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Some problems and results on the local behaviour of arithmetical functions

I. KÁTAI

1. Let $s(n)=\sum(n-p)^{-1}$ where p runs over the set of primes less than n. ERDŐS and DE BRUIJN (see [1]) proved that

$$(1.1)—(1.2) \quad c_1 < \frac{1}{N}\sum_{n<N} s^2(n) < c_2, \quad c_1 < \frac{1}{\pi(N)}\sum_{p<N} s^j(p) < c_2 \quad (j=1,2)$$

hold with suitable positive constants $c_1, c_2 > 0$.

In [2] it was proved that

$$(1.3)—(1.4) \qquad \sum_{n<N} s^k(n) = O(N), \quad \sum_{p<N} s^k(p) = O(\pi(N))$$

hold for every fixed $k \in \mathbf{N}$. The sums

$$E(N) := \sum_{n \leqq N} (s(n)-1)^2, \quad F(N) = \sum_{p<N} (s(p)-1)^2$$

have been considered in [3], [4], [5] under the unproved density hypothesis assumed in the form

$$(1.5) \qquad N(\sigma, T) < cT^{2(1-\sigma)}(\log T)^2$$

where $N(\sigma, T)$ denotes the number of zeros ϱ of $\zeta(s)$ in the rectangle $\operatorname{Re}\varrho \geqq \sigma$, $|\operatorname{Im}\varrho| \leqq T$.

In [5] we deduced from (1.5) that

$$(1.6)—(1.7) \qquad E(N) \ll \frac{N}{\log N}(\log\log N)^2, \quad F(N) \ll \frac{N(\log\log N)}{(\log N)^{3/2}}.$$

The proof was based on the inequality

$$\int_{-\Delta}^{\Delta} |T(z)|^2\, dy \ll N(\log N)^{-1}, \quad z = \frac{1}{N} + iy,$$

$$T(z) = \sum_{\varrho} z^{-\varrho}\Gamma(\varrho), \quad \Delta = (\log N)^{-7}$$

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due to YU. V. LINNIK [6], [7]. We can prove that $s(n)=1+o(1)$ holds for almost all n under the assumption

$$\sup_{X^{\varepsilon}\leqq h<X}\frac{1}{Xh^2}\int_X^{2X}\Big|\sum_{y\leqq m\leqq y+h}\Lambda(m)-h\Big|^2dy=:\varrho_{X,\varepsilon}\to 0\quad (X\to\infty)\tag{1.8}$$

for every $\varepsilon>0$.

It is not known whether (1.8) is true, but it can be deduced from the density-hypothesis

$$N(\sigma,T)\ll_{\varepsilon}T^{2(1-\sigma)+\varepsilon},\quad 1/2\leqq\sigma\leqq T.\tag{1.9}$$

Let $a(m)$, $m\in\mathbf{N}$, be any sequence of complex numbers,

$$A(y):=\sum_{m\leqq y}a(m),\quad B(y)=\sum_{m\leqq y}|a(m)|,$$

$$\alpha(X,h)=X^{-1}\sum_{y=X}^{2X-1}|A(y+h)-A(y)|^2,\quad \beta(X,h)=X^{-1}\sum_{y=X}^{2X-1}|B(y+h)-B(y)|^2.$$

Assume that

$$\sup_{X^{\varepsilon}\leqq h\leqq X}\frac{\alpha(X,h)}{\beta(X,h)}=:\varrho_{X,\varepsilon}\to 0\quad\text{as}\quad X\to\infty\tag{1.10$_\varepsilon$}$$

holds for all $\varepsilon>0$. Let

$$S_H(n):=\sum_{1\leqq m\leqq n-H}\frac{a(m)}{n-m},\tag{1.11}$$

$$E(X,H):=\sum_{X\leqq n<2X}|S_H(n)|^2.\tag{1.12}$$

We should like to give an upper-estimate for the sum (1.12) in terms of $\beta(x,\cdot)$ under assumption $(1.10)_\varepsilon$. It is clear that

$$|S_{X/4}(n)|\leqq(4/3X)B(2X).\tag{1.13}$$

Let $1/2>\delta>0$ be fixed, $H=H_0=X^{\delta}$. Let the sequence H_k be defined as follows:

$$H_k=(1+\delta)^kH_0,\quad k=1,2,\dots,k_0,$$

where $H_{k_0}<X/4<(1+\delta)H_{k_0}$, $H_{k_0+1}=X/4$, let

$$S^{(k)}(n)=\sum_{H_k\leqq n-m<H_{k+1}}\frac{a(m)}{n-m}\quad(k\leqq k_0).\tag{1.14}$$

Then

$$S_{H_0}(n)=\sum_{k=0}^{k_0}S^{(k)}(n)+S_{X/4}(n).$$

By the Cauchy—Schwarz inequality we get that

(1.15) $$|S_{H_0}(n)|^2 \leqq (k_0+2)\Big(\sum_{k=0}^{k_0} |S^{(k)}(n)|^2 + |S_{X/4}(n)|^2\Big).$$

Furthermore, we have

$$|S^{(k)}(n)| \leqq \left(\frac{1}{H_k} - \frac{1}{H_{k+1}}\right)\big(B(n-H_k) - B(n-H_{k+1})\big) + \\ + \frac{1}{H_k}\big|\big(A(n-H_k) - A(n-H_{k+1})\big)\big|,$$

and hence we get that

(1.16)

$$|S^{(k)}(n)|^2 \leqq \frac{2\delta^2}{H_{k+1}^2}|B(n-H_k)-B(n-H_{k+1})|^2 + \frac{2}{H_k^2}|A(n-H_k)-A(n-H_{k+1})|^2.$$

Summing for $n \in [X, 2X]$, from (1.13), (1.15), (1.16) we deduce that

(1.17) $$E(X, H_0) \leqq (k_0+2)\cdot 2\delta^2 \sum_{k=0}^{k_0} \frac{1}{H_{k+1}^2} \sum_{X \leqq n < 2X} |B(n-H_k)-B(n-H_{k+1})|^2 + \\ + 2(k_0+2)\sum_{k=0}^{k_0}\frac{1}{H_k^2}\sum_{X \leqq n < 2X}|A(n-H_k)-A(n-H_{k+1})|^2 + \frac{16(k_0+2)}{9X}B^2(2X).$$

Let us assume now that X is large enough. Then

$$k_0 \leqq \frac{\log X}{\log(1+\delta)} < \frac{2}{\delta}\log X.$$

Furthermore $X^\delta \leqq H_k < X/3$, $H_{k+1} - H_k = \delta H_k \geqq \delta X^\delta$. So by $(1.10)_{\delta/2}$ we get for every large X,

(1.18) $$E(X, H_0) \leqq \left[4\delta + \frac{4}{\delta}\varrho_{X/2,\,\delta/2}\right](\log X)\cdot T + \frac{64}{9\delta}\frac{\log X}{X}B^2(2X),$$

where

(1.19) $$T = X\sum_{k=0}^{k_0} H_k^{-2}\big(\beta(X-H_{k+1}),\, H_{k+1}-H_k\big) + \beta\big(2(X-H_{k+1}),\, H_{k+1}-H_k\big).$$

Assume that (1.9) is satisfied. A theorem due to K. RAMACHANDRA [8] gives that

(1.20) $$\sum_{y=X}^{2X-1}\Big[\sum_{y \leqq m < y+h} A(m) - h\Big]^2 \ll h^2 e^{-(\log X)^{1/6}} + X^\varepsilon,$$

(1.21) $$\sum_{y=X}^{2X-1}\Big[\sum_{y \leqq m < y+h} \lambda(m)\Big]^2 \ll h^2 e^{-(\log X)^{1/6}} + X^\varepsilon,$$

$$\sum_{y=X}^{2X-1}\Big[\sum_{y \leqq m < y+h} \mu(m)\Big]^2 \ll h^2 e^{-(\log X)^{1/6}} + X^\varepsilon$$

hold for all $\varepsilon>0$, uniformly as $1\leq h\leq X$. The constant implied by $\ll$, may depend on ε. Let $a(m)=\Lambda(m)-1$, and consider the function

$$S_\Lambda(n):=\sum_{m<n}\frac{a(m)}{n-m}=U(n)+S_H(n),$$

where

$$U(n)=\sum_{1\leq n-m<H}\frac{a(m)}{n-m},$$

and $S_H(n)$ is defined by (1.11). Since in our case $B(y+h)-B(y)\geq h$, from (1.20) we get immediately that $(1.10)_\varepsilon$ holds. Furthermore $B(X)\ll X$, and from standard sieve result we get that

$$B(y+h)-B(y)\ll h+h\log y/h.$$

So by (1.18), (1.19) we get easily that

$$E(X,H)/X(\log X)^2\to 0,\quad H=X^\delta. \tag{1.22}$$

Let us consider the sum

$$F(X,H):=\sum_{X\leq n<2X}U^2(n).$$

Since

$$|U(n)|\leq V(n)+\sum_{1\leq n-m<H}\frac{1}{n-m},\quad V(n)=\sum_{1\leq n-m<H}\frac{\Lambda(m)}{n-m},$$

we have

$$F(X,H)\leq 2F_1(X,H)+X(\log H+O(1))^2, \tag{1.23}$$

$$F_1(X,H)=\sum_{X\leq n<2X}V^2(n). \tag{1.24}$$

To estimate (1.24) we can use standard sieve result, namely that

$$\sum_{X\leq m<2X}\Lambda(m)\Lambda(m+k)\ll X\prod_{p|k}\left(1+\frac{1}{p}\right),$$

squaring out $V^2(n)$, we get easily that

$$\frac{F_1(X,H)}{X\log^2 X}\leq r(\delta), \tag{1.25}$$

where $r(\delta)\to 0$ as $\delta\to 0$. Hence and from (1.22) we get that

$$\sum_{X\leq n<2X}S_\Lambda^2(n)=o(X\log^2 X). \tag{1.26}$$

After some obvious observation, hence we get that

(1.27) $$\sum_{n<x} (s(n)-1)^2 = o(x), \quad s(n) = \sum_{p<n} \frac{1}{n-p}.$$

Let us consider now

$$S_\lambda(n) := \sum_{m<n} \frac{\lambda(m)}{n-m}.$$

Let

$$R(n) = \sum_{n-m<H} \frac{\lambda(m)}{n-m}, \quad H = X^\delta.$$

Since $R(n) \leq \log H + O(1)$, from (1.21), (1.18), (1.19), we get that

(1.28) $$\sum_{n\leq X} S_\lambda^2(n) = o(X \log^2 X).$$

Similarly, we can deduce from (1.9) that

(1.29) $$\sum_{n\leq X} S_\mu^2(n) = o(X \log^2 X), \quad S_\mu(n) := \sum_{m<n} \frac{\mu(m)}{n-m}.$$

So we have proved the following

Theorem 1. *Under the unproved hypothesis* (1.9) *the inequalities* (1.27), (1.28), (1.29) *hold.*

From the unproved hypothesis (1.9) and the Main Theorem of RAMACHANDRA [8], from (1.18), (1.19) we can deduce nontrivial estimate for some other functions as well. We shall state without proof

Theorem 2. *Assume that* (1.9) *holds.*

Let P_k run over the integers having exactly k prime factors. Then for each fixed k,

$$\sum_{n\leq X} \left(S_k(n) - \frac{(\log\log n)^{k-1}}{(k-1)!} \right)^2 = o\left(X (\log\log X)^{2(k-1)}\right)$$

where

$$S_k(n) := \sum_{P_k<n} (n-P_k)^{-1}.$$

Let z be any nonpositive complex number, $|z| = R$,

$$S_z(n) := \sum_{m<n} \frac{z^{\omega(m)}}{n-m}.$$

Then

$$\sum_{n\leq X} |S_z(n)|^2 = o\left(X (\log X)^{2R}\right) \quad as \quad X \to \infty.$$

Let $0 \leqq l < k$, *let* $\mathcal{B}_{k,l}$ *be the set of integers n satisfying the condition* $\omega(n) \equiv l \pmod k$. *Let*

$$S_{k,l}(n) = \sum_{\substack{m<n \\ m \in \mathcal{B}_{k,l}}} (n-m)^{-1}.$$

Then

$$\sum_{n \leqq X} (S_{k,l_1}(n) - S_{k,l_2}(n))^2 = o(X(\log X)^2).$$

2. Let $e(m)=1$ if m is a sum of two squares, and let $e(m)=0$ otherwise. Let us consider the sum

$$f(n) := \sum_{m<n} \frac{e(m)}{n-m}. \tag{2.1}$$

A classical result of Landau gives that

$$\sum_{m<x} e(m) \sim cx/\sqrt{\log x} \tag{2.2}$$

with a suitable constant. Hence one can deduce that

$$\sum_{n \leqq x} f(n) = \sum_{m<x} e(m) \log(x-m) \sim cx\sqrt{\log x}.$$

The Dirichlet-series

$$F(s) = \sum_{m=1}^{\infty} \frac{e(m)}{m^s}$$

can be written as

$$F(s) = \frac{1}{1-1/2^s} \prod_{p \equiv 1 \pmod 4} \frac{1}{1-1/p^s} \prod_{p \equiv -1 \pmod 4} \frac{1}{1-1/p^{2s}} = \sqrt{\xi(s) L(s,\chi)}\, v(s),$$

where $L(s,\chi)$ is the Dirichlet L-function corresponding to the nonprincipal character mod 4, $v(s)$ is a function which has a Dirichlet-series expansion

$$v(s) = \sum_n a_n/n^s,$$

that is absolute convergent in the halfplane $\operatorname{Re} s > 1/2$.

Let $N(\sigma, T, \chi)$ denote the number of roots ϱ of $L(s,\chi)$ in the domain $\operatorname{Re} \varrho \geqq \sigma$, $|\operatorname{Im} \varrho| \leqq T$.

Theorem 3. *Assume that* (1.9) *holds, and that*

$$N(\sigma, T, \chi) \ll_\varepsilon T^{2(1-\sigma)+\varepsilon} \tag{2.3}$$

holds for all $\varepsilon > 0$. *Then*

$$\sum_{n \leqq x} (f(n) - c\sqrt{\log n})^2 = o(x \log x) \quad (x \to \infty), \tag{2.4}$$

with the c occuring in (2.2).

Proof. The Main Theorem [8] under the conditions (1.9), (2.3) guarantee the fulfilment of the inequality $(1.10)_\varepsilon$ with $a(m)=e(m)-C(\log x)^{-1/2}$. By using the inequalities (1.18), (1.19) and some standard sieve results, namely that

$$\sum_{X\leqq m<2X} e(m)e(m+k) \ll \frac{x}{\log x} \prod_{p|k}\left(1+\frac{2}{p}\right)$$

(see Halberstam—Richert [10]), we get (2.4) easily.

As an example of further conditional results that can be deduced similarly, we mention without proof the next

Theorem 4. *Let* $k\in\mathrm{N}$ *be fixed and assume that*

$$N(\sigma, T, \chi) \ll_\varepsilon T^{2(1-\sigma)+\varepsilon} \quad (\sigma \geqq 1/2)$$

holds for all character $\chi\,(\mathrm{mod}\ k)$ *for all* $\varepsilon>0$. *Let* $l_1<l_2<\ldots<l_t$ *be distinct residues* mod k, $(l_j, k)=1$ $(j=1, \ldots, t)$. *Let* $\mathscr{E}$ *denote the set of the integers n the prime factors p of which belong to the residue classes* $\equiv l_j\,(\mathrm{mod}\ k)$ $(j=1, \ldots, k)$. *Let* $e(n)=1$ *if* $n\in\mathscr{E}$, *and* $e(n)=0$ *if* $n\notin\mathscr{E}$. *Let*

$$f(n) := \sum_{m<n} \frac{e(m)}{n-m}.$$

Then

$$\sum_{n\leqq x} \left(f(n)-c(\log n)^s\right)^2 = o\left(x(\log x)^{2s}\right), \quad s := t/\varphi(k)$$

with a suitable constant c.

3. One can prove similar theorems for the sums

$$g(n) := \sum_{1\leqq h<n} \frac{a(n+h)}{h}.$$

For example, from (1.9) we can deduce that

$$\sum_{n<x}\left(\sum_{n<p<2n} \frac{1}{p-n} - 1\right)^2 = o(x),$$

and so we have

Theorem 5. *If* (1.9) *is true, then*

$$\sum_{n<x} t^2(n) = o(x), \quad t(n) = \sum_{n<p<2n} (p-n)^{-1} - \sum_{p<n} (n-p)^{-1}.$$

Let us consider now the function

$$S(n) = \sum_{m\leqq n-2} \frac{\Lambda(m)}{(n-m)\log(n-m)}.$$

One can get easily that

$$\sum_{n\leq x} S(n) = x\log\log x + O(x).$$

Assuming that with some suitable ϱ_X, $\varepsilon_X > 0$, $\varrho_X \to 0$, $\varepsilon_X \to 0$,

(3.1) $$\sup_{Z_X \leq h \leq X} \frac{1}{h^2 X} \int_X^{2X} \Big[\sum_{y<m\leq y+h} \Lambda(m) - h\Big]^2 dy \leq \varrho_X, \quad \log\log Z_X = \varepsilon_X \log\log X$$

holds, we can get that

(3.2) $$\sum_{n<x} (S(n) - \log\log n)^2 = o\big(x(\log\log x)^2\big).$$

The inequality (3.1) holds if we assume somewhat more than (1.9), namely that

(3.3) $$N(\sigma, T) \leq cT^{2(1-\sigma)} \log^2 T.$$

In [9] we proved that under the condition (3.3) the inequality (3.1) holds with $Z_x = (\log x)^{7.5} h(x)$, $h(x) \to \infty$. So we have

Theorem 6. *If* (3.3) *is true then* (3.2) *holds.*

At present we are unable to prove that

$$\sum_{n<x} \Big(\sum_{m\leq n-2} \frac{\lambda(n)}{(n-m)\log(n-m)}\Big)^2 = o\big(x(\log\log x)^2\big)$$

even under the Riemann conjecture.

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EÖTVÖS LORÁND UNIVERSITY
COMPUTER CENTER
BOGDÁNFY U. 10/B
1117 BUDAPEST, HUNGARY