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Models for infinite sequences of noncommuting operators

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In [1], J. W. BUNCE developed a model theory for n-tuples of not necessarily commuting operators, extending the work of A. E. FRAZHO [4] for pairs of operators. He proved, for a finite number of operators on Hilbert space, versions of the Rota model theorem, the de Branges—Rovnyak model theorem, and the coisometric extension theorem.

The aim of this paper is to extend these results for infinite sequences of noncommuting operators, to generalize some results due to B. SZ.-NAGY [8] and G. C. ROTA [7] [5, Problem 121] and to give some necessary and sufficient conditions for simultaneous similarity. We shall prove all these results without using the theorems above mentioned (for a finite number of operators).

1. Let $\mathscr{H}$ be a Hilbert space and $B(\mathscr{H})$ be the algebra of all bounded operators in $\mathscr{H}$. We recall that a coisometry $V\in B(\mathscr{H})$ is called pure if $V^n\to 0$ (strongly) as $n\to\infty$.

In [1, Proposition 2] J. W. Bunce proved that for any finite family $\{A_i;\ 1\leqq i\leqq n\}$ ($n\in\mathbf{N}$) of operators such that $r(A_i)<1$ for each i, ($r(T)$ denoting the spectral radius of an operator $T\in B(\mathscr{H})$), and $\sum_{i=1}^{n} A_i^*A_i\leqq I_{\mathscr{H}}$ ($I_{\mathscr{H}}$ is the identity on $\mathscr{H}$), there are a Hilbert space $\mathscr{K}\supset\mathscr{H}$ and pure coisometries $\{S_i;\ 1\leqq i\leqq n\}$ acting on $\mathscr{K}$ such that $S_i(\mathscr{H})\subset\mathscr{H}$, $S_i|_{\mathscr{H}}=A_i$ for each i and $S_iS_j^*=0$ for $i\neq j$.

We begin with a theorem which generalizes Proposition 1 of [1] and the above mentioned result, replacing the condition that $r(A_i)<1$ by the condition $A_i^n\to 0$ (strongly) as $n\to\infty$. Let us consider Λ and J to be sets of indices such that card $\Lambda\leqq\aleph_0$.

Proposition 1.1. *Let* $\mathscr{A}_\alpha=\{A_{\alpha,i}\colon\ i\in\Lambda\}\subset B(\mathscr{H})$ *for each* $\alpha\in J$. *Then the following two conditions are equivalent:*

a) $\sum_{i\in\Lambda} A_{\alpha,i}^*A_{\alpha,i}\leqq I_{\mathscr{H}}$ *for each* $\alpha\in J$.

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b) *There exist a Hilbert space* $\mathcal{K}\supset\mathcal{H}$ *and families of coisometries* $\mathcal{V}_\alpha=\{V_{\alpha,i};\ i\in\Lambda\}\subset B(\mathcal{K})$ $(\alpha\in J)$ *such that*

$$\sum_{i\in\Lambda} V_{\alpha,i}^* V_{\alpha,i}\leqq I_{\mathcal{K}},\quad V_{\alpha,i}(\mathcal{H})\subset\mathcal{H}\quad \text{and}\quad V_{\alpha,i}|_{\mathcal{H}}=A_{\alpha,i}\quad \text{for each}\quad \alpha\in J,\quad i\in\Lambda.$$

One can even require that $V_{\alpha,i}$ *be a pure coisometry for every* $\alpha\in J$ *and* $i\in\Lambda$ *for which* $A_{\alpha,i}^n\to 0$ *(strongly) as* $n\to\infty$.

Proof. It is easy to see that b) implies a).

Conversely, assume that for each $\alpha\in J$, $\sum_{i\in\Lambda} A_{\alpha,i}^* A_{\alpha,i}\leqq I_{\mathcal{H}}$. Consider the Hilbert space $\mathbf{H}=\bigoplus_{\alpha\in J, i\in\Lambda}\mathcal{H}_{\alpha,i}$, where $\mathcal{H}_{\alpha,i}$ is a copy of the Hilbert space $\mathcal{H}$, and the operator $T\in B(\mathbf{H})$ defined by

$$T\Big(\bigoplus_{\alpha\in J, i\in\Lambda} h_{\alpha,i}\Big)=\bigoplus_{\alpha\in J, i\in\Lambda} A_{\alpha,i}h_{\alpha,1}.$$

Note that T is a contraction. Indeed,

$$\Big\|T\Big(\bigoplus_{\alpha\in J, i\in\Lambda} h_{\alpha,i}\Big)\Big\|^2=\sum_{\alpha\in J}\Big(\sum_{i\in\Lambda} A_{\alpha,i}^* A_{\alpha,i}h_{\alpha,1}, h_{\alpha,1}\Big)\leqq\sum_{\alpha\in J}\|h_{\alpha,1}\|^2\leqq\Big\|\bigoplus_{\alpha\in J, i\in\Lambda} h_{\alpha,i}\Big\|^2,$$

for every $\bigoplus_{\alpha\in J, i\in\Lambda} h_{\alpha,i}\in\mathbf{H}$.

Let us determine a Hilbert space $\mathbf{K}\supset\mathbf{H}$ and a coisometry $V\in B(\mathbf{K})$ such that $V(\mathbf{H})\subset\mathbf{H}$ and $V|_{\mathbf{H}}=T$. Let $\mathbf{K}=\mathbf{H}\oplus\mathcal{M}$, where $\mathcal{M}$ is a Hilbert space which we shall determine. With respect to this decomposition of $\mathbf{K}$ the matrix of V is

$$V=\begin{pmatrix} T & X\\ 0 & Y\end{pmatrix}$$

where $X\colon \mathcal{M}\to\mathbf{H}$ and $Y\colon\mathcal{M}\to\mathcal{M}$ satisfy the relations:

(1.1) $$TT^*+XX^*=I_{\mathbf{H}},\quad XY^*=0,\quad YY^*=I_{\mathcal{M}}.$$

Since Y is a coisometry, the Wold decomposition of the Hilbert space $\mathcal{M}$ with respect to Y^* is $\mathcal{M}=\mathcal{M}_0\oplus\mathcal{M}_1$ and $Y=S^*\oplus U$, where S is the unilateral shift acting on $\mathcal{M}_0=\bigoplus_{n=0}^{\infty} Y^{*n}(\mathcal{L})$, $\mathcal{L}=\mathcal{M}\ominus Y^*(\mathcal{M})$ is the wandering subspace of S, and U is a unitary operator acting on $\mathcal{M}_1=\mathcal{M}\ominus\mathcal{M}_0=\bigcap_{n=0}^{\infty} Y^{*n}(\mathcal{M})$.

The relation $XY^*=0$ implies $X|_{\mathcal{M}_1}=0$. Therefore, with respect to the decomposition $\mathbf{K}=\mathbf{H}\oplus\mathcal{M}_0\oplus\mathcal{M}_1$, the matrix of V is

$$V=\begin{pmatrix} T & X & 0\\ 0 & S^* & 0\\ 0 & 0 & U\end{pmatrix}$$

where X stands for $X|_{\mathcal{M}_0}$. Therefore $X\colon \mathcal{M}_0\to\mathbf{H}$ and the relations (1.1) become

(1.2) $$TT^*+XX^* = I_{\mathbf{H}}, \quad XS = 0.$$

Obviously, we can consider

$$\mathcal{M}_0 = \ell^2(\mathcal{L}) = \{(x_1, x_2, \ldots);\ x_i\in\mathcal{L},\ \sum_{i=1}^{\infty} \|x_i\|^2 <\infty\}$$

and $S\colon \mathcal{M}_0\to\mathcal{M}_0$, $S(x_1, x_2, \ldots)=(0, x_1, x_2, \ldots)$. Since $XS=0$, it follows that $X(0, y_1, y_2, \ldots)=0$ for every $y_i\in\mathcal{L}$ satisfying $\sum_{i=1}^{\infty} \|y_i\|^2<\infty$. We embed $\mathcal{L}$ in $\ell^2(\mathcal{L})$ by identifying the element $\ell\in\mathcal{L}$ with the element $(\ell, 0, 0, \ldots)\in\ell^2(\mathcal{L})$. In the sequel X stands for $X|_{\mathcal{L}}$. Thus the relations (1.2) become

(1.3) $$TT^*+XX^* = I_{\mathbf{H}},$$

where $X\colon \mathcal{L}\to\mathbf{H}$.

The relation (1.3) holds for $\mathcal{L}=\mathbf{H}$ and $X=(I_{\mathbf{H}}-TT^*)^{1/2}$. With respect to the decomposition $\mathbf{H}=\bigoplus_{\alpha\in J,\, i\in\Lambda} \mathcal{H}_{\alpha,i}$ we have $X=\bigoplus_{\alpha\in J,\, i\in\Lambda} X_{\alpha,i}$, where $X_{\alpha,i}\colon \mathbf{H}\to\mathcal{H}$. Taking into account (1.3), the following relations hold:

(1.4) $$\begin{cases} A_{\alpha,i}A^*_{\alpha,i}+X_{\alpha,i}X^*_{\alpha,i} = I_{\mathcal{H}}, & \text{for } \alpha\in J,\ i\in\Lambda,\\ A_{\alpha,i}A^*_{\alpha,j}+X_{\alpha,i}X^*_{\alpha,j} = 0, & \text{for } \alpha\in J,\ i\neq j,\ i, j\in\Lambda.\end{cases}$$

Let $\{1, 2, \ldots\}=\bigcup_{i\in\Lambda} N_i$ such that $N_i\cap N_j=\emptyset$ $(i\neq j)$ and card $N_i=\aleph_0$ for each $i\in\Lambda$. Setting $N_i=\{n_1^{(i)}, n_2^{(i)}, \ldots\}$, where $n_1^{(i)}<n_2^{(i)}<\ldots$ for each $i\in\Lambda$, we define $Z_i\in B(\ell^2(\mathbf{H}))$ by

$$Z_i(h_1, h_2, \ldots) = (h_{n_1^{(i)}}, h_{n_2^{(i)}}, \ldots), \quad h_j\in\mathbf{H}, \quad (\sum_{j\in\Lambda} \|h_j\|^2 <\infty).$$

Note that $Z_iZ_i^*=I_{\ell^2(\mathbf{H})}$ $(i\in\Lambda)$, and $Z_iZ_j^*=0$ $(i\neq j)$.

Consider the Hilbert space $\mathcal{K}=\mathcal{H}\oplus\ell^2(\mathbf{H})$ and define $V_{\alpha,i}\in B(\mathcal{K})$ $(\alpha\in J,\ i\in\Lambda)$ by the matrix

$$V_{\alpha,i} = \begin{pmatrix} A_{\alpha,i} & W_{\alpha,i}\\ 0 & Z_iS^*\end{pmatrix}$$

where $W_{\alpha,i}$ $(\alpha\in J,\ i\in\Lambda)$ stands for the operator $X_{\alpha,i}\oplus0\oplus0\oplus\ldots$ By (1.4) a simple computation shows that for every $\alpha\in J$, $i\in\Lambda$ we have

$$V_{\alpha,i}V^*_{\alpha,i} = I_{\mathcal{K}}, \quad V_{\alpha,i}V^*_{\alpha,j} = 0 \ (i\neq j), \quad V_{\alpha,i}(\mathcal{H})\subset\mathcal{H} \ \text{ and } \ V_{\alpha,i}|_{\mathcal{H}} = A_{\alpha,i}.$$

Let us prove that, if $A^n_{\alpha,i}\to0$ (strongly) as $n\to\infty$ for some $\alpha\in J$, $i\in\Lambda$, then $V^n_{\alpha,i}\to0$ (strongly) as $n\to\infty$ for the same $\alpha\in J$, $i\in\Lambda$. Note that, with respect to the decomposition $\mathcal{K}=\mathcal{H}\oplus\ell^2(\mathbf{H})$, we have

$$V^n_{\alpha,i} = \begin{pmatrix} A^n_{\alpha,i} & \sum_{j=0}^{n-1} A^j_{\alpha,i}W_{\alpha,i}(Z_iS^*)^{n-j-1}\\ 0 & (Z_iS^*)^n\end{pmatrix}.$$

Let $y=(\underbrace{0, \ldots, 0}_{m-1 \text{ times}}, y_m, 0, 0, \ldots)\in\ell^2(\mathbf{H})$, where $m\geqq 1$, and let $n>m$. Since $W_{\alpha,i}(0, h_2, h_3, \ldots)=0$ for every $(0, h_2, h_3, \ldots)\in\ell^2(\mathbf{H})$, it follows that, if there exists $1\leqq p\leqq m$ such that $(Z_iS^*)^p y=(y_m, 0, 0, \ldots)$, then

$$\sum_{j=0}^{n} A_{\alpha,i}^{j} W_{\alpha,i}(Z_iS^*)^{n-j} y = A_{\alpha,i}^{n-p} W_{\alpha,i}(Z_iS^*)^p y,$$

otherwise we have

$$\sum_{j=0}^{n} A_{\alpha,i}^{j} W_{\alpha,i}(Z_iS^*)^{n-j} y = A_{\alpha,i}^{n} W_{\alpha,i} y.$$

In both the cases, since $A_{\alpha,i}^n\to 0$ (strongly) as $n\to\infty$, it follows that

$$\sum_{j=0}^{n} A_{\alpha,i}^{j} W_{\alpha,i}(Z_iS^*)^{n-j} y \to 0 \quad \text{as} \quad n\to\infty.$$

Since $(Z_iS^*)^n\to 0$ (strongly) too, it follows that $V_{\alpha,i}^n(h\oplus y)\to 0$ as $n\to\infty$, for every $h\in\mathcal{H}$ and all y of the form above mentioned. But, the span of all the vectors y of considered types is the Hilbert space $\ell^2(\mathbf{H})$ and $\|V_{\alpha,i}^n\|\leqq 1$ for each $n\in\{1, 2, \ldots\}$. Thus, we have that $V_{\alpha,i}^n\to 0$ (strongly) as $n\to\infty$ for the same $\alpha\in J$, $i\in\Lambda$ for which $A_{\alpha,i}^n\to 0$ (strongly) as $n\to\infty$. This completes the proof.

We remark that, if card Λ = card $J=1$, we find again the coisometric extension theorem and the de Branges—Rovnyak model theorem (see [9], [5]). The result of E. Durszt and B. Sz.-Nagy [2] is contained also in Proposition 1.1.

2. We say that a family $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ is simultaneously similar to a family $\mathcal{B}=\{B_i\}_{i\in\Lambda}\subset B(\mathcal{K})$ if there exists an invertible operator $R\in B(\mathcal{H}, \mathcal{K})$ so that $A_i=R^{-1}B_iR$ for every $i\in\Lambda$.

In what follows we shall obtain a generalization of a result due to B. Sz.-Nagy [8], that is, an operator $A\in B(\mathcal{H})$ is similar to an isometry if and only if there exist $a\geqq b>0$ such that

$$b\|h\|^2 \leqq \|A^n h\|^2 \leqq a\|h\|^2$$

for every $h\in\mathcal{H}$, $n\in\mathbf{N}$.

We shall also obtain a generalization of Rota's model theorem, for infinite sequences of operators, and we shall give some necessary and sufficient conditions for a family $\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ to be simultaneously similar to a family $\{T_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ of contractions with $\sum_{i\in\Lambda} T_i^*T_i\leqq I_{\mathcal{H}}$.

Let us denote by $F(k, \Lambda)$ the set of all functions from the set $\{1, 2, \ldots, k\}$ to the set Λ. For $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ and $f\in F(k, \Lambda)$, let $A_f=A_{f(1)}A_{f(2)}\ldots A_{f(k)}$.

The following two lemmas are simple extensions of Lemmas 4 and 5 from [1]. We omit the proofs.

Lemma 2.1. *Let* $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ *such that the series* $\sum_{i\in\Lambda} A_i^* A_i$ *is convergent in the strong operator topology (if* card $\Lambda=\aleph_0$*).*

a) *If* $1\leqq m<n$, *then* $\sum_{f\in F(n,\Lambda)} A_f^* A_f=\sum_{q\in F(n-m,\Lambda)} A_q^*\big(\sum_{g\in F(m,\Lambda)} A_g^* A_g\big) A_q$.

b) *For any* $m,n\geqq 1$ $\|\sum_{f\in F(mn,\Lambda)} A_f^* A_f\|\leqq\|\sum_{g\in F(n,\Lambda)} A_g^* A_g\|^m$.

Lemma 2.2. *Let* $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ *with* $\sum_{i\in\Lambda} A_i^* A_i$ *strongly convergent. Then*

$$\lim_{m\to\infty}\|\sum_{f\in F(m,\Lambda)} A_f^* A_f\|^{1/m}=\inf_m\{\|\sum_{f\in F(m,\Lambda)} A_f^* A_f\|^{1/m}\}.$$

Define

$$r(\mathcal{A})=\inf_m\{\|\sum_{f\in F(m,\Lambda)} A_f^* A_f\|^{1/2m}\}.$$

For Λ with card $\Lambda=1$ we find again the well-known formula for the spectral radius of an operator. The case when card $\Lambda<\aleph_0$ is considered in [1].

Proposition 2.3. *Let* $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$. *The following statements are equivalent:*

a) *There exists a family of contractions* $\mathcal{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ *with* $\sum_{i\in\Lambda} T_i^* T_i\leqq I_{\mathcal{H}}$ *such that* $\mathcal{A}$ *is simultaneously similar to* $\mathcal{T}$.

b) *There exists a positive invertible operator* $P\in B(\mathcal{H})$ *such that* $\sum_{i\in\Lambda} A_i^* P A_i\leqq P$.

Proof. Assume a) and let $R\in B(\mathcal{H})$ be an invertible operator such that $A_i=R^{-1}T_i R$ for each $i\in\Lambda$. Since

$$\sum_{i\in\Lambda} T_i^* T_i=(R^{-1})^*\big(\sum_{i\in\Lambda} A_i^* R^* R A_i\big)R^{-1}\leqq I_{\mathcal{H}}$$

it follows that $\sum_{i\in\Lambda} A_i^* P A_i\leqq P$, where $P=R^*R$. Conversely, assume b) and consider $T_i=RA_iR^{-1}$, where $R=P^{1/2}$. Thus, $\sum_{i\in\Lambda} T_i^* T_i\leqq I_{\mathcal{H}}$ and the proof is complete.

A necessary condition for simultaneous similarity is the following

Proposition 2.4. *If a family* $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ *is simultaneously similar to a family* $\mathcal{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ *of contractions with* $\sum_{i\in\Lambda} T_i^* T_i\leqq I_{\mathcal{H}}$, *then there exists* $M>0$ *such that*

$$\sum_{f\in F(n,\Lambda)}\|A_f h\|^2\leqq M\|h\|^2$$

for every $h\in\mathcal{H}$ *and* $n\in\mathbf{N}$. *In particular it follows that* $r(\mathcal{A})\leqq 1$.

Proof. According to Proposition 2.3 there is a positive invertible operator $R\in B(\mathscr{H})$ such that $T_i=RA_iR^{-1}$ for each $i\in\Lambda$. By Lemma 2.1 we have

$$\Big\|\sum_{f\in F(k,\Lambda)} T_f^*T_f\Big\| \leqq \Big\|\sum_{i\in\Lambda} T_i^*T_i\Big\|^k \leqq 1 \quad \text{for any} \quad k\in\mathbf{N}.$$

Since $A_f=R^{-1}T_fR$ $(f\in F(k,\Lambda))$, it follows that

$$\Big\|\sum_{f\in F(k,\Lambda)} A_f^*A_f\Big\| \leqq \|R\|^2\Big\|\sum_{f\in F(k,\Lambda)} T_f^*(R^{-1})^2T_f\Big\| \leqq \|R\|^2\|R^{-1}\|^2$$

for any $k\in\mathbf{N}$. By Lemma 2.2 it is simple to deduce that $r(\mathscr{A})\leqq 1$.

The following proposition is a generalization of the result due to B. Sz.-Nagy [8] (for single operators) and the proof is on the same line.

Proposition 2.5. *Let $\mathscr{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathscr{H})$. The following two conditions are equivalent:*

a) *There exists $\mathscr{V}=\{V_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ with $\sum_{i\in\Lambda} V_i^*V_i=I_{\mathscr{H}}$ such that $\mathscr{A}$ is simultaneously similar to $\mathscr{V}$.*

b) *There exist $a\geqq b>0$ such that*

$$b\|h\|^2 \leqq \sum_{f\in F(n,\Lambda)} \|A_fh\|^2 \leqq a\|h\|^2$$

for any $h\in\mathscr{H}$ and $n\in\mathbf{N}$.

Proof. Assume a) and let $R\in B(\mathscr{H})$ be a positive invertible operator such that $A_i=R^{-1}V_iR$ for each $i\in\Lambda$. Since $\sum_{i\in\Lambda} V_i^*V_i=I_{\mathscr{H}}$, we have, using Lemma 2.1,

$$\sum_{f\in F(n,\Lambda)} V_f^*V_f = I_{\mathscr{H}} \quad \text{for any} \quad n\in\mathbf{N}. \tag{2.1}$$

As in the proof of Proposition 2.4, we obtain that

$$\sum_{f\in F(n,\Lambda)} \|A_fh\|^2 \leqq \|R\|^2\|R^{-1}\|^2\|h\|^2$$

for any $h\in\mathscr{H}$, $n\in\mathbf{N}$. On the other hand using (2.1), we have

$$\sum_{f\in F(n,\Lambda)} \|A_fh\|^2 \geqq \frac{1}{\|R\|^2}\sum_{f\in F(n,\Lambda)} \|V_fRh\|^2 = \frac{1}{\|R\|^2}\|Rh\|^2 \geqq \frac{\|h\|^2}{\|R\|^2\|R^{-1}\|^2}$$

for any $h\in\mathscr{H}$, $n\in\mathbf{N}$.

Conversely, assume condition b) is true. If $\operatorname{card}\Lambda=\aleph_0$, we can take $\Lambda=\mathbf{N}=\{1,2,\ldots\}$. $\sum_{f\in F(n,\Lambda)} \|A_fh\|^2$ is convergent for $h\in\mathscr{H}$ and $n\in\mathbf{N}$, whence $\sum_{f\in F(n,\Lambda)} (A_fh_1, A_fh_2)$ is convergent for every $h_1,h_2\in\mathscr{H}$ and $n\in\mathbf{N}$. Since for any $h_1,h_2\in\mathscr{H}$ we have that

$$\Big|\sum_{f\in F(n,\Lambda)} (A_fh_1, A_fh_2)\Big| \leqq (a/2)(\|h_1\|^2+\|h_2\|^2),$$

for every $n\in\mathbf{N}$, we can define for any $h_1, h_2\in\mathscr{H}$

$$\langle h_1, h_2\rangle = \operatorname{LIM}_{n\to\infty} \sum_{f\in F(n,A)} (A_f h_1, A_f h_2),$$

where LIM means a Banach limit.

Taking into account the properties of the Banach limit we see that $\langle \cdot, \cdot \rangle$ is a hermitian bilinear form and

$$\text{(2.2)} \qquad b\|h\|^2 \leqq \langle h, h\rangle = \operatorname{LIM}_{n\to\infty} \sum_{f\in F(n,A)} \|A_f h\|^2 \leqq a\|h\|^2 \quad \text{for each} \quad h\in\mathscr{H}.$$

By a well-known theorem on the bounded hermitian bilinear form, there exists a selfadjoint operator $P\in B(\mathscr{H})$ such that

$$\langle h_1, h_2\rangle = (Ph_1, h_2) \quad \text{for all} \quad h_1, h_2\in\mathscr{H}.$$

From (2.2) it follows that $bI_{\mathscr{H}}\leqq P\leqq aI_{\mathscr{H}}$, therefore P is a positive invertible operator.

Now we shall show that $P=\sum_{i=1}^{\infty} A_i^* P A_i$. Since the series $\sum_{i\in A} \|A_i h\|^2$ is convergent, for every $\varepsilon>0$, there exists $k_0\in\mathbf{N}$ such that $\sum_{i=k+1}^{\infty} \|A_i h\|^2\leqq\varepsilon/a$ for any $k\geqq k_0$. Thus, for every $k\geqq k_0$ and $n\in\mathbf{N}$ we have:

$$0 \leqq \sum_{i=1}^{\infty} \sum_{f\in F(n,A)} \|A_f A_i h\|^2 - \sum_{i=1}^{k} \sum_{f\in F(n,A)} \|A_f A_i h\|^2 =$$

$$= \sum_{i=k+1}^{\infty} \sum_{f\in F(n,A)} \|A_f A_i h\|^2 \leqq a \sum_{i=k+1}^{\infty} \|A_i h\|^2 \leqq \varepsilon,$$

whence it follows

$$0 \leqq \operatorname{LIM}_{n\to\infty} \Big(\sum_{i=1}^{\infty} \sum_{f\in F(n,A)} \|A_f A_i h\|^2\Big) - \operatorname{LIM}_{n\to\infty} \Big(\sum_{i=1}^{k} \sum_{f\in F(n,A)} \|A_f A_i h\|^2\Big) \leqq \varepsilon$$

for any $k\geqq k_0$. Since

$$(Ph, h) = \operatorname{LIM}_{n\to\infty} \sum_{f\in F(n+1,A)} \|A_f h\|^2 = \operatorname{LIM}_{n\to\infty} \Big(\sum_{i=1}^{\infty} \sum_{f\in F(n,A)} \|A_f A_i h\|^2\Big)$$

for any $h\in\mathscr{H}$, we have that

$$0 \leqq (Ph, h) - \sum_{i=1}^{k} \Big(\operatorname{LIM}_{n\to\infty} \sum_{f\in F(n,A)} \|A_f A_i h\|^2\Big) \leqq \varepsilon \quad \text{for any} \quad k \geqq k_0.$$

In other words

$$(Ph, h) = \lim_{k\to\infty} \sum_{i=1}^{k} \Big(\operatorname{LIM}_{n\to\infty} \sum_{f\in F(n,A)} \|A_f A_i h\|^2\Big) =$$

$$= \sum_{i=1}^{\infty} \langle A_i h, A_i h\rangle = \Big(\Big(\sum_{i=1}^{\infty} A_i^* P A_i\Big)h, h\Big) \quad \text{for every} \quad h\in\mathscr{H}.$$

Therefore, $\sum_{i=1}^{\infty} A_i^* P A_i = P$ and setting $R = P^{1/2}$, $V_i = R A_i R^{-1}$ for each $i \in \Lambda$, we deduce that $\sum_{i \in \Lambda} V_i^* V_i = I_{\mathscr{H}}$. The case when Λ is a finite set is even simpler to deduce. The proof is complete.

We now give a necessary and sufficient condition for simultaneous similarity.

Proposition 2.6. *Let* $\mathscr{A} = \{A_i\}_{i \in \Lambda} \subset B(\mathscr{H})$. *Then the following conditions are equivalent.*

a) *There exists* $\mathscr{T} = \{T_i\}_{i \in \Lambda} \subset B(\mathscr{H})$ *with* $\sum_{i \in \Lambda} T_i^* T_i \leqq I_{\mathscr{H}}$ *such that* $\mathscr{A}$ *is simultaneously similar to* $\mathscr{T}$.

b) *There exist* $D \in B(\mathscr{H})$ *and* $a \geqq b > 0$ *such that*

$$b\|h\|^2 \leqq \sum_{f \in F(n,\Lambda)} \|A_f h\|^2 + \sum_{f \in F(n-1,\Lambda)} \|DA_f h\|^2 + \ldots + \sum_{f \in F(1,\Lambda)} \|DA_f h\|^2 + \|Dh\|^2 \leqq a\|h\|^2$$

for every $h \in \mathscr{H}$, $n \in \mathbf{N}$.

Proof. Assume condition a) is true. Then, according to Proposition 2.3, there exists a positive invertible operator $P \in B(\mathscr{H})$ such that $\sum_{i \in \Lambda} A_i^* P A_i \leqq P$ (we can assume that $\|P\| \leqq 1$). Let

$$D = \Big(P - \sum_{i \in \Lambda} A_i^* P A_i\Big)^{1/2}$$

and for each $h \in \mathscr{H}$ and $n \in \mathbf{N}$ let

$$S_{n,h} = \sum_{f \in F(n,\Lambda)} \|A_f h\|^2 + \sum_{f \in F(n-1,\Lambda)} \|DA_f h\|^2 + \ldots + \sum_{f \in F(1,\Lambda)} \|DA_f h\|^2 + \|Dh\|^2.$$

An easy computation shows that

$$S_{n,h} = \sum_{f \in F(n,\Lambda)} (A_f^* A_f h, h) - \sum_{f \in F(n,\Lambda)} (A_f^* P A_f h, h) + (Ph, h).$$

By Proposition 2.4 there exists $M > 0$ such that

$$S_{n,h} \leqq \sum_{f \in F(n,\Lambda)} (A_f^* A_f h, h) + (Ph, h) \leqq (M+1)\|h\|^2$$

for any $h \in \mathscr{H}$ and $n \in \mathbf{N}$.

On the other hand, since $P \leqq I_{\mathscr{H}}$, we have

$$\sum_{f \in F(n,\Lambda)} (A_f^* A_f h, h) \geqq \sum_{f \in F(n,\Lambda)} (A_f^* P A_f h, h),$$

therefore $S_{n,h} \geqq (Ph, h) \geqq \|R^{-1}\|^{-1}\|h\|^2$ (where $R = P^{1/2}$) for any $h \in \mathscr{H}$ and $n \in \mathbf{N}$.

Now we shall prove that b) implies a). Let us consider the Hilbert space

$$\mathbf{K} = \mathscr{H} \oplus \mathscr{D} \oplus \mathscr{D} \oplus \ldots \quad \text{where} \quad \mathscr{D} = \overline{D\mathscr{H}},$$

and embed $\mathscr{H}$ in $\mathbf{K}$ by identifying the element $h\in\mathscr{H}$ with the element $(h, 0, 0, \ldots)\in\mathbf{K}$. Let $\lambda_i\in\mathbf{C}$ $(i\in\Lambda)$ with $\sum_{i\in\Lambda}|\lambda_i|^2=1$ and define the operators $B_i\in B(\mathbf{K})$ $(i\in\Lambda)$ by

$$B_i(h_0, h_1, \ldots) = (A_i h_0, \lambda_i D h_0, \lambda_i h_1, \ldots).$$

For each $f\in F(n, \Lambda)$ $(n\in\mathbf{N})$ we have:

$$B_f(h_0, h_1, \ldots) = (A_{f(1)} \ldots A_{f(n)} h_0, \lambda_{f(1)} D A_{f(2)} \ldots A_{f(n)} h_0,$$
$$\lambda_{f(1)}\lambda_{f(2)} D A_{f(3)} \ldots A_{f(n)} h_0, \ldots, \lambda_{f(1)} \ldots \lambda_{f(n-1)} D A_{f(n)} h_0,$$
$$\lambda_{f(1)} \ldots \lambda_{f(n)} D h_0, \lambda_f h_1, \ldots)$$

where λ_f stands for $\lambda_{f(1)} \ldots \lambda_{f(n)}$.

Since $\sum_{f\in F(n,\Lambda)} |\lambda_f|^2=1$, for any $n\in\mathbf{N}$, it is easy to show (by induction) that, for each $k\in\{1, 2, \ldots, n-1\}$,

$$\sum_{f\in F(n,\Lambda)} \|\lambda_{f(1)} \ldots \lambda_{f(k)} D A_{f(k+1)} \ldots A_{f(n)} h\|^2 = \sum_{g\in F(n-k,\Lambda)} \|D A_g h\|^2$$

for any $h\in\mathscr{H}$ and $n\in\mathbf{N}$. Therefore

$$\sum_{f\in F(n,\Lambda)} \|B_f(h_0, h_1, \ldots)\|^2 = \sum_{f\in F(n,\Lambda)} \|A_f h_0\|^2 +$$
$$+ \sum_{f\in F(n-1,\Lambda)} \|D A_f h_0\|^2 + \ldots + \sum_{f\in F(1,\Lambda)} \|D A_f h_0\|^2 + \|h_1\|^2 + \ldots$$

for any $(h_0, h_1, \ldots)\in\mathbf{K}$ and $n\in\mathbf{N}$. Thus, by the assumption b), there exist $a\geqq h>0$ such that

$$b\|k\|^2 \leqq \sum_{f\in F(n,\Lambda)} \|B_f k\|^2 \leqq a\|k\|^2 \quad \text{for any} \quad k\in\mathbf{K}, \quad n\in\mathbf{N}.$$

According to Proposition 2.5, there exists $\mathscr{V}=\{V_i\}_{i\in\Lambda}\subset B(\mathbf{K})$ with $\sum_{i\in\Lambda} V_i^* V_i = I_{\mathbf{K}}$ such that the family $\mathscr{B}=\{B_i\}_{i\in\Lambda}$ is simultaneously similar to $\mathscr{V}$. Let us notice that $B_i^*|_{\mathscr{H}}=A_i^*$, $B_i^*(\mathscr{H})\subset\mathscr{H}$. Let $Q\in B(\mathbf{K})$ an invertible operator such that $B_i^*=QV_i^*Q^{-1}$ $(i\in\Lambda)$ and consider the invertible operator $Q_0\colon \mathscr{H}\to\mathscr{H}_0$, $Q_0=Q^{-1}|_{\mathscr{H}}$, where $\mathscr{H}_0$ stands for $Q^{-1}\mathscr{H}$. Since $B_i^*(\mathscr{H})\subset\mathscr{H}$ we have that $V_i^*(\mathscr{H}_0)\subset\mathscr{H}_0$ $(i\in\Lambda)$ and

$$A_i^* = B_i^*|_{\mathscr{H}} = Q_0^{-1}(V_i^*|_{\mathscr{H}_0})Q_0.$$

Using the polar decomposition of Q_0, that is $Q_0=U|Q_0|$ where $|Q_0|=(Q_0^* Q_0)^{1/2}$ and $U=Q_0|Q_0|^{-1}$ we obtain $A_i^*=RT_i^*R^{-1}$, where $R=|Q_0|^{-1}$ and $T_i^*=U^*(V_i^*|_{\mathscr{H}_0})U$. Now $\sum_{i\in\Lambda} V_i^* V_i = I_K$ implies that $\sum_{i\in\Lambda} T_i^* T_i \leqq I_{\mathscr{H}}$. The proof is complete.

Remark 2.7. For Λ with card $\Lambda=1$ the above proposition was proved in [3].

Corollary 2.8. *Let* $D_{\mathscr{A}}=\left|I_{\mathscr{H}}-\sum_{i\in\Lambda}A_i^*A_i\right|^{1/2}$. *If there exists* $a>0$ *such that*

$$\sum_{f\in F(n,\Lambda)}\|A_fh\|^2+\sum_{f\in F(n-1,\Lambda)}\|D_{\mathscr{A}}A_fh\|^2+\ldots+\|D_{\mathscr{A}}h\|^2\leq a\|h\|^2 \tag{2.3}$$

for any $h\in\mathscr{H}$ *and* $n\in\mathbf{N}$, *then the condition* b) *of Proposition* 2.6 *is fulfilled.*

Proof. The upper estimation is trivial. Since for every $h\in\mathscr{H}$, $k\in\mathbf{N}$

$$\sum_{f\in F(k,\Lambda)}(A_f^*D_{\mathscr{A}}A_fh,h)\geq\sum_{f\in F(k,\Lambda)}(A_f^*A_fh,h)-\sum_{f\in F(k+1,\Lambda)}(A_f^*A_fh,h),$$

we have the lower estimation with $b=1$.

The following corollary is a generalization of Rota's model theorem [7].

Corollary 2.9. *Let* $\mathscr{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ *and suppose that there exists* $a>0$ *such that*

$$\sum_{n=1}^{\infty}\Big(\sum_{f\in F(n,\Lambda)}\|A_fh\|^2\Big)\leq a\|h\|^2 \tag{2.4}$$

for any $h\in\mathscr{H}$. *Then there exist a Hilbert space* $\mathscr{K}\supset\mathscr{H}$, *a family* $\mathscr{S}=\{S_i\}_{i\in\Lambda}\subset B(\mathscr{K})$ *of pure coisometries satistying* $S_i(\mathscr{H})\subset\mathscr{H}$ $(i\in\Lambda)$ *with orthogonal initial spaces and an invertible operator* $R\in B(\mathscr{H})$ *such that*

$$A_i=R^{-1}(S_i|_{\mathscr{H}})R\quad\textit{for each}\quad i\in\Lambda.$$

Proof. According to (2.4) and Proposition 2.6 (with $D=I_{\mathscr{H}}$), there exists $\mathscr{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ with $\sum_{i\in\Lambda}T_i^*T_i\leq I_{\mathscr{H}}$ and an invertible operator $R\in B(\mathscr{H})$ such that

$$A_i=R^{-1}T_iR\quad(i\in\Lambda). \tag{2.5}$$

On the other hand, for each $i\in\Lambda$,

$$\sum_{n=1}^{\infty}\|A_i^nh\|^2\leq\sum_{n=1}^{\infty}\Big(\sum_{f\in F(n,\Lambda)}\|A_fh\|^2\Big)\leq a\|h\|^2\quad\text{for any}\quad h\in\mathscr{H}.$$

Hence $A_i^n\to0$ (strongly) as $n\to\infty$ and by (2.5) $T_i^n\to0$ (strongly) for any $i\in\Lambda$. Applying Proposition 1.1, there exist a Hilbert space $\mathscr{K}\supset\mathscr{H}$, a family $\mathscr{S}=\{S_i\}_{i\in\Lambda}\subset B(\mathscr{K})$ of pure coisometries with $\sum_{i\in\Lambda}S_i^*S_i\leq I_{\mathscr{K}}$, $S_i(\mathscr{H})\subset\mathscr{H}$ and $S_i|_{\mathscr{H}}=T_i$ for each $i\in\Lambda$. Thus, $A_i=R^{-1}(S_i|_{\mathscr{H}})R$ $(i\in\Lambda)$ and the proof is complete.

Remark 2.10. For Λ with card $\Lambda=1$ we find again the Rota model theorem, namely, if there exists $a>0$ such that $\sum_{n=1}^{\infty}\|A^nh\|^2\leq a\|h\|^2$ for any $h\in\mathscr{H}$ (equivalent to $r(A)<1$), then A is similar to a part of a backward shift.

We now give some conditions equivalent to condition (2.4) in Corollary 2.9. The proof of the following proposition is almost identical to that of [1, Proposition 6]. We omit the proof.

Proposition 2.11. *Let* $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$. *The following conditions are equivalent.*

a) *There is a positive operator* $P\in B(\mathcal{H})$ *such that* $\left(\sum_{i\in\Lambda} A_i^*PA_i\right)+I_{\mathcal{H}}=P$.

b) *The series* $\sum_{i\in\Lambda} A_i^*A_i$ *and* $\sum_{n=1}^{\infty}\left(\sum_{f\in F(n,\Lambda)} A_f^*A_f\right)$ *are strongly convergent.*

c) *The series* $\sum_{i\in\Lambda} A_i^*A_i$ *is strongly convergent and* $r(\mathcal{A})<1$.

d) *There is* $a>0$ *such that* $\sum_{n=1}^{\infty}\left(\sum_{f\in F(n,\Lambda)} \|A_fh\|^2\right)\leqq a\|h\|^2$ *for any* $h\in\mathcal{H}$.

Remark 2.12. If $\sum_{i\in\Lambda} A_i^*A_i\leqq rI_{\mathcal{H}}$, where $r<1$, then the family $\mathcal{A}=\{A_i\}_{i\in\Lambda}$ satisfies the condition d) of Proposition 2.11.

3. In this section we generalize the result from [5], namely, an operator $A\in B(\mathcal{H})$ is similar to a contraction if and only if there exists $k\in\mathbf{N}$ such that A^k is similar to a contraction. We shall also obtain a formula for $r(\mathcal{A})$, where $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ with $\sum_{i\in\Lambda} A_i^*A_i$ strongly convergent, which generalizes the well-known formula for the spectral radius of an operator $A\in B(\mathcal{H})$, that is,

$$r(A)=\inf_S \|S^{-1}AS\|,$$

where the infimum is taken for all invertible operators $S\in B(\mathcal{H})$ (see [5, Problem 122]).

Proposition 3.1. *Let* $\mathcal{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathcal{H})$. *The following statements are equivalent:*

a) *There is a family* $\mathcal{C}=\{C_i\}_{i\in\Lambda}\subset B(\mathcal{H})$ *with* $\sum_{i\in\Lambda} C_i^*C_i\leqq I_{\mathcal{H}}$ *such that* $\mathcal{A}$ *is simultaneously similar to* $\mathcal{C}$.

b) *There are* $k\in\mathbf{N}$ *and a family* $\mathcal{T}_k=\{T_{(f)}\}_{f\in F(k,\Lambda)}\subset B(\mathcal{H})$ *with* $\sum_{f\in F(k,\Lambda)} T_{(f)}^*T_{(f)}\leqq$ $\leqq I_{\mathcal{H}}$ *such that the family* $\mathcal{A}_k=\{A_f\}_{f\in F(k,\Lambda)}$ *is simultaneously similar to* $\mathcal{T}_k$.

Moreover, a) *implies* b) *for all* $k\in\mathbf{N}$.

Proof. Assume condition a) is true. Let $R\in B(\mathcal{H})$ an invertible operator such that $A_i=R^{-1}C_iR$ $(i\in\Lambda)$. Hence $A_f=R^{-1}C_fR$ for $f\in F(k,\Lambda)$, $k\in\mathbf{N}$. Setting $T_{(f)}=C_f$ for $f\in F(k,\Lambda)$, $k\in\mathbf{N}$, we have that for each $k\in\mathbf{N}$ the family $\mathcal{A}_k$ is simultaneously similar to $\mathcal{T}_k$.

Conversely, assume b) is true. By Proposition 2.3 there is a positive invertible operator $P\in B(\mathcal{H})$ such that

$$\sum_{f\in F(k,\Lambda)} A_f^*PA_f\leqq P. \tag{3.1}$$

Let us consider the positive invertible operator $Q\in B(\mathscr{H})$ given by the relation

$$Q=P+\sum_{n=1}^{k-1}\Big(\sum_{f\in F(n,\Lambda)}A_f^*PA_f\Big).$$

Taking into account (3.1) we have

$$\sum_{i\in\Lambda}A_i^*QA_i=\sum_{n=1}^{k}\Big(\sum_{f\in F(n,\Lambda)}A_f^*PA_f\Big)\leqq P+\sum_{n=1}^{k-1}\Big(\sum_{f\in F(n,\Lambda)}A_f^*PA_f\Big)=Q.$$

It then follows from Proposition 2.3 that a) is true, so the proof is complete.

Corollary 3.2. *Let $\mathscr{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ be such that there exist $k\in\mathbf{N}$, $0<r\leqq 1$ and*

(3.2) $$\sum_{f\in F(k,\Lambda)}\|A_fh\|^2\leqq r\|h\|^2 \quad \textit{for any} \quad h\in\mathscr{H}.$$

Then there exists $\mathscr{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ with $\sum_{i\in\Lambda}T_i^*T_i\leqq I_{\mathscr{H}}$ and such that $\mathscr{A}$ is simultaneously similar to $\mathscr{T}$.

If $0<r<1$, one can even require that $\|T_i\|<1$ for any $i\in\Lambda$.

Proof. Note that the condition (3.2) is equivalent to the condition

(3.3) $$\sum_{f\in F(k,\Lambda)}A_f^*A_f\leqq rI_{\mathscr{H}}.$$

By Proposition 3.1 there exists a family $\mathscr{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ with $\sum_{i\in\Lambda}T_i^*T_i\leqq I_{\mathscr{H}}$ and such that $\mathscr{A}$ is simultaneously similar to $\mathscr{T}$.

If $0<r<1$, then there is $\varepsilon>1$ such that $\Big\|\varepsilon^{2k}\sum_{f\in F(k,\Lambda)}A_f^*A_f\Big\|\leqq 1$. Considering $B_i=\varepsilon A_i$ $(i\in\Lambda)$ we have $\sum_{f\in F(k,\Lambda)}B_f^*B_f\leqq I_{\mathscr{H}}$ and by Proposition 3.1 there exists a family $\mathscr{C}=\{C_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ with $\sum_{i\in\Lambda}C_i^*C_i\leqq I_{\mathscr{H}}$ such that the family $\mathscr{B}=\{B_i\}_{i\in\Lambda}$ is simultaneously similar to $\mathscr{C}$. Hence, $\mathscr{A}$ is simultaneously similar to the family $\mathscr{T}=\{T_i\}_{i\in\Lambda}$, where $T_i=(1/\varepsilon)C_i$ $(i\in\Lambda)$.

Remark 3.3. If $\mathscr{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ and $r(\mathscr{A})<1$, then the condition (3.2) of Corollary 3.2 is fulfilled.

Corollary 3.4. *If $\mathscr{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ and $r(\mathscr{A})=0$ then for every $\varepsilon>0$, there is a family $\mathscr{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ with $\sum_{i\in\Lambda}T_i^*T_i\leqq\varepsilon^2I_{\mathscr{H}}$ $(i\in\Lambda)$ such that $\mathscr{A}$ is simultaneously similar to $\mathscr{T}$.*

Proof. For any $\varepsilon>0$ we have $r(\varepsilon^{-1}\mathscr{A})=\varepsilon^{-1}r(\mathscr{A})=0$, where $\varepsilon^{-1}\mathscr{A}=\{\varepsilon^{-1}A_i\}_{i\in\Lambda}$. Hence, $r(\varepsilon^{-1}\mathscr{A})<1$ and by Remark 3.3 and Corollary 3.2 there is a family $\mathscr{C}=\{C_i\}_{i\in\Lambda}$ with $\sum_{i\in\Lambda}C_i^*C_i\leqq I_{\mathscr{H}}$ such that $\varepsilon^{-1}\mathscr{A}$ is simultaneously similar to $\mathscr{C}$. Setting $\mathscr{T}=\{T_i\}_{i\in\Lambda}$ where $T_i=\varepsilon C_i$ $(i\in\Lambda)$ the proof is complete.

We now use these results for proving the following

Proposition 3.5. *If* $\mathscr{A}=\{A_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ *and* $\sum_{i\in\Lambda} A_i^*A_i$ *is strongly convergent, then*

$$r(\mathscr{A}) = \inf_S \{\|\sum_{i\in\Lambda}(S^{-1}A_iS)^*(S^{-1}A_iS)\|^{1/2}\},$$

where the infimum is taken for all invertible operators $S\in B(\mathscr{H})$.

Proof. First we show that for each invertible operator $S\in B(\mathscr{H})$, $r(\mathscr{A})=r(S^{-1}\mathscr{A}S)$, where $S^{-1}\mathscr{A}S$ stands for the family $\{S^{-1}A_iS\}_{i\in\Lambda}$. By the definition of $r(\mathscr{A})$ we have

$$r(S^{-1}\mathscr{A}S) \leqq \inf_k \{\|S\|^{1/k}\|S^{-1}\|^{1/k}\|\sum_{f\in F(k,\Lambda)} A_f^*A_f\|^{1/2k}\} \leqq r(\mathscr{A}).$$

Hence, $r(\mathscr{A})=r(S(S^{-1}\mathscr{A}S)S^{-1})\leqq r(S^{-1}\mathscr{A}S)$. Therefore,

$$r(\mathscr{A}) = r(S^{-1}\mathscr{A}S). \tag{3.4}$$

Using Lemma 2.1 and (3.4) we obtain

$$r(\mathscr{A}) \leqq \inf_S \{\|\sum_{i\in\Lambda}(S^{-1}A_iS)^*(S^{-1}A_iS)\|^{1/2}\}. \tag{3.5}$$

According to Corollary 3.4, if $r(\mathscr{A})=0$ and $0<\varepsilon<1$, then there is a family $\mathscr{T}=\{T_i\}_{i\in\Lambda}\subset B(\mathscr{H})$ with $\sum_{i\in\Lambda} T_i^*T_i\leqq\varepsilon^2 I_{\mathscr{H}}$, $(i\in\Lambda)$ such that $A_i=R^{-1}T_iR$ $(i\in\Lambda)$ for an invertible operator $R\in B(\mathscr{H})$. Therefore,

$$\|\sum_{i\in\Lambda}(RA_iR^{-1})^*(RA_iR^{-1})\|^{1/2} \leqq \varepsilon$$

whence

$$\inf_S \{\|\sum_{i\in\Lambda}(S^{-1}A_iS)^*(S^{-1}A_iS)\|^{1/2}\} = 0.$$

If $r(\mathscr{A})\neq 0$, let us consider the family $\mathscr{B}=\{B_i\}_{i\in\Lambda}$ where $B_i=(\varepsilon/r(\mathscr{A}))A_i$, $0<\varepsilon<1$, $i\in\Lambda$. Since $r(\mathscr{B})<1$, by Remark 3.3 and Corollary 3.2 there exist a family $\mathscr{C}=\{C_i\}_{i\in\Lambda}$ with $\sum_{i\in\Lambda} C_i^*C_i\leqq I_{\mathscr{H}}$ and a positive invertible operator $P\in B(\mathscr{H})$ such that $B_i=P^{-1}C_iP$ $(i\in\Lambda)$. An easy computation shows that

$$\sum_{i\in\Lambda}(PA_iP^{-1})^*(PA_iP^{-1}) = (r(\mathscr{A})/\varepsilon)^2 \sum_{i\in\Lambda} C_i^*C_i,$$

whence

$$r(\mathscr{A}) \geqq \varepsilon\|\sum_{i\in\Lambda}(PA_iP^{-1})^*(PA_iP^{-1})\|^{1/2} \geqq \varepsilon \inf_S \{\|\sum_{i\in\Lambda}(SA_iS^{-1})^*(SA_iS^{-1})\|^{1/2}\}$$

for every $0<\varepsilon<1$. Setting $\varepsilon\to 1$ it follows that

$$r(\mathscr{A}) \geqq \inf_S \{\|\sum_{i\in\Lambda}(SA_iS^{-1})^*(SA_iS^{-1})\|^{1/2}\}. \tag{3.6}$$

The proof is complete.

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