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The numerical ranges and the smooth points of the unit sphere

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I. Let S_p be the unit sphere of a complex Banach space (E, p). The set of all smooth points on S_p will be denoted by F_p. The element $x \in S_p$ is a smooth point if and only if the Gâteaux derivative p' at x exists. We denote by $V_{D_p}(T)$ the spatial numerical range of T. If the unit sphere is smooth, then the relation

$$V_{D_p}(T) = \{p'(x, Tx) - ip'(x, iTx) \colon x \in S_p\}$$

holds. We assume that the set F_p is dense in the unit sphere S_p, e.g. this holds for separable or reflexive Banach spaces. We prove that for continuous operators T the closure of the set

$$\{p'(x, Tx) - ip'(x, iTx) \colon x \in F_p\}$$

is the closure of a Lumer numerical range of T.

II. Let D_p be the mapping of S_p into the power set of the dual E' of E defined by

$$D_p(x) = \{f \in E' \colon f(x) = 1, \ |f(y)| \leqq p(y), \ (y \in E)\}.$$

We consider the continuous operator $G \colon E \to E$ with the domain $D(G) \subseteqq S_p$. For a mapping Q_p of $D(G)$ into the power set of E' with

$$\emptyset \neq Q_p(x) \subseteqq D_p(x) \quad (x \in D(G))$$

the set

$$V_{Q_p}(G) = \{f(Gx) \colon f \in Q_p(x), \ x \in D(G)\}$$

is called the numerical range of G corresponding to Q_p. (See [7].) If card $Q_p(x) = 1$ $(x \in D(G))$ holds, then $V_{Q_p}(G)$ is a Lumer numerical range. $V_{D_p}(G)$ is called the spatical numerical range of G.

Theorem 1. *Let T be a continuous operator of S_p into E. If $V_{\tilde{Q}_p}(T|A)$ is a numerical range of the restriction of T to the subset A of S_p with* cl $A = S_p$, *then there exists an extension Q_p of $\tilde{Q}_p$ to the unit sphere S_p such that*

$$\mathrm{cl}\, V_{\tilde{Q}_p}(T|A) = \mathrm{cl}\, V_{Q_p}(T).$$

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Proof. Let $x \in S_p \setminus A$. Then there are sequences (x_n) in A and (f_n) with $f_n \in \tilde{Q}_p(x_n)$ and $p(x_n - x) \to 0$. Since the unit ball of E' is weak*-compact, we can choose subnets $(f_\beta)_{\beta \in B}$ of (f_n) and $(x_\beta)_{\beta \in B}$ of (x_n) and an $f_x \in E'$ such that

$$(f_\beta)_{\beta \in B} \text{ is weak *-convergent to } f_x \text{ and } p(x_\beta - x) \to 0.$$

The inequalities

$$|f_n(y)| \leqq p(y) \quad (y \in E,\ n \in N)$$

imply

$$|f_x(y)| \leqq p(y) \quad (y \in E).$$

But since

$$f_\beta(x_\beta) = f_\beta(x_\beta - x) + f_\beta(x); \quad |f_\beta(x_\beta - x)| \leqq p(x_\beta - x)$$

we deduce $f_\beta(x_\beta) \to f_x(x)$ and $f_x(x) = 1$. So we have $f_x \in D_p(x)$. Now we extend the mapping $\tilde{Q}_p$ by the definition

$$Q_p(z) = \begin{cases} \tilde{Q}_p(z) & \text{for } z \in A, \\ \{f_z\} & \text{for } z \in S_p \setminus A. \end{cases}$$

It is clear that the relation $\operatorname{cl} V_{\tilde{Q}_p}(T|A) \subseteqq \operatorname{cl} V_{Q_p}(T)$ holds. It remains to show that the scalar $f_x(Tx)$ is a cluster point of $V_{\tilde{Q}_p}(T|A)$ $(x \in S_p \setminus A)$. By the construction there are nets $(x_\beta)_{\beta \in B}$ of A and $(f_\beta)_{\beta \in B}$ with $f_\beta \in \tilde{Q}_p(x_\beta)$ such that

$$f_\beta(y) \to f_x(y) \ (y \in E) \quad \text{and} \quad p(x_\beta - x) \to 0.$$

The inequality $|f_\beta(Tx_\beta - Tx)| \leqq p(Tx_\beta - Tx)$ and the continuity of T imply $f_\beta(Tx_\beta - Tx) \to 0$. Hence from the relation

$$f_\beta(Tx_\beta) = f_\beta(Tx) + f_\beta(Tx_\beta - Tx)$$

follows $f_\beta(Tx_\beta) \to f_x(Tx)$.

Remark 1. The proof of Theorem 1 shows that there exists an extension Q_p of $\tilde{Q}_p$ satisfying the condition $\operatorname{card} Q_p(x) = 1$ $(x \in S_p \setminus A)$.

Theorem 2. *Let T be a continuous operator of S_p into E. If F_p is dense in S_p, then the set*

$$\operatorname{cl} \{p'(x, Tx) - ip'(x, iTx) \colon x \in F_p\}$$

is the closure of a Lumer numerical range of T corresponding to a mapping Q_p defined on the whole S_p.

Proof. We applicate Theorem 1 putting $A = F_p$. There exists exactly one mapping $\tilde{Q}_p$ of F_p into the power set of E' with $\emptyset \neq \tilde{Q}_p(x) \subseteqq D_p(x)$ $(x \in F_p)$. By [6] holds

$$V_{\tilde{Q}_p}(T|F_p) = \{p'(x, Tx) - ip'(x, iTx) \colon x \in F_p\}.$$

Hence by Theorem 1 the conclusion follows.

Corollary 1. *Let T be a linear continuous operator of S_p into E. If F_p is dense in S_p, then for the numerical radius $v_p(T)$ the following relation holds:*

$$v_p(T) = \sup_{x \in F_p} |p'(x, Tx) - ip'(x, iTx)|.$$

Remark 2. The condition $\mathrm{cl}\, F_p = S_p$ is fulfilled for separable Banach spaces (see [5]), and for reflexive Banach spaces (see [3]).

Remark 3. Let E be a separable Banach space and let T be a linear continuous operator of S_p into E. While the set

$$\mathrm{cl}\, \{p'(x, Tx) - ip'(x, iTx) \colon x \in F_p\}$$

is the closure of a Lumer numerical range of T defined on the whole S_p, in general it is not the closure of the spatial numerical range of T. We consider the following example.

Let c_0 be the Banach space of all complex null sequences $x = (x_i)$ equipped with the norm $p(x) = \max |x_i|$. Then $x \in S_p$ is a smooth point on S_p if and only if the relation $|x_i| = 1$ holds for exactly one coordinate x_i of x; let be $|x_{i(x)}| = 1$. Using the functional f_x defined by

$$f_x(y) = y_{i(x)} \bar{x}_{i(x)} \quad (y = (y_i)),$$

it follows $\tilde{Q}_p(x) = \{f_x\}$ $(x \in F_p)$. For the operator T with

$$Tx = (x_1, 1/2 x_2, 1/3 x_2, \ldots, 1/n x_n, \ldots) \quad (x \in c_0)$$

one obtains $V_{\tilde{Q}_p}(T|F_p) = \{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$. Therefore the set $\{1/n \colon n \in \mathbf{N}\} \cup \{0\}$ is the closure of a Lumer numerical range of T defined on the whole S_p. The closure of the spatial numerical range of T is the interval $\{\lambda \in \mathbf{R} \colon 0 \leq \lambda \leq 1\}$.

References

[1] F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, London Math. Soc. Lect. Note series, 2 (Campbridge, 1971).

[2] F. F. Bonsall and J. Duncan, *Numerical Ranges. II*, London Math. Soc. Lect. Note series, 10 (Cambridge, 1973).

[3] J. Lindenstrauß, On nonseparable reflexive Banach spaces, *Bull. Amer. Math. Soc.*, **72** (1966), 967—970.

[4] G. Lumer, Semi-inner-product spaces, *Trans. Amer. Math. Soc.*, **100** (1961), 29—43.

[5] S. Mazur, Über konvexe Mengen in linearen normierten Räumen, *Studia Math.*, **4** (1933), 70—84.

[6] A. Rhodius, Der numerische Wertebereich für nicht notwendig lineare Abbildungen in lokalkonvexen Räumen, *Math. Nachrichten*, **72** (1976), 169—180.

[7] A. Rhodius, Über zu Halbnormen gehörende numerische Wertebereiche linearer Operatoren, *Math. Nachrichten*, **86** (1978), 181—185.

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