Then there is a norm p on A so that (A, p) is a C^*-algebra and a constant C such that $\|a\| \leqq C \cdot p(a)$ for all $a \in A$. It is known that a C^*-algebra is Hermitian, skew-Hermitian and its norm equals the spectral radius on normal elements (this is well known for $\mathbf{F}=\mathbf{C}$, and for $\mathbf{F}=\mathbf{R}$ we can canonically embed the subalgebra of $B(\mathfrak{H})$ into $B(\mathfrak{H}_{\mathbf{C}})$ where $\mathfrak{H}_{\mathbf{C}}$ is the complexification of the real Hilbert space $\mathfrak{H}$, and thus we can infer the statement). Therefore if $h \in A_H$ then $\operatorname{Sp}(A, h) \subset \mathbf{R}$, and so $\operatorname{Sp}(A, \sin(h)) \subset [-1, 1]$ (see [1], § 7), further $\sin(h) \in A_H$ for the * is norm-preserving in a C^*-algebra, and hence $p(\sin(h)) = r(\sin(h)) \leqq 1$, $\|\sin(h)\| \leqq C \cdot p(\sin(h)) \leqq C$. Similarly, if $k \in A_J$ then $\operatorname{Sp}(A, k) \subset i \cdot \mathbf{R}$, $\operatorname{Sp}(A, \sinh(k)) \subset i \cdot [-1, 1]$, $\sinh(k) \in A_J$, and hence $\|\sinh(k)\| \leqq C$.

Now we assume that A satisfies (i) and (ii) with a suitable constant C. First we show that

(1) A is Hermitian and skew-Hermitian.

Observe that if $z \in \mathbf{C} \setminus \mathbf{R}$, then the set $\{\sin(tz);\ t \in \mathbf{R}\}$ is not bounded. This fact implies that $\{r(\sin(th));\ t \in \mathbf{R}\}$ is not bounded if $\operatorname{Sp}(A, h) \not\subset \mathbf{R}$, and similarly $\{r(\sinh(tk));\ t \in \mathbf{R}\}$ is not bounded if $\operatorname{Sp}(A, k) \not\subset i \cdot \mathbf{R}$ for $\sinh(z) = -i \cdot \sin(iz)$. Since $r(a) \leqq \|a\|$, thus (i) and (ii) clearly imply (1).

Now we want to show that

(2) there is a constant M such that $\|a\| \leqq M \cdot r(a)$ for all $a \in A_H \cup A_J$.

We have by Lemma 1.2 and (i) a constant m_1 such that

(3) $$\|a\| \leqq m_1 \cdot r(a) \quad \text{for all} \quad a \in A_H$$

and we have by Lemma 1.1 and (ii) a constant m_2 such that

(4) $$\|a\|^2 \leqq m_2 \|a^2\| \quad \text{for all} \quad a \in A_J.$$

But $a^2 \in A_H$ for $a \in A_I$, thus $\|a^2\| \leqq m_1 \cdot r(a^2) = m_1 \cdot r(a)^2$, and hence (2) is true with $M = \max(m_1, \sqrt{m_1 \cdot m_2})$.

We can apply Lemma 0.1 to A because (1) holds; let π be the corresponding representation. Since $\|\pi(a)\|=r(a^*a)^{1/2}$, we have

(5) $$\|\pi(a)\| = r(a) \quad \text{for all} \quad a\in A_H\cup A_J,$$

and so by (2) we get $\|a\|\leqq M\cdot\|\pi(a)\|$ for all $a\in A_H\cup A_J$. Thus if a is an arbitrary element in A and $h=\frac{a+a^*}{2}$, $k=\frac{a-a^*}{2}$, then $\|a\|\leqq\|h\|+\|k\|\leqq M(\|\pi(h)\|+\|\pi(k)\|)$ and $\|\pi(h)\|\leqq\|\pi(a)\|$, $\|\pi(k)\|\leqq\|\pi(a)\|$ for the * is norm-preserving on $B(\mathfrak{H})$. Thus we get

(6) $$\|a\| \leqq 2M\cdot\|\pi(a)\| \quad \text{for all} \quad a\in A.$$

We have $\|\pi(a)\|^2=r(a^*a)\leqq\|a^*a\|\leqq\|a^*\|\cdot\|a\|$, and hence by (6) we infer $\|a\|\leqq \leqq 4M^2\cdot\|a^*\|$. Thus $\|a^*\|\leqq 4M^2\cdot\|a\|$ for $a^{**}=a$, and hence

(7) $$\|\pi(a)\|^2 \leqq 4M^2\cdot\|a\|^2 \quad \text{for all} \quad a\in A.$$

It follows from (6) and (7) that π is homeomorphic and $\pi(A)$ is complete. Therefore A is an equivalent C^*-algebra.

2. A characterization of Hermitian algebras

Lemma 2.1. *Let A and B be Banach generalized *-algebras over* $\mathbf{F}$. *Assume that $p\colon A\mapsto B$ is a *-homomorphism satisfying $r(h)\leqq r(p(h))$ for all $h\in A_H$. Then A is Hermitian (resp. skew-Hermitian) whenever B is.*

Remark. The condition $r(h)\leqq r(p(h))$ is equivalent to $r(h)=r(p(h))$ for $\mathrm{Sp}\,(B, p(h))\subset \mathrm{Sp}\,(A, h)\cup\{0\}$.

Proof. Suppose that A is not Hermitian (resp. skew-Hermitian) but B is. Then there is an element $h_1\in A_H$ (resp. $k_1\in A_J$) such that $\mathrm{Sp}\,(A, h_1)\not\subset\mathbf{R}$ (resp. $\mathrm{Sp}\,(A, k_1)\not\subset i\cdot\mathbf{R}$). If $z\in\mathbf{C}\setminus(\mathbf{R}\cup i\cdot\mathbf{R})$ then $z^2\notin\mathbf{R}$ and hence $\{tz+sz^3;\ t, s\in\mathbf{R}\}=\mathbf{C}$. This implies that there is an element $h\in\{th_1+sh_1^3;\ t, s\in\mathbf{R}\}\subset A_H$ (resp. $k\in\{tk_1+sk_1^3;\ t, s\in\mathbf{R}\}\subset A_J$) such that $i\in\mathrm{Sp}\,(A, h)$ (resp. $1\in\mathrm{Sp}\,(A, k)$). Let $c=h^2$ (resp. $c=-k^2$). Then

(1) $$-1\in\mathrm{Sp}\,(A, c) \quad \text{and} \quad c\in A_H.$$

Further, $p(c)=p(h)^2$ (resp. $p(c)=-p(k)^2$), p is a *-homomorphism, and B is Hermitian (resp. skew-Hermitian); thus we get

(2) $$\mathrm{Sp}\,(B, p(c))\subset\mathbf{R}_+.$$

Since A is a Banach-algebra, $\mathrm{Sp}\,(A, c)$ is bounded and hence there is a real number λ such that

(3) $$\lambda > 1 \quad \text{and} \quad -\lambda^{-1}\cdot c \text{ has a quasi-inverse } d \text{ in } A.$$

Moreover, $d \in A_H$, because $-\lambda^{-1} \cdot c \in A_H$. Since p is homomorphic, thus $p(d)$ is the quasi-inverse of $-\lambda^{-1} \cdot p(c)$. It is known that if b is the quasi-inverse of a in an arbitrary algebra then $\{t(t-1)^{-1};\ t \in \mathrm{Sp}(a)\} = \mathrm{Sp}(b)$. (Sketch of the proof: b is the quasi-inverse of a if and only if $1-b$ is the inverse of $1-a$, where $1-a, 1-b \in \in A+\mathbf{F}$ if A does not have a unit in which case $\mathrm{Sp}(A, x) = \mathrm{Sp}(A+\mathbf{F}, x)$ for all $x \in A$; and hence it is easy to deduce the statement.) Thus we get from (1), (2) and (3) that

(4) there is a negative number (namely $(1-\lambda)^{-1}$) in $\mathrm{Sp}(A, d)$

and

(5) $$\mathrm{Sp}(B, p(d)) \subset [0, 1).$$

Consider the polynomials $P_n(X) = X(1-X)^n$. Then $P_n(d) \in A_H$, and since $\mathrm{Sp}(P_n(a)) = P_n(\mathrm{Sp}(a))$ in an arbitrary algebra, thus $r(P_n(d)) > 1$ for sufficient large n by (4), while $r(P_n(p(d))) < 1$ for all n by (5). Thus we have got a contradiction to the assumption of our lemma.

Lemma 2.2. *Let A and B be Banach algebras over* $\mathbf{F}$ *and* $p\colon A \mapsto B$ *be a homomorphism. Then the following conditions are equivalent:*

(i) $r(a) = r(p(a))$ *for all* $a \in A$,

(ii) $\partial \mathrm{Sp}(A, a) \subset \partial \mathrm{Sp}(B, p(a)) \cup \{0\}$ *for all* $a \in A$.

Proof. First we assume (ii). Let $a \in A$ be fixed and let S be the closed disc about zero in $\mathbf{C}$ with radius $r(p(a))$. Then $\partial \mathrm{Sp}(A, a) \subset S$, and $\mathrm{Sp}(A, a)$ is a bounded set in $\mathbf{C}$, thus $\mathrm{Sp}(A, a) \subset S$, $r(a) \leqq r(p(a))$. Therefore (i) holds, for $r(a) \geqq r(p(a))$ is true for any homomorphism p.

Now we assume (i). Fix an element $a \in A$ and a complex number $z \in \partial \mathrm{Sp}(A, a) \setminus \{0\}$. Suppose that $z \notin \partial \mathrm{Sp}(B, p(a))$. Since $\mathrm{Sp}(B, p(a)) \subset \mathrm{Sp}(A, a) \cup \cup \{0\}$, we get $z \notin \mathrm{Sp}(B, p(a))$. Choose a sequence of complex numbers $z_n \to z$ such that $z_n \notin \mathrm{Sp}(A, a)$. We may assume $z_n \neq 0$ for all n. If $\mathbf{F} = \mathbf{R}$ then let

$$u_n = |z_n|^{-2} \cdot (2 \cdot \mathrm{Re}(z_n) a - a^2) \quad \text{and} \quad u = |z|^{-2} \cdot (2 \cdot \mathrm{Re}(z) a - a^2),$$

while in case $\mathbf{F} = \mathbf{C}$ let

$$u_n = z_n^{-1} \cdot a \quad \text{and} \quad u = z^{-1} \cdot a.$$

Then we have by [1] (see p. 70):

(1) $$u_n \to u \text{ in } A \text{ and } p(u_n) \to p(u) \text{ in } B,$$

(2) $$u_n \text{ has a quasi-inverse in } A,$$

(3) $$u \text{ does not have a quasi-inverse in } A,$$

(4) $$p(u) \text{ has a quasi-inverse in } B.$$

Further on, u_n and u are polynomials of a, and hence there is a maximal commutative subalgebra A' of A containing u and u_n for all n, and similarly a maximal commutative subalgebra B' of B containing $p(A')$. By (3) there is a character φ on A' such that $\varphi(u)=1$. Thus $\varphi(u_n)\to 1$, and hence, denoting the quasi-inverse of u_n by v_n, $|\varphi(v_n)|\to\infty$. Therefore $r(v_n)\to\infty$ and thus (i) yields

$$r(p(v_n))\to\infty. \tag{5}$$

On the other hand, $1\notin \mathrm{Sp}(B', p(u))$, and hence there is an $\varepsilon>0$ such that $|\psi(p(u))-1|>\varepsilon$ for all characters ψ of B'. Thus if $\|p(u_n)-p(u)\|\leqq\varepsilon/2$ then $|\psi(p(u_n))-1|\geqq\varepsilon/2$ for all ψ, and $|\psi(p(u_n))|\leqq\|p(u_n)\|\leqq$constant, because $p(u_n)\to p(u)$. Hence $r_n:=\max\{|\lambda(\lambda-1)^{-1}|;\ \lambda\in\mathrm{Sp}(B, p(u_n))\}\not\to\infty$, while $r_n=r(p(v_n))$ for $p(v_n)$ is the quasi-inverse of $p(u_n)$. This contradiction to (5) proves our lemma.

Theorem 2. *Let A be a Banach generalized *-algebra over* $\mathbf{F}$. *Then the following conditions are equivalent:*

(i) *A is Hermitian and skew-Hermitian,*

(ii) *there is a Hilbert space $\mathfrak{H}$ and a *-homomorphism $\pi\colon A\mapsto B(\mathfrak{H})$ satisfying $\|\pi(a)\|=r(a^*a)^{1/2}$ for all $a\in A$,*

(iii) *there is a π as in* (ii) *and satisfying $r(\pi(a))=r(a)$ for all $a\in A$,*

(iv) *there is a π as in* (ii) *and satisfying*

$$\partial\,\mathrm{Sp}(A, a)\subset\partial\,\mathrm{Sp}(B(\mathfrak{H}), \pi(a))\cup\{0\}\quad \textit{for all}\quad a\in A.$$

Proof. First we prove (i)$\Rightarrow$(iii). Consider the homomorphism π obtained from Lemma 0.1. Then for any $a\in A$ $r(a)^n=r(a^n)\leqq\|\pi(a^n)\|$ for all n, and hence $r(a)\leqq$ $\leqq r(\pi(a))$, thus $r(a)=r(\pi(a))$.

Now we prove (ii)$\Rightarrow$(i). If $h\in A_H$ then $r(h)^2=r(h^2)=r(h^*h)=\|\pi(h)\|^2=r(\pi(h))^2$ and hence by Lemma 2.1 we get (i), because $B(\mathfrak{H})$ is Hermitian and skew-Hermitian.

Since (iii)$\Rightarrow$(ii) is trivial and (iii)$\Leftrightarrow$(iv) was proved in Lemma 2.2, the proof of Theorem 2 is complete.

3. Relation between real and complex Hermitian algebras

Lemma 3.1. *Let A and B be Banach-algebras with unit over* $\mathbf{F}$, *and $p\colon A\mapsto B$ be a homomorphism satisfying $p(1)=1$ and $r(p(a))=r(a)$ for all $a\in A$. Assume that* $\mathrm{Sp}(B, p(x))\subset\mathbf{R}\setminus\{0\}$ *for some $x\in A$. Then x is invertible in A.*

Proof. Since A is a Banach algebra with unit, there is a real number $\lambda>0$ so that $a=(\lambda+x^2)^{-1}$ exists in A. Then $p(a)=(\lambda+p(x)^2)^{-1}$, and hence $\mathrm{Sp}(B, p(a))\subset(0, 1/\lambda)$, $r(p(a))<\lambda^{-1}$. Thus $r(a)<\lambda^{-1}$, and therefore $\lambda^{-1}\notin\mathrm{Sp}(A, a)$, $\lambda\notin\mathrm{Sp}(A, \lambda+x^2)$, and we see that x^2 is invertible in A. Hence x is invertible in A.

Lemma 3.2. *Let A be a generalized *-algebra over $\mathbf{R}$, and $A_{\mathbf{C}}$ be its complexification. Then $(A/\mathrm{rad}(A))_{\mathbf{C}}$ is *-isomorphic to $A_{\mathbf{C}}/\mathrm{rad}(A_{\mathbf{C}})$.*

Proof. We want to prove that

$$\mathrm{rad}(A_{\mathbf{C}}) = \{(a, b) \in A_{\mathbf{C}};\ a, b \in \mathrm{rad}(A)\}. \tag{1}$$

(We use the symbols of [1], see p. 68.) Let $N = \{a \in A;\ (a, 0) \in \mathrm{rad}(A_{\mathbf{C}})\}$. Clearly N is an ideal of A. If $a \in N$, then $(a, 0)$ is quasi-invertible in $A_{\mathbf{C}}$, hence a is quasi-invertible in A. Thus we obtain

$$N \subset \mathrm{rad}(A). \tag{2}$$

Now we fix an element $b \in \mathrm{rad}(A)$ and an irreducible representation p of $A_{\mathbf{C}}$ over the complex linear space X. Suppose that L is a real subspace of X, invariant for the operators $p((a, 0))$ for all $a \in A$. Then $L + i \cdot L$ and $L \cup i \cdot L$ are complex subspaces, invariant for $p(A_{\mathbf{C}})$, and hence, being p an irreducible representation, if L is non-trivial then $X = L \oplus i \cdot L$ as a real linear space. Hence if L_1 is another such subspace then $\{0\} \subsetneqq L_1 \subsetneqq L$ is not possible, that is $a \to p((a, 0))|_L$ is an irreducible representation of A on L. Thus $p((b, 0))|_L = 0$ for $b \in \mathrm{rad}(A)$, and hence $p((b, 0)) = 0$ because X is the complex hull of L. If such L does not exist then $a \to p((a, 0))$ gives an irreducible representation and $p((b, 0)) = 0$, too. Having this for any irreducible representation p of $A_{\mathbf{C}}$ we see that $b \in N$, $\mathrm{rad}(A) \subset N$, and hence by (2) we get

$$N = \mathrm{rad}(A). \tag{3}$$

Now consider the mapping $(a, b)' := (a, -b)$ on $A_{\mathbf{C}}$. This is conjugate linear automorphism, hence it preserves the quasi-invertibility, and therefore maps $\mathrm{rad}(A_{\mathbf{C}})$ onto itself. We can infer from this:

$$\mathrm{rad}(A_{\mathbf{C}}) = \{(a, b);\ (a, 0), (0, b) \in \mathrm{rad}(A_{\mathbf{C}})\}.$$

But $-i \cdot (0, b) = (b, 0)$ and hence $\mathrm{rad}(A_{\mathbf{C}}) = \{(a, b);\ a, b \in N\}$, that is, by (3), we can see that (1) holds.

It is easy to deduce from (1) that $(A/\mathrm{rad}(A))_{\mathbf{C}}$ is *-isomorphic to $A_{\mathbf{C}}/\mathrm{rad}(A_{\mathbf{C}})$.

Theorem 3. *Let A be a Banach generalized *-algebra over $\mathbf{R}$. Then A is Hermitian and skew-Hermitian if an only if its complexification $A_{\mathbf{C}}$ is a complex Hermitian algebra.*

Proof. Since the spectrum in a real algebra is defined through its complexification, one of the directions is trivial. To prove the other direction let A be Hermitian and skew-Hermitian as well. We may assume A has a unit, because otherwise $A + \mathbf{R}$ is Hermitian and skew-Hermitian while $(A + \mathbf{R})_{\mathbf{C}}$ is *-isomorphic to $A_{\mathbf{C}} + \mathbf{C}$. Then we may also assume A is semi-simple by Lemma 3.2.

Thus by Lemma 0.1 we have a *-homomorphism $\pi\colon A\mapsto B(\mathfrak{H})$, which is now *injective*. Moreover, it is easy to show (see e.g. the proof of Theorem 2) that

(1) π satisfies the conditions of Lemma 3.1.

We want to prove that $A_{\mathbf{C}}$ is Hermitian. Since $1\in A$, it is enough to show that $1+x^2$ is invertible in $A_{\mathbf{C}}$ whenever $x\in(A_{\mathbf{C}})_H$. Fix an $x=(a,b)\in(A_{\mathbf{C}})_H$, then $a\in A_H$ and $b\in A_I$. Let $c=1+a^2-b^2$, $d=ab+ba$, then $1+x^2=(c,d)$. Since the complexification of $B(\mathfrak{H})$ is clearly *-isomorphic to $B(\mathfrak{H}_{\mathbf{C}})$, which is Hermitian, thus $(\pi(c),\pi(d))$ is invertible in $B(\mathfrak{H})_{\mathbf{C}}$, so we have $u, v\in B(\mathfrak{H})$ satisfying

$$u\cdot\pi(c)-v\cdot\pi(d)=1,\quad u\cdot\pi(d)+v\cdot\pi(c)=0 \tag{2}$$

and

$$\pi(c)\cdot u-\pi(d)\cdot v=1,\quad \pi(d)\cdot u+\pi(c)\cdot v=0. \tag{3}$$

It is known that the set $A_p=\{h\in A_H;\ \mathrm{Sp}\,(A,h)\subset\mathbf{R}_+\}$ is a cone (see [5]), and hence $a^2-b^2\in A_p$ because $a^2, -b^2\in A_p$. Thus we can infer

(4) c has an inverse h in A_H.

We see from (2) that $v=-u\cdot\pi(dh)$ and so $u\cdot\pi(c+dhd)=1$. Similarly, we can see from (3) that $\pi(c+dhd)\cdot u=1$. Observe that $m=c+dhd\in A_H$ because $d\in A_I$ and $c, h\in A_H$, and hence $\mathrm{Sp}\,(B(\mathfrak{H}),\pi(m))\subset\mathbf{R}$. Applying Lemma 3.1 we get a $k=m^{-1}$ in A, moreover, $\pi(k)=u$. Hence $v=\pi(j)$, where $j=-kdh$. Now by the injectivity of π we can infer that $(k,j)=(1+x^2)^{-1}$ in $A_{\mathbf{C}}$. The proof is complete.

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