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On reducing subspaces of composition operators

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If φ is an analytic function mapping the unit disk into itself, RYFF [10] has shown that φ induces a bounded linear operator C_φ on Hardy space H^2 defined by $C_\varphi f = f \circ \varphi$. Many of the basic properties of C_φ depend on the fixed points of φ in the closure of the disk (see [4], [9] for references). If φ is not a rotation about a fixed point, then by the Denjoy—Wolff theorem ([6], [13]) φ has a unique fixed point α such that $|\varphi'(\alpha)| \leq 1$. In this paper, the reducing subspaces of classes of C_φ are characterized when $|\alpha| < 1$ and either φ is univalent or some positive integral power of C_φ is compact. The complementary case when $\alpha = 0$ and φ is inner follows from results of NORDGREN [8] and BROWN [2].

Notion. We will assume henceforth that φ is neither a constant nor a Möbius transformation of the disk onto itself, that α is the Denjoy—Wolff fixed point of φ, and that $|\alpha| < 1$. Then $|\varphi'(\alpha)| < 1$, and there is a natural basis of H^2 with respect to which C_φ is lower triangular with diagonal $[1, \varphi'(\alpha), \varphi'(\alpha)^2, \ldots]$. Indeed, let

$$b_n(\alpha, z) = \frac{(1-|\alpha|^2)^{1/2}}{1-z\bar{\alpha}} \left[\frac{z-\alpha}{1-z\bar{\alpha}}\right]^n \quad (n = 0, 1, \ldots),$$

then for $i \leq j$,

$$\langle C_\varphi b_j, b_i \rangle = \left\langle \frac{1-|\alpha|^2}{1-\varphi(z)\bar{\alpha}} \left[\frac{\varphi(z)-\varphi(\alpha)}{z-\alpha}\right]^j \left[\frac{1-z\bar{\alpha}}{1-\varphi(z)\bar{\alpha}}\right]^j \left[\frac{z-\alpha}{1-z\bar{\alpha}}\right]^{j-i}, \frac{1}{1-z\bar{\alpha}} \right\rangle,$$

which is $\varphi'(\alpha)^i$ whenever $i = j$, and is 0 when $i < j$.

Moreover, if f is in H^2, then $f = \sum \langle f, b_n \rangle b_n$ where

$$\langle f, b_n \rangle = \sum_{k=0}^{n} \binom{n}{k} \frac{f^{(k)}(\alpha)}{k!} (-\bar{\alpha})^{n-k} (1-|\alpha|^2)^{k+(1/2)}.$$

This follows directly by writing f in terms of its Taylor series, expanding

$$\left[\frac{z-\alpha}{1-z\bar{\alpha}}\right]^n = \left[-\alpha + (1-|\alpha|^2)\frac{z}{1-z\bar{\alpha}}\right]^n$$

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by the binomial theorem, and by observing that $\langle (z-\alpha)^k, z^n/(1-z\bar{\alpha})^{n+1}\rangle$ is 0 whenever $n\neq k$, and is 1 when $n=k$, since the adjoint of multiplication by $z/(1-z\bar{\alpha})$ on H^2 maps $h(z)$ into $(h(z)-h(\alpha))/(z-\alpha)$.

We recall that a subspace $\mathcal{M}$ reduces an operator T on a Hilbert space $\mathcal{H}$ if both $\mathcal{M}$ and $\mathcal{H}\ominus\mathcal{M}$ are invariant under T, or equivalently, if the orthogonal projection onto $\mathcal{M}$ commutes with T. If the only subspaces that reduce T are $\{0\}$ and $\mathcal{H}$ itself, then T is said to be irreducible (otherwise T is reducible).

Theorem 1. *If φ is univalent and $\alpha\neq 0$, then C_φ is irreducible.*

When $\alpha=0$, the constants reduce C_φ. In fact, by [4, Theorem 4.1], the kernel of $1-C_\varphi$ contains only the constant functions, so it follows in this case that $\mathcal{M}$ reduces C_φ if and only if $\mathcal{M}=\mathcal{M}_0\oplus\mathcal{M}_1$, where $\mathcal{M}_0$ is either $\{0\}$ or the space of constants, and $\mathcal{M}_1$ reduces the restriction of C_φ to zH^2. A complete description (Theorem 2) of the subspaces $\mathcal{M}_1$ may be obtained under a compactness condition, with univalence weakened to $\varphi'(\alpha)\neq 0$. The study of compact composition operators was initiated by Schwartz in [11], and continued by several authors ([3], [5], [12]). In particular, Caughran and Schwartz [3, Theorem 2] have shown that when some positive integral power of C_φ is compact, the Denjoy—Wolff point always lies inside the disk. Note that $C_\varphi^N=C_{\varphi_N}$ where φ_N is defined inductively by $\varphi_1=\varphi$ and $\varphi_{n+1}=\varphi\circ\varphi_n$.

Theorem 2. *Suppose that C_φ^N is compact for some positive integer N, and that $\varphi'(\alpha)\neq 0$. Then C_φ is reducible if and only if $\alpha=0$. Moreover, if $\alpha=0$, then the restriction of C_φ to zH^2 is reducible if and only if there exists an H^∞ function Ψ which is bounded by one, and a nonnegative integer $p\neq 1$, such that $\varphi(z)=z\Psi(z^p)$; in this case, a subspace $\mathcal{M}$ reduces C_φ restricted to zH^2 if and only if $\mathcal{M}=\bigvee\{b_{ip+j}\colon i\geqq 0, j\in\Gamma\}$ where Γ is an arbitrary subset of $\{1,\dots,p\}$ ($\{1,2,\dots\}$ if $p=0$).*

The reducing subspaces of more general composition operators are formed from cyclic subsets of basis vectors as follows. Let $j\geqq 1$, $p\geqq 0$, and $r\geqq 1$ be integers such that if $p>0$, then $j\leqq p$ and p is relatively prime to r. Let $j_0=j$ and $j_{n+1}=rj_n-i_np$ $(n=0,1,\dots)$ where i_n is the unique integer which is 0 if $p=0$, and satisfies $i_np<rj_n\leqq(i_n+1)p$ if $p>0$. The set $\{j_n\colon n\geqq 0\}$ will be called the (r,p)-cycle generated by j. Let $p>0$. Then since $1\leqq j_n\leqq p$ for all n, the terms of the sequence j_n repeat. It follows easily that if $j_{m+1}=j_{n+1}$ $(m>n)$, then $j_m=j_n$; and hence $j_{m-n}=j$. Therefore, j is the first term to reappear. Moreover, the set $\{1,\dots,p\}$ ($\{1,2,\dots\}$ if $p=0$) may be written as a disjoint union of (r,p)-cycles. With no additional conditions on φ, we have

Theorem 3. *If $\alpha\neq 0$, then no nontrivial closed span of basis vectors b_n $(n\geqq 0)$ reduces C_φ. If $\alpha=0$, and is of order r, then a nontrivial closed span $\mathcal{M}$ of vectors b_n*

$(n\geqq 1)$ reduces the restriction of C_φ to zH^2 if and only if there exists a nonnegative integer $p\neq 1$, which is relatively prime to r whenever $p\neq 0$, such that $\varphi(z)=z^r\Psi(z^p)$ for some H^∞ function Ψ which is bounded by one, and $\mathcal{M}=\vee\{b_{ip+j}\colon i\geqq 0,\ j\in\Gamma\}$ where Γ is a union of (r,p)-cycles.

In view of the above results, a natural question is: when either φ is univalent or C_φ^N is compact, are all the reducing subspaces of C_φ closed spans of basis vectors? The related step in the proof of Theorem 2 follows by expressing the span of the first n basis vectors $(n=0, 1, \ldots)$ in terms of the kernel of some element of the von Neumann algebra generated by 1, C_φ and C_φ^*. A similar argument may be used in the following example.

Example 1. Let $\varphi=\lambda\theta$ where λ is a constant $(0<|\lambda|<1)$ and θ is an inner function such that $\theta(0)=0$. By ([1, Theorem 20], [8, Theorem 1], or [10, Theorem 3]), C_θ is an isometry, so that $C_\varphi^* C_\varphi$ is a diagonal operator with diagonal $(1, |\lambda|, |\lambda|^2, \ldots)$. Therefore, $\bigvee_0^n b_i=\ker\prod_0^n (C_\varphi^* C_\varphi-|\lambda|^i)$ $(n=0, 1, \ldots)$, and it follows that the reducing subspaces are closed spans of b_n's and are thus described by Theorem 3.

Further evidence is provided by the following result which implies that reducing subspaces are (at least) closed spans of *finite* linear combinations of basis vectors.

Theorem 4. *Suppose that $\|\varphi\|_\infty<1$ and $\varphi'(\alpha)=0$. If X commutes with C_φ and (λ_{ij}) is the matrix of X with respect to $\{b_n\}$, then $\lambda_{0j}=0$ $(j=1, 2, \ldots)$ and there exists an integer M such that $\lambda_{ij}=0$ $(i=1, 2, \ldots)$ for every $j\geqq Mi$.*

Theorem 4 suggests an alternative approach to answering the above question in the affirmative, as illustrated by

Example 2. Let $\alpha=0$ be of order $r>1$, and suppose that φ is a polynomial of degree r^M such that $\|\varphi_N\|_\infty<1$ for some positive integers M and N. Then the reducing subspaces of C_φ are given by Theorem 3: Let P be the projection onto a reducing subspace. Since P commutes with $C_{\varphi_N}^*$, it follows from Theorem 4 that Pb_n is a polynomial for every n; thus, it suffices to show that the degree of Pb_n does not exceed n for every n. Suppose that $n<\deg Pb_n$ for some n, and let i be the least such integer. Setting $j=\deg Pb_i$, we have that

$$\langle PC_{\varphi_N}^{*M} C_{\varphi_N} b_i, b_j\rangle = \langle C_{\varphi_N}^{*M} C_{\varphi_N} Pb_i, b_j\rangle,$$

and hence by straightforward calculations, $\mu^i=\mu^j$ where

$$\mu = [\langle b_r, \varphi\rangle^{1/(r-1)}\langle \varphi, b_{r^M}\rangle^{1/(r^M-1)}]^{r^{MN}-1}$$

(so that $0<|\mu|<1$). Therefore $i=j$, a contradiction.

The verification of Theorem 3 depends upon a reformulation of the usual multinomial theorem, which subsequently determines how often powers of φ have nonzero coefficients.

Lemma. *Let*

$$f(z) = a_{10}+\sum_{i=1}^{\infty}\sum_{j=0}^{p-1} a_{1,ip+j}z^{ip+j}$$

be a formal power series where a_{10} is nonzero, and for $j=0, 1, \ldots,$ define

$$\hat{a}_{1j} = a_{1,p+j} \quad \textit{and} \quad \hat{a}_{mj} = \sum_{k=0}^{j} a_{1,p+k}\hat{a}_{m-1,j-k} \quad (m > 1).$$

Then for every positive integer n,

$$f(z)^n = a_{10}^n+\sum_{i=1}^{\infty}\sum_{j=0}^{p-1} a_{n,ip+j}z^{ip+j}$$

where

$$a_{n,ip+j} = \sum_{m=1}^{i}\binom{n}{m} a_{10}^{n-m}\hat{a}_{m,(i-m)p+j}.$$

In particular, for fixed i and j, either $a_{n,ip+j}=0$ for all $n\geqq 1$, or $a_{n,ip+j}=0$ for at most $i-1$ values of $n\geqq 1$. If $a_{1p}\neq 0$, then $a_{n,ip}=0$ for at most $i-1$ values of $n\geqq 1$.

The following estimate in H^∞ is essential to the proof of Theorem 4, and may be of independent interest.

Proposition. *If $\varphi^{(i)}(\alpha)=0$ for every $i=1, \ldots, r-1$, then*

$$\|C_\varphi^n b_m\|_\infty \leqq \left\|\frac{\varphi-\alpha}{1-\varphi\bar{\alpha}}\right\|_\infty^{m(r^n-1)/(r-1)}$$

for all nonnegative integers m and n.

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Proof of Theorem 1. Let $\mathcal{M}$ reduce C_φ. By [4, Theorem 4.1], the kernel of $1-C_\varphi$ consists of just the constant functions. Thus we may assume that constants belong to $\mathcal{M}$, and hence so does $C_\varphi^{*n}1=[1-z\overline{\varphi_n(0)}]^{-1}$ $(n=1, 2, \ldots)$.

Let f be orthogonal to $\mathcal{M}$. Then f vanishes on the set $\Omega=\{\varphi_n(0)\colon n\geqq 1\}$. If $\varphi_m(0)=\varphi_{n+m}(0)$ for some positive integers m and n, then $\varphi_m(0)$ is a fixed point of φ_n. But α is also a fixed point of φ_n, and φ_n is not a rotation about a fixed point since otherwise φ would be inner and hence by [7, Theorem 3.17] φ would be a Möbius transformation. Thus it follows that $\varphi_m(0)=\alpha=\varphi_m(\alpha)$, and since φ_m is univalent, we have that $\alpha=0$, a contradiction. Hence, the set Ω consists of distinct points which must cluster in the closure of the disk. However, by Schwarz's lemma,

$$\left|\frac{\varphi_n(z)-\alpha}{1-\varphi_n(z)\bar{\alpha}}\right| \leqq \left|\frac{z-\alpha}{1-z\bar{\alpha}}\right|$$

for every z in the disk. Setting $z=0$, we conclude that Ω must cluster inside the disk. Therefore, f is identically zero.

Proof of the lemma. The formula is obvious for $n=1$, so by induction, we assume it is valid for some n. Multiplying $f(z)^{n+1}=f(z)\cdot f(z)^n$, we have that

$$a_{n+1,ip+j} = a_{10}\,a_{n,ip+j}+\Big[\sum_{k=0}^{(i-2)p+j} a_{1,p+k}\,a_{n,(i-1)p+j-k}\Big]+a_{1,ip+j}\,a_{10}^{n} =$$

$$= \left[\sum_{m=1}^{i}\binom{n}{m} a_{10}^{n-m+1}\hat{a}_{m,(i-m)p+j}\right]+\left[\sum_{m=2}^{i}\binom{n}{m-1} a_{10}^{n-m+1}\hat{a}_{m,(i-m)p+j}\right]+a_{10}^{n}a_{1,ip+j} =$$

$$= \sum_{m=1}^{i}\left[\binom{n}{m}+\binom{n}{m-1}\right] a_{10}^{n-m+1}\hat{a}_{m,(i-m)p+j} = \sum_{m=1}^{i}\binom{n+1}{m} a_{10}^{(n+1)-m}\hat{a}_{m,(i-m)p+j}.$$

Thus, the form of $f(z)^n$ follows for every n.

Fix i and j, and let $n\geqq i$. Then $(na_{10}^{n})^{-1}a_{n,ip+j}$ is a polynomial in n of degree at most $i-1$. Suppose that $a_{k,ip+j}=0$ for some k such that $1\leqq k\leqq i-1$. It follows that the sum of the first k terms of $(na_{10}^{n})^{-1}a_{n,ip+j}$ is equal to

$$\sum_{m=1}^{k}\frac{1}{m!}\left[\frac{(n-1)!}{(n-m)!}-\frac{(k-1)!}{(k-m)!}\right] a_{10}^{-m}\hat{a}_{m,(i-m)p+j},$$

which is divisible by $n-k$. Since each of the last $i-k$ terms contains a factor of $n-k$, we have that $n-k$ divides $(na_{10}^{n})^{-1}a_{n,ip+j}$. Therefore, either all of the coefficients of $(na_{10}^{n})^{-1}a_{n,ip+j}$ $(n\geqq i)$ are zero, or $a_{n,ip+j}=0$ for at most $i-1$ values of $n\geqq 1$.

Finally, suppose that $a_{1p}\neq 0$. Then the leading coefficient of $(na_{10}^{n})^{-1}a_{n,ip}$ $(n\geqq i)$ is $(i!\,a_{10}^{i})^{-1}\hat{a}_{i0}=(i!\,a_{10}^{i})^{-1}a_{1p}^{i}\neq 0$. Hence, $a_{n,ip}=0$ for at most $i-1$ values of $n\geqq 1$.

Proof of Theorem 3. Let $\alpha\neq 0$, and suppose that $\mathcal{M}$ is a nontrivial closed span of b_n's $(n\geqq 0)$ which reduces C_φ. Since $\mathcal{M}^\perp$ is of the same form, we may assume that b_0 is in $\mathcal{M}$. Let n be the greatest integer such that b_m belongs to $\mathcal{M}$ for every $m=0, \ldots, n$. Since $\mathcal{M}$ is invariant under C_φ, we have that $f=C_\varphi\sum_{0}^{n}(-\bar{\alpha})^m\times$

$\times(1-|\alpha|^2)^{-1/2}b_m$ is in $\mathcal{M}$ and $\langle f, b_{n+1}\rangle=0$. However, by induction,

$$f'(z) = -(n+1)(-\bar{\alpha})^{n+1}\varphi'(z)[1-\varphi(z)\bar{\alpha}]^{-n-2}[\varphi(z)-\alpha]^n$$

and

$$f^{(n+1)}(\alpha) = -(n+1)!(-\bar{\alpha})^{n+1}(1-|\alpha|^2)^{-n-2}\varphi'(\alpha)^{n+1}.$$

Therefore,

$$\langle f, b_{n+1}\rangle = (-\bar{\alpha})^{n+1}(1-|\alpha|^2)^{-1/2}[1-\varphi'(\alpha)^{n+1}] \neq 0,$$

a contradiction. Thus, $\mathcal{M}$ must be trivial.

Next, let $\alpha=0$ be a zero of φ of order r, and let $\mathcal{M}$ be a nontrivial closed span of vectors $b_n=z^n$ $(n\geqq 1)$ which reduces the restriction of C_φ to zH^2. Let us write $\varphi(z)=z^r\Psi(z)$ for some H^∞ function Ψ. If Ψ is a constant function, then the proposed forms of φ and $\mathcal{M}$ clearly follow with $p=0$. Henceforth, using the notation of the lemma, we assume that $\Psi(z)=\sum\langle\Psi, b_n\rangle b_n=a_{10}+a_{1q}z^q+a_{1,q+1}z^{q+1}+\dots$ where a_{10} and a_{1q} are nonzero. Observe that if $\mathcal{M}$ contains z^n, and $a_{n,rq}\neq 0$, then $\mathcal{M}$ also contains z^{n+q}. Indeed, $\varphi(z)^n=z^{rn}\Psi(z)^n$ is in $\mathcal{M}$, and $\langle\varphi(z)^n, z^{r(n+q)}\rangle\neq 0$; therefore, by the given form of $\mathcal{M}$, $z^{r(n+q)}$ belongs to $\mathcal{M}$. Since z^{rm} is orthogonal to $\mathcal{M}$ whenever z^m is, we have that $\mathcal{M}$ contains z^{n+q}.

Let z^n be in $\mathcal{M}$. By the lemma, there exists an integer K such that $a_{k,rq}\neq 0$ for every $k\geqq r^K n$. Now, $z^{r^K n}$ is in $\mathcal{M}$. And, if $z^{r^K n+mq}$ is in $\mathcal{M}$ for some $m\geqq 0$, then $a_{r^K n+mq,rq}\neq 0$, and, by the above argument, $\mathcal{M}$ contains $z^{r^K n+(m+1)q}$. Thus, by induction, $z^{r^K n+mq}$ is in $\mathcal{M}$ for every $m=0, 1, \dots$, and hence, in particular, $z^{r^K(n+q)}$ is in $\mathcal{M}$. Consequently, $\mathcal{M}$ contains z^{n+q} whenever it contains z^n.

For integers i and j, let $i\wedge j$ denote the greatest common divisor of i and j. Let $q(1)=q$, and for $t=2, 3, \dots$, define $q(t)=[r\wedge q(t-1)]^{-1}q(t-1)$. Since $\{q(t)\}$ is a monotonically decreasing sequence of positive integers, there exists a least integer T such that $q(T+1)=q(T)$, i.e., $r\wedge q(T)=1$. Note that $q(T)=r^{-T}\varrho q$ where $\varrho=\prod_1^T[r\wedge q(t)]^{-1}r$. If z^n belongs to $\mathcal{M}$, it follows that $z^{n+q(T)}=z^{r^{-T}(r^T n+\varrho q)}$ belongs to $\mathcal{M}$. Similarly, the orthogonal complement of $\mathcal{M}$ in zH^2 is invariant under multiplication by $z^{q(T)}$. Therefore, there exists a subset Γ_1 of $\{1, \dots, q(T)\}$ such that $\mathcal{M}$ is the closed span of vectors of the form $z^{iq(T)+j}$ $(i\geqq 0;\ j\in\Gamma_1)$. Furthermore, if j is in Γ_1, then $iq(T)<rj\leqq(i+1)q(T)$ for some integer i, and $rj-iq(T)$ is in Γ_1. Hence, Γ_1 is a union of $[r, q(T)]$-cycles.

If $\Psi=\Psi(z^{q(T)})$, let $p=q(T)$ and $\Gamma=\Gamma_1$; otherwise, let $p(1)=q(T)$. Suppose that for some integer $s\geqq 1$, a positive integer $p(s)$, relatively prime to r, is defined such that $\mathcal{M}=\bigvee\{z^{ip(s)+j}\colon i\geqq 0,\ j\in\Gamma_s\}$ for some union Γ_s of $[r, p(s)]$-cycles, and $\Psi\neq\Psi(z^{p(s)})$. Let $I=\min\{i\colon a_{1,ip(s)+j}\neq 0$ for some j such that $0<j<p(s)\}$, and let $J=\min\{j>0\colon a_{1,Ip(s)+j}\neq 0\}$. By the lemma, $a_{n,Ip(s)+J}=na_{10}^{n-1}a_{1,Ip(s)+J}\neq 0$ for every $n=1, 2, \dots$. As above, $z^{rn+Ip(s)+J}$, and hence z^{rn+J}, belong to $\mathcal{M}$ whenever z^n belongs to $\mathcal{M}$.

Let z^n be in $\mathcal{M}$. Then $n=ip(s)+j$ where j is in Γ_s and $i\geqq 0$. Since Γ_s is a union of cycles, there exists an element j' in Γ_s and an integer i' such that $j=rj'-i'p(s)$. Thus, $n+J=(i-i')p(s)+(rj'+J)$, so that z^{n+J} is in $\mathcal{M}$. Similarly, $zH^2\ominus\mathcal{M}$ is invariant under multiplication by z^J.

Let $p(s+1)=p(s)\wedge J$. Then $p(s+1)$ is relatively prime to r, and there exist integers u and v such that $p(s+1)=up(s)+vJ$. It follows that $\mathcal{M}$ and the orthogonal complement of $\mathcal{M}$ in zH^2 are invariant under multiplication by $z^{p(s+1)}$, and hence, as above, there exists a union Γ_{s+1} of $[r, p(s+1)]$-cycles such that $\mathcal{M}=\bigvee\{z^{ip(s+1)+j}:$ $i\geqq 0,\ j\in\Gamma_{s+1}\}$. Therefore, $p(s+1)$ $(s=1, 2, \ldots)$ may be defined recursively provided $\Psi\neq\Psi(z^{p(s)})$ for every s. But this is impossible since $\{p(s)\}$ is a strictly decreasing sequence of positive integers. Consequently, there exists an integer S such that $\Psi=\Psi(z^{p(S)})$, and the forms of φ and $\mathcal{M}$ follow by setting $p=p(S)$ and $\Gamma=\Gamma_S$. Note that $p\neq 1$ since $\mathcal{M}$ is nontrivial.

Conversely, suppose that $\varphi(z)=z^r\Psi(z^p)$ where $r\wedge p=1$ if $p\neq 0$, and that $\mathcal{M}=\bigvee\{z^{ip+j}:\ i\geqq 0,\ j\in\Gamma\}$ for some union Γ of (r,p)-cycles. If $p=0$, then clearly $\mathcal{M}$ is invariant under C_φ; so we assume that $p>1$. If z^n belongs to $\mathcal{M}$, then so does z^{rn+mp} for every $m\geqq 0$. Indeed $n=ip+j$ where j is in Γ and $i\geqq 0$ and there exists an integer i' such that $i'p<rj\leqq(i'+1)p$. Hence, $rj=i'p+j'$, where j' is in Γ by the definition of (r,p)-cycle. Therefore, $z^{rn+mp}=z^{(ri+i'+m)p}z^{j'}$ is in $\mathcal{M}$, and thus, so is $\varphi(z)^n=z^{rn}\Psi(z^p)^n$. It follows that $\mathcal{M}$ is invariant under C_φ.

Finally, $\mathcal{M}^\perp$ is invariant under C_φ since it is the closed span of vectors of the form z^{ip+j} $(i\geqq 0;\ j\in\Gamma')$, where Γ' is the complement in $\{1, \ldots, p\}$ ($\{1, 2, \ldots\}$, if $p=0$) of Γ and is hence the union of (r,p)-cycles.

Proof of Theorem 2. Since $C_\varphi^{*^N}=C_{\varphi_N}^*$ is compact with nonzero eigenvalues $\overline{\varphi'(\alpha)}^{mN}$ $(m=0, 1, \ldots)$, it follows from [4, Theorem 4.1] that $\overline{\varphi'(\alpha)}^m$ is an eigenvalue of C_φ^* of multiplicity one for every m. Thus, by the matrix of C_φ^* with respect to $\{b_n\}$, we have that $\bigvee_0^n b_m=\ker\prod_0^n[C_\varphi^*-\overline{\varphi'(\alpha)}^m]$ for every $n=0, 1, \ldots$. Therefore, by induction, either b_n belongs to $\mathcal{M}$ or is orthogonal to $\mathcal{M}$ for each n, and hence the form of $\mathcal{M}$ is given by Theorem 3.

Proof of the proposition. Using induction on n with m fixed, the case $n=0$ is obvious, so we assume that the inequality holds for some n. Since $\varphi(\alpha)=\alpha$ and $\varphi^{(i)}(\alpha)=0$ $(i=1, \ldots, r-1)$, we have that $\varphi_n^{(i)}(\alpha)=0$ for every $i=1, \ldots, r^n-1$. Hence, $C_\varphi^{n+1}b_m=C_\varphi f$ where $f=b_m(\alpha, \varphi_n)=[(z-\alpha)/(1-z\bar\alpha)]^{mr^n}g$ for some H^∞ function g. Therefore, $\|C_\varphi^{n+1}b_m\|_\infty=\|(\varphi-\alpha)/(1-\varphi\bar\alpha)\|_\infty^{mr^n}\|g(\varphi)\|_\infty$ where $\|g(\varphi)\|_\infty\leqq$ $\leqq\|g\|_\infty=\|f\|_\infty\leqq\|(\varphi-\alpha)/(1-\varphi\bar\alpha)\|_\infty^{m(r^n-1)/(r-1)}$ by the induction hypothesis. The case $n+1$ now follows by combining the above inequalities.

Proof of Theorem 4. Since $\|\varphi\|_\infty<1$, C_φ is compact by [11, Theorem 5.2].

Therefore, by [4, Theorem 4.1], the kernel of $1-C_\varphi^*$ is one-dimensional, and since it is invariant under X^*, we have that $\lambda_{0j}=0$ $(j\geqq 1)$.

Suppose that $\varphi^{(m)}(\alpha)=0$ $(m=1, ..., r-1)$ and $\varphi^{(r)}(\alpha)\neq 0$. By direct computations, there exist constants $\mu_{i'}=\mu_{i'}(n)$ such that for every $i\geqq 0$,

$$b_i = \bar{\mu}^{-i(r^n-1)/(r-1)} C_\varphi^{*^n} b_{ir^n} + \sum_{i'<i} \mu_{i'} b_{i'}$$

where $\mu=(1-|\alpha|^2)^{r-1}\varphi^{(r)}(\alpha)(r!)^{-1}$. Moreover, since $\|\varphi\|_\infty<1$, it follows that $\|(\varphi-\alpha)/(1-\varphi\bar{\alpha})\|_\infty<1$, and hence there exists an integer $M\geqq 1$ such that $\|(\varphi-\alpha)/(1-\varphi\bar{\alpha})\|_\infty^M<|\mu|$. Thus,

$$\lambda_{ij} = \langle Xb_j, b_i\rangle_2 = \mu^{-i(r^n-1)/(r-1)} \langle XC_\varphi^n b_j, b_{ir^n}\rangle_2 + \sum_{i'<i} \bar{\mu}_{i'} \lambda_{i'j}$$

and consequently by the proposition, for $j\geqq Mi$ we have that

$$\Big|\lambda_{ij} - \sum_{i'<i} \bar{\mu}_{i'} \lambda_{i'j}\Big| \leqq \|X\|_2 \left(\left\|\frac{\varphi-\alpha}{1-\varphi\bar{\alpha}}\right\|_\infty^M |\mu|^{-1}\right)^{i(r^n-1)/(r-1)}.$$

Therefore, the theorem follows by induction on $i\geqq 1$, and the separate case $i=0$, since the right hand side converges to zero as n tends to infinity.

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