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Boolean algebras of factor congruences

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We want to examine algebras **A** whose factor congruences form a distributive sublattice of the congruence lattice of **A** (and hence the factor congruences can be thought of as forming a Boolean algebra). This concept appeared in the paper [8] of CHANG, JÓNSSON and TARSKI as an equivalent formulation of the strict refinement property (which, they noted, implies the refinement property, and hence the unique factorization property). Later this same concept was introduced by COMER in [10] to generalize the Pierce sheaf construction for rings. Comer applied his results to the study of cylindric algebras, and later BULMAN-FLEMING, KEIMEL and WERNER [4], [16] extended this to arbitrary discriminator varieties.

In §1 we state the basic characterizations of algebras with Boolean factor congruences which appear in CHEN [9] and SWAMY and MURTI [18], and note that every variety with the Fraser—Horn—Hu property[2]) has Boolean factor congruences. Next we look at an example of Chen and show that the only nontrivial variety of semigroups with Boolean factor congruences is the variety of semilattices. It turns out that the property of having Boolean factor congruences is a Mal'cev property of varieties — but we have been unable to find a corresponding Mal'cev condition. § 2 contains some technical lemmas about factor congruences of Boolean products which are used in § 3 where we characterize the Boolean products which arise as Pierce sheaves of algebras with Boolean factor congruences. PIERCE [17] was particularly interested in sheaf representations where the stalks were directly indecomposable. We show that if all the Pierce sheaves in a variety with Boolean factor congruences have directly indecomposable stalks then the directly indecomposable members of V form a universal class. In conclusion we show that the complete description of finite B-separating groups by APPS [1] carries over to the finite B-separating algebras in a variety with Boolean factor congruences.

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[2]) We have used the phrase 'Fraser—Horn property' prior to this paper, and we are indebted to Professor Walter Taylor for pointing out Hu's independent work on this property.

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1. Basic results

Our terminology and notation follow that of BURRIS and SANKAPPANAVAR [6]. For $\mathbf{A}$ an algebra, a pair of congruences $\theta, \bar{\theta}$ of $\mathbf{A}$ which satisfies $\theta \cap \bar{\theta} = \Delta_A$ and $\theta \circ \bar{\theta} = \nabla_A$ is called a pair of factor congruences. FC(**A**) denotes the set of factor congruences of **A**. An algebra **A** has Boolean factor congruences (BFC) if the factor congruences of **A** form a distributive sublattice of the congruence lattice Con (**A**) of **A**. A class of algebras has BFC if every member does.

Given congruences $\theta_i \in \mathrm{Con}(\mathbf{A}_i)$, $i=1,2$, the product congruence $\theta_1 \times \theta_2$ on $\mathbf{A}_1 \times \mathbf{A}_2$ is defined by:

$$\theta_1 \times \theta_2 = \{\langle\langle a_1, a_2\rangle, \langle b_1, b_2\rangle\rangle : a_i \theta_i b_i,\ i = 1, 2\}.$$

A congruence θ of $\mathbf{A}_1 \times \mathbf{A}_2$ is skew if it is not a product congruence. A variety V has the Fraser—Horn—Hu property if each $\mathbf{A}_1 \times \mathbf{A}_2 \in V$ has no skew congruences.

Given a homomorphism $\alpha: \mathbf{A} \to \mathbf{B}$ and $\theta \in \mathrm{Con}(\mathbf{A})$ let

$$\alpha(\theta) = \{\langle \alpha(a_1), \alpha(a_2)\rangle : \langle a_1, a_2\rangle \in \theta\}.$$

We note that for α surjective, $\alpha(\theta)$ is a congruence iff it is a transitive relation on B.

Given a product of algebras $\prod_{i\in I} \mathbf{A}_i$ let π_i denote the projection homomorphism from $\prod_{i\in I} \mathbf{A}_i$ to $\mathbf{A}_i$. More generally, for $J \subseteq I$ let π_J be the projection homomorphism from $\prod_{i\in I} \mathbf{A}_i$ to $\prod_{i\in J} \mathbf{A}_i$. The notation $\mathbf{A} \leqq_{sd} \prod_{i\in I} \mathbf{A}_i$ means $\mathbf{A}$ is a subdirect product of the $\mathbf{A}_i$, i.e., each projection map π_i maps A onto A_i.

Lemma 1.1. *Let* $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *be algebras of the same type. Then for* φ_i, $\theta_i \in \mathrm{Con}(\mathbf{A}_i)$, $i=1,2$ *we have:*

$$(\theta_1 \times \theta_2) \circ (\varphi_1 \times \varphi_2) = (\theta_1 \circ \varphi_1) \times (\theta_2 \circ \varphi_2),$$
$$(\theta_1 \times \theta_2) \wedge (\varphi_1 \times \varphi_2) = (\theta_1 \wedge \varphi_1) \times (\theta_2 \wedge \varphi_2),$$
$$(\theta_1 \times \theta_2) \vee (\varphi_1 \times \varphi_2) = (\theta_1 \vee \varphi_1) \times (\theta_2 \vee \varphi_2).$$

Proof. (Routine.)

Lemma 1.2. *Let* $\theta_i \in \mathrm{Con}(\mathbf{A}_i)$, $i=1,2$, *and suppose* $\mathbf{A}_1 \times \mathbf{A}_2$ *has no skew factor congruences. Then*

$$\theta_1 \times \theta_2 \in \mathrm{FC}(\mathbf{A}_1 \times \mathbf{A}_2) \quad \textit{iff} \quad \theta_i \in \mathrm{FC}(\mathbf{A}_i), \quad i = 1, 2.$$

Proof. The direction ($\Leftarrow$) follows immediately from Lemma 1.1. So suppose $\theta_1 \times \theta_2$ is in FC($\mathbf{A}_1 \times \mathbf{A}_2$). Choose $\varphi_1 \times \varphi_2$ such that $\theta_1 \times \theta_2$ and $\varphi_1 \times \varphi_2$ form a pair of factor congruences of $\mathbf{A}_1 \times \mathbf{A}_2$. Then by Lemma 1.1 it easily follows that θ_i, φ_i form a pair of factor congruences of $\mathbf{A}_i$, $i=1,2$.

Proposition 1.3. *For* **A** *an algebra the following are equivalent:*

(a) **A** *has* BFC.

(b) $\varphi, \theta \in \mathrm{FC}(\mathbf{A}) \Rightarrow \varphi = (\varphi \vee \theta) \wedge (\varphi \vee \bar{\theta})$ *where* $\theta, \bar{\theta}$ *is a pair of factor congruences of* **A**.

(c) $\mathbf{A} \cong \mathbf{A}_1 \times \mathbf{A}_2 \Rightarrow \mathbf{A}_1 \times \mathbf{A}_2$ *has no skew factor congruences.*

(d) $\mathbf{A} \cong \mathbf{A}_1 \times \mathbf{A}_2$ *and* $\theta \in \mathrm{FC}(\mathbf{A}_1 \times \mathbf{A}_2)$ *imply* $\pi_1(\theta) \times \Delta_{A_2} \subseteq \theta$.

Furthermore, **A** *has* BFC *implies that the factor congruences of* **A** *permute.*

Proof. The details for the equivalence of (a)—(c) and the last sentence can be found in CHEN [9] and SWAMY and MURTI [18]. Then it is easy to see that (d) is equivalent to (c).

Corollary 1.4. *A variety with the Fraser—Horn—Hu property has* BFC.

Proof. Use 1.3 (c).

This corollary covers the well known cases of congruence distributive varieties and the variety of rings with 1.

Corollary 1.5. *Any expansion of an algebra with* BFC *has* BFC.

Proof. Use 1.3 (c).

Proposition 1.6. *The only nontrivial variety of semigroups with* BFC *is the variety of semilattices.*

Proof. First we know from CHEN [9] that semilattices have BFC. Then using the description of the minimal varieties of semigroups given by Theorem 2.6 of KALICKI and SCOTT [15], as well as the description of the lattice of varieties of idempotent semigroups due (independently) to BIRYUKOV [3], FENNEMORE [11] and GERHARD [13], one can easily show that any nontrivial variety of semigroups which is not the variety of semilattices contains an algebra **A** which is isomorphic to either $\langle Z_p, +\rangle$, for p a prime, or to one of the following:

a two-element semigroup satisfying $x \cdot y = y$,

a two-element semigroup satisfying $x \cdot y = x$,

a two-element semigroup satisfying $x \cdot y = u \cdot v$.

In any of these cases $\mathbf{A} \times \mathbf{A}$ does not have a Boolean algebra of factor congruences.

Remark. Since semilattices do not have the Fraser—Horn—Hu property, we see that the latter property of varieties is different from having BFC.

Theorem 1.7. *The property of having* BFC *is a Mal'cev property of varieties.*

Proof. Let K be the class of varieties having BFC. We will verify the four conditions of TAYLOR in Theorem 4.2 of [19]:

(i) Clearly K is closed under the formation of equivalent varieties.

(ii) Clearly K is closed under the formation of subvarieties.

(iii) Let $V_1, V_2 \in K$, and let $\mathbf{A}=\mathbf{A}_1 \otimes \mathbf{A}_2$, where $\mathbf{A}_i \in V_i$, $i=1,2$. Let $\theta, \bar{\theta}$ be a pair of factor congruences of $\mathbf{A}$. Then, since all congruences on $\mathbf{A}_1 \otimes \mathbf{A}_2$ decompose into a product of congruences, we have $\theta=\theta_1 \times \theta_2$, $\bar{\theta}=\bar{\theta}_1 \times \bar{\theta}_2$. Then it is easy to see that $\theta_i, \bar{\theta}_i$ is a pair of factor congruences of $\mathbf{A}_i$, $i=1,2$. Since the calculations of $\vee$ and $\wedge$ of congruences of $\mathbf{A}_1 \otimes \mathbf{A}_2$ are done coordinatewise, it follows that $\mathbf{A}_1 \otimes \mathbf{A}_2$ has BFC.

(iv) Let Σ be a set of equations defining a variety in K. We need to show that some finite subset of Σ defines a variety (with the same language $\mathscr{L}$) in K. To do this we mimic an argument in Taylor's paper. Let $R_1, \bar{R}_1, R_2, \bar{R}_2$ be four binary relation symbols, and let Φ be a set of sentences which assert that $R_i, \bar{R}_i$ defines a pair of factor congruences, for $i=1,2$, on an $\mathscr{L}$-algebra. Let σ be a sentence which says that R_1 permutes with R_2 and $\bar{R}_2$, and

$$R_1 = (R_1 \circ R_2) \cap (R_1 \circ \bar{R}_2).$$

By 1.3 (b) and the last sentence of 1.3 we have $\Sigma \cup \Phi \models \sigma$, so for some finite $\Sigma_0 \subseteq \Sigma$, $\Sigma_0 \cup \Phi \models \sigma$. Thus Σ_0 defines an $\mathscr{L}$-variety with BFC.

2. Factor congruences of Boolean products

In this section we present some technical results concerning the impact of BFC on Boolean products. The Boolean product operator Γ on a class of algebras K is defined by: $\mathbf{A} \in \Gamma(K)$ iff for some Boolean space X and indexed family $(\mathbf{A}_x)_{x \in X}$ of algebras from K,

(i) $\mathbf{A} \leqq_{sd} \prod_{x \in X} \mathbf{A}_x$,

(ii) (equalizers are open) for $f, g \in A$ the set $[\![f=g]\!] =: \{x \in X : f(x)=g(x)\}$ is an open subset of X, and

(iii) (patchwork property) for N a clopen subset of X and $f, g \in A$ we have $f|_N \cup g|_{X-N} \in A$.

For $\mathbf{A} \in \Gamma(K)$ we denote the base space X by $X(\mathbf{A})$. We obtain the operator Γ^a if we replace (ii) by

(ii^a) (equalizers are clopen) for $f, g \in A$, $[\![f=g]\!]$ is clopen.

The Boolean product was introduced by Burris and Werner [7] as an (equivalent) alternative to Boolean sheaves. Given an algebra $\mathbf{A}$ we say that $\mathbf{B}$ is a Boolean product representation of $\mathbf{A}$ with stalks from K if $\mathbf{A} \cong \mathbf{B} \in \Gamma(K)$.

Lemma 2.1. *For* $\mathbf{A}\in\Gamma(K)$ *such that* $\mathbf{A}$ *has* BFC *let* θ *be a factor congruence of* $\mathbf{A}$. *Then for* $\langle f,g\rangle\in\theta$, *for* $h\in A$, *and for* N *a clopen subset of* $X=X(\mathbf{A})$ *we have*

$$\langle f|_N\cup h|_{X-N},\ g|_N\cup h|_{X-N}\rangle\in\theta.$$

Proof. If N is $\emptyset$ or X then the claim is obvious. So suppose $\emptyset\neq N\neq X$. Then let $\mathbf{A}_1=\mathbf{A}|_N$, $\mathbf{A}_2=\mathbf{A}|_{X-N}$, and let $\alpha:\mathbf{A}\to\mathbf{A}_1\times\mathbf{A}_2$ be the natural isomorphism. From 1.3 (c) there are congruences $\theta_i\in\mathrm{Con}\,(\mathbf{A}_i)$, $i=1,2$, with $\alpha(\theta)=\theta_1\times\theta_2$, and elements $f_i, g_i, i=1,2$, with

$$\alpha(f)=\langle f_1, f_2\rangle\alpha(\theta)\langle g_1, g_2\rangle=\alpha(g).$$

Let $\alpha(h)=\langle h_1, h_2\rangle$. Then $f_1\theta_1 g_1$ and $h_2\theta_2 h_2$, so

$$\langle f_1, h_2\rangle\alpha(\theta)\langle g_1, h_2\rangle.$$

Applying α^{-1} we get the desired conclusion.

Lemma 2.2. *For* $\mathbf{A}\in\Gamma(K)$ *such that* $\mathbf{A}$ *has* BFC *let* $\theta, \bar{\theta}$ *be a pair of factor congruences of* $\mathbf{A}$. *Then* $\pi_x(\theta), \pi_x(\bar{\theta})$ *is a pair of factor congruences of* $\mathbf{A}_x$, *for each* $x\in X(\mathbf{A})$.

Proof. Let $x\in X(\mathbf{A})$ be given. First we need to show that $\pi_x(\theta)$ is transitive for $\theta\in\mathrm{Con}\,(\mathbf{A})$ — for then $\pi_x(\theta)\in\mathrm{Con}\,(\mathbf{A}_x)$. (The same argument will apply to $\pi_x(\bar{\theta})$.) Let $\langle a, b\rangle$, $\langle c, d\rangle\in\theta$ with $\pi_x(b)=\pi_x(c)$. We want to show $\langle\pi_x(a), \pi_x(d)\rangle\in\pi_x(\theta)$. Choose a clopen set N such that

$$(*)\qquad x\in N\subseteq [\![b=c]\!].$$

Then, selecting an element e of A, we have by 2.1

$$\langle a|_N\cup e|_{X-N},\ b|_N\cup e|_{X-N}\rangle\in\theta,\quad \langle c|_N\cup e|_{X-N},\ d|_N\cup e|_{X-N}\rangle\in\theta.$$

Since $b|_N\cup e|_{X-N}=c|_N\cup e|_{X-N}$ we have

$$\langle a|_N\cup e|_{X-N},\ d|_N\cup e|_{X-N}\rangle\in\theta,$$

and thus $\langle\pi_x(a), \pi_x(d)\rangle\in\pi_x(\theta)$. Thus $\pi_x(\theta)$ is transitive.

Now choose $\bar{\theta}$ to be the complement of θ in $\mathrm{FC}(\mathbf{A})$. Then $\theta\circ\bar{\theta}=\nabla_A$ easily yields

$$\pi_x(\theta)\circ\pi_x(\bar{\theta})=\nabla_{A_x};$$

and using 2.1 we can also show

$$\pi_x(\theta)\wedge\pi_x(\bar{\theta})=\Delta_{A_x}.$$

This proves the lemma.

For $\mathbf{A}\in\Gamma(K)$ and $\varphi, \theta\in\mathrm{Con}\,(\mathbf{A})$ we define

$$[\![\varphi=\theta]\!]=\{x\in X(\mathbf{A})\colon \pi_x(\varphi)=\pi_x(\theta)\},\quad [\![\varphi\neq\theta]\!]=\{x\in X(\mathbf{A})\colon \pi_x(\varphi)\neq\pi_x(\theta)\}.$$

Lemma 2.3. *Suppose* $\mathbf{A} \in \Gamma^a(K)$. *If* $\theta \in \mathrm{Con}(\mathbf{A})$ *then* $[\![\theta \neq \Delta]\!]$ *is an open set.*

Proof. For $x \in [\![\theta \neq \Delta]\!]$ choose $\langle f, g\rangle \in \theta$ such that $f(x) \neq g(x)$. As $x \in [\![f \neq g]\!] \subseteq \subseteq [\![\theta \neq \Delta]\!]$, it follows that $[\![\theta \neq \Delta]\!]$ is open.

For $\mathbf{A} \in \Gamma(K)$ and N a clopen subset of $X(\mathbf{A})$ let $\pi_N^{\mathbf{A}}: \mathbf{A} \to \mathbf{A}|_N$ be the projection map from $\prod_{x \in X} \mathbf{A}_x$ to $\prod_{x \in N} \mathbf{A}_x$ restricted to $\mathbf{A}$. Then we have a factor congruence of $\mathbf{A}$ given by

$$\ker(\pi_N^{\mathbf{A}}) = \{\langle f, g\rangle \in A \times A\colon\ N \subseteq [\![f = g]\!]\}.$$

If every factor congruence of $\mathbf{A}$ is of this form then we say $\mathbf{A}$ is factor transparent.

Proposition 2.4. *Suppose* $\mathbf{A} \in \Gamma^a(K)$ *is such that* $\mathbf{A}$ *has* BFC *and no trivial stalks. If* $\theta \in \mathrm{FC}(\mathbf{A})$ *is such that on a dense subset* D *of* $X(\mathbf{A})$ *we have* $\pi_x(\theta) \in \{\Delta_{A_x}, \nabla_{A_x}\}$ *then* $\theta = \ker(\pi_N^{\mathbf{A}})$ *for some clopen subset* N.

Proof. Let $\bar{\theta}$ be the complement of θ in $\mathrm{FC}(\mathbf{A})$. If, for some x, $\pi_x(\theta) \notin \{\Delta_{A_x}, \nabla_{A_x}\}$, then by 2.2 we have $\pi_x(\bar{\theta}) \notin \{\Delta_{A_x}, \nabla_{A_x}\}$. Thus by 2.3 we can find a clopen set N such that

$$x \in N \subseteq [\![\theta \neq \Delta]\!] \cap [\![\bar{\theta} \neq \Delta]\!].$$

However, for $y \in D \cap N$ we have $\pi_y(\theta) \in \{\Delta_{A_y}, \nabla_{A_y}\}$, which gives a contradiction. Thus

$$[\![\theta = \Delta]\!] \cup [\![\theta = \nabla]\!] = X(\mathbf{A}).$$

Then by 2.2 $[\![\theta = \Delta]\!] = [\![\bar{\theta} \neq \Delta]\!]$, so $[\![\theta = \Delta]\!]$ is open by 2.1. Consequently $[\![\theta = \Delta]\!]$ is clopen. Letting $N = [\![\theta = \Delta]\!]$ we have

$$\theta = \{\langle f, g\rangle \in A \times A\colon\ N \subseteq [\![f = g]\!]\}.$$

3. The Pierce sheaf

Pierce [17] utilized the fact that the central idempotents of a ring form a Boolean ring to represent the ring as a Boolean sheaf, or, in our terminology, as a Boolean product. Comer's generalization, again in our terminology, says that if an algebra has BFC then one can let X be the Stone space of maximal ideals $\mathcal{M}$ of $\mathrm{FC}(\mathbf{A})$, with a basis of clopen sets obtained by taking all subsets of X of the form $N_\varphi = \{\mathcal{M} \in X\colon \varphi \in \mathcal{M}\}$, for $\varphi \in \mathrm{FC}(\mathbf{A})$. Then the natural map $\nu\colon \mathbf{A} \to \prod_{\mathcal{M} \in X} \mathbf{A}/\bigcup\mathcal{M}$ is an embedding which gives a Boolean product representation $\nu(\mathbf{A})$ of $\mathbf{A}$. A detailed proof that $\nu(\mathbf{A})$ is in $\Gamma(\{\mathbf{A}/\bigcup\mathcal{M}\colon \mathcal{M} \in X\})$ can be found in Chap. IV § 8 of [6], along with the fact that $\nu(\mathbf{A})$ gives a Γ^a-representation iff each $\langle a, b\rangle$ in $A \times A$ belongs to a smallest member of $\mathrm{FC}(\mathbf{A})$. In the following we denote $\nu(\mathbf{A})$, the Pierce sheaf of $\mathbf{A}$, by PSh(A). (This is an abuse of the word 'sheaf' since we really mean the Boolean product associated with the Pierce sheaf.) Pierce was particularly interested in obtaining a Boolean sheaf representation with directly indecomposable stalks. As we shall see this imposes a strong condition on a variety with BFC.

In order to study the Pierce sheaf of a Boolean product we introduce the notion of two Boolean products being essentially the same, namely if $\mathbf{A}, \mathbf{B} \in \Gamma(K)$ we write $\mathbf{A} \underset{\mathrm{a}}{=} \mathbf{B}$ if there is an isomorphism $\alpha\colon \mathbf{A} \to \mathbf{B}$ and a homeomorphism $\beta\colon X(\mathbf{A}) \to X(\mathbf{B})$ such that for each $x \in X(\mathbf{A})$ the relation $\alpha_x = \{\langle f(x), \alpha f(\beta x)\rangle\colon f \in A\}$ defines an isomorphism from $\mathbf{A}_x$ to $\mathbf{B}_{\beta x}$.

If $\mathbf{A} \in \Gamma(K)$ let $\mathrm{Triv}(\mathbf{A}) = \{x \in X(\mathbf{A})\colon |A_x| = 1\}$, the trivial part of $\mathbf{A}$. $\mathbf{A}$ is reduced if it is factor transparent and $\mathrm{Int}(\mathrm{Triv}(\mathbf{A})) = \emptyset$, i.e., the interior of the trivial part of $\mathbf{A}$ is empty. (This is more general than Comer's definition in [10] — he requires that the trivial part of $\mathbf{A}$ be empty.) The following characterization of Pierce sheaves is an extension of Theorems 3.7 and 4.2 of Comer [10] since it includes algebras for which the Pierce sheaf has some trivial stalks.

Lemma 3.1. *Suppose* $\mathbf{A}$ *has* BFC. *If* $\varphi \in \mathrm{FC}(\mathbf{A})$ *and* $\langle a, b\rangle \in \nabla_A - \varphi$ *then there is an* $\mathcal{M} \in N_\varphi$ *such that* $\langle a, b\rangle \notin \bigcup \mathcal{M}$.

Proof. Let $\mathcal{F} = \{\theta \in \mathrm{FC}(\mathbf{A})\colon \langle a, b\rangle \in \theta\}$. Then $\mathcal{F}$ is a filter in the Boolean algebra of factor congruences of $\mathbf{A}$, and $\varphi \notin \mathcal{F}$. Thus we can extend $\mathcal{F}$ to an ultrafilter $\mathcal{U}$ with $\varphi \notin \mathcal{U}$. Consequently $\mathcal{M} = \mathrm{FC}(\mathbf{A}) - \mathcal{U}$ is a maximal ideal, and $\mathcal{M} \in N_\varphi$. Clearly $\langle a, b\rangle \notin \bigcup \mathcal{M}$.

Proposition 3.2. (a) *If* $\mathbf{A}$ *has* BFC *then* PSh(A) *is reduced.*

(b) *Let* $\mathbf{A}$ *be a reduced Boolean product. Then* $\mathbf{A} \underset{\mathrm{a}}{=} \mathrm{PSh}(\mathbf{A})$.

Proof. For (a) let $\nu\colon \mathbf{A} \to \mathrm{PSh}(\mathbf{A})$ be the canonical isomorphism. Then, using 3.1, we see that for $\varphi \in \mathrm{FC}(\mathbf{A})$,

$$\langle a, b\rangle \in \varphi \Leftrightarrow N_\varphi \subseteq [\![\nu(a) = \nu(b)]\!].$$

Thus $\nu(\varphi) = \ker(\pi^{\mathbf{A}}_{N_\varphi})$, so $\mathbf{A}$ is factor transparent. And for $\varphi \in \mathrm{FC}(\mathbf{A})$ with $\varphi \neq \nabla$ there is, by 3.1, an $\mathcal{M} \in X$ with $\varphi \in \mathcal{M}$ such that $|\mathbf{A}/\bigcup \mathcal{M}| > 1$, so $N_\varphi \subsetneqq \mathrm{Triv}(\mathbf{A})$. It follows that $\mathrm{Int}(\mathrm{Triv}(\mathbf{A})) = \emptyset$.

(b) Let $\mathbf{A}$ be a reduced Boolean product. Then, since the only factor congruences of $\mathbf{A}$ are of the form $\ker(\pi^{\mathbf{A}}_N)$ for N clopen, they form a Boolean algebra. Since $\mathrm{Int}(\mathrm{Triv}(\mathbf{A})) = \emptyset$, distinct clopen sets N give rise to distinct factor congruences $\ker(\pi^{\mathbf{A}}_N)$. Thus there is a bijection between the maximal ideals $\mathcal{M}$ of FC(A) and ultrafilters $\mathcal{U}$ of clopen subsets of $X(\mathbf{A})$ such that if $\mathcal{M}$ and $\mathcal{U}$ correspond then $\mathcal{M} = \{\ker(\pi^{\mathbf{A}}_N)\colon N \in \mathcal{U}\}$. Consequently $\bigcup \mathcal{M}$ gives the stalk congruence $\ker(\pi^{\mathbf{A}}_x)$, where $\bigcap \mathcal{U} = \{x\}$. Then with $\alpha\colon \mathbf{A} \to \mathrm{PSh}(\mathbf{A})$ the natural map and $\beta\colon X(\mathbf{A}) \to$

$\to X(\mathrm{PSh}(\mathbf{A}))$ given by $\beta(x)=\{\ker(\pi_N^{\mathbf{A}}): x\in N, N \text{ clopen}\}$, it is routine to check that $\mathbf{A} \underset{a}{=} \mathrm{PSh}(\mathbf{A})$.

Recall that an algebra $\mathbf{A}$ is directly indecomposable if $|\mathrm{FC}(\mathbf{A})|\leq 2$.

Corollary 3.3. *Suppose* $\mathbf{A}\in\Gamma^a(K)$ *is such that* $\mathbf{A}$ *has* BFC, *the stalks of* $\mathbf{A}$ *are directly indecomposable on a dense subset of* $X(\mathbf{A})$, *and* $\mathrm{Triv}(\mathbf{A})=\emptyset$. *Then* $\mathbf{A} \underset{a}{=} \mathrm{PSh}(\mathbf{A})$.

Proof. From 2.2 and 2.4 we see that $\mathbf{A}$ is reduced, so 3.2 applies.

Given an algebra $\mathbf{A}$ and a Stone space X let $\mathbf{A}[X]^*$ be the subalgebra of $\mathbf{A}^X$ consisting of all continuous functions from X to A, giving A the discrete topology. $\mathbf{A}[X]^*$ is called a Boolean power of $\mathbf{A}$. If $\mathbf{B}$ is a Boolean algebra, let $\mathbf{A}[\mathbf{B}]^*$ be $\mathbf{A}[X]^*$ where $X=\mathbf{B}^*$, the Stone space of $\mathbf{B}$.

Theorem 3.4. *Let V be a variety with* BFC *such that the Pierce sheaf of each member of V has directly indecomposable stalks. Then V_{DI}, the class of directly indecomposable members of V, is a universal class, i.e., it is an elementary class which can be axiomatized by universal sentences.*

Proof. Suppose V_{DI} is not closed under subalgebras. Choose $\mathbf{A}_0\leq\mathbf{A}\in V_{DI}$ with $\mathbf{A}_0\notin V_{DI}$. Let x_0 be a point in the Cantor discontinuum C, and let $\mathbf{D}$ be the subalgebra of $\mathbf{A}[C]^*$ with

$$\mathbf{D} = \{f\in\mathbf{A}[C]^*: f(x_0)\in\mathbf{A}_0\}.$$

Then $\mathbf{D}_x\cong\mathbf{A}$ for $x\neq x_0$, and $\mathbf{D}_{x_0}\cong\mathbf{A}_0$. Thus by 3.3 it follows that a stalk of the Pierce sheaf of $\mathbf{D}$ will be isomorphic to $\mathbf{A}_0$. This contradicts our assumption on the Pierce sheaves in V, so V_{DI} must be closed under subalgebras. (We note that a similar argument was applied to the case of rings by Burgess and Stephenson [5].)

Next we want to show that V_{DI} is closed under ultraproducts. Let $\mathbf{A}_i\in V_{DI}$ for $i\in I$, and let θ, $\bar\theta$ be a pair of factor congruences of $\prod_{i\in I}\mathbf{A}_i$. Then, using 1.2, we conclude that each $\pi_i(\theta)$, $\pi_i(\bar\theta)$ is a pair of factor congruences of $\mathbf{A}_i$, $i\in I$. Consequently $\theta\doteq\ker(\pi_J)$ where $J=[\![\theta=\Delta]\!]$. This leads to a bijection between the maximal ideals $\mathcal{M}$ of $\mathrm{FC}(\prod_{i\in I}\mathbf{A}_i)$ and the ultrafilters $\mathcal{U}$ on I such that for corresponding $\mathcal{M}$ and $\mathcal{U}$ we have $\mathcal{M}=\{\ker(\pi_J): J\in\mathcal{U}\}$. Since $\prod_{i\in I}\mathbf{A}_i/\bigcup\mathcal{M}=\prod_{i\in I}\mathbf{A}_i/\mathcal{U}$, the stalks of $\mathrm{PSh}(\prod_{i\in I}\mathbf{A}_i)$ are the ultraproducts $\prod_{i\in I}\mathbf{A}_i/\mathcal{U}$. Thus, since we are assuming the stalks of the Pierce sheaves in V to be directly indecomposable, it follows that V_{DI} is closed under ultraproducts. This suffices to prove that V_{DI} is a universal class.

An algebra $\mathbf{A}$ is B-separating if for any Boolean algebras $\mathbf{B}_1$ and $\mathbf{B}_2$ we have

$$\mathbf{A}[\mathbf{B}_1]^* \cong \mathbf{A}[\mathbf{B}_2]^* \Rightarrow \mathbf{B}_1 \cong \mathbf{B}_2.$$

In the following we show that the characterization of the finite B-separating members in a variety with BFC is the same as that obtained by APPS [1] for groups. (For a through survey of the results on B-separating algebras see BIGELOW [2].)

Theorem 3.5. *Let V be a variety with* BFC. *Then*

(a) *for* $\mathbf{A}_1, \dots, \mathbf{A}_k$ *nontrivial directly indecomposable members of V which are pairwise nonisomorphic we have* $\mathbf{A}_1 \times \mathbf{A}_2^{n_2} \times \dots \times A_k^{n_k}$ *is B-separating for any choice of natural numbers* $n_2, \dots, n_k$; *and*

(b) *for* $\mathbf{A}$ *a finite member of V we have* $\mathbf{A}$ *is B-separating iff* $\mathbf{A}$ *is isomorphic to* $\mathbf{A}_1 \times \mathbf{A}_2^{n_2} \times \dots \times \mathbf{A}_k^{n_k}$, *where the* $\mathbf{A}_i$ *are, for* $1 \leqq i \leqq k$, *directly indecomposable pairwise non-isomorphic members of V.*

Proof. For (a) let $\mathbf{A}_1, \dots, \mathbf{A}_k$ be nontrivial directly indecomposable pairwise nonisomorphic members of V, and given natural numbers $n_2, \dots, n_k$ let $\mathbf{A}$ be the direct product $\mathbf{A}_1 \times \mathbf{A}_2^{n_2} \times \dots \times \mathbf{A}_k^{n_k}$. Then for $\mathbf{B}$ a Boolean algebra we have $\mathbf{A}[\mathbf{B}]^*$ is isomorphic to

$$\mathbf{A}_1[\mathbf{B}]^* \times \mathbf{A}_2[\mathbf{B}^{n_2}]^* \times \dots \times \mathbf{A}_k[\mathbf{B}^{n_k}]^*.$$

Thus we can express $\mathbf{A}[\mathbf{B}]^*$ as a Γ^a-Boolean product $\mathbf{C}$ over the base space

$$\mathbf{B}^* \cup (\mathbf{B}^{n_2})^* \cup \dots \cup (\mathbf{B}^{n_k})^*$$

with stalks $\mathbf{A}_1, \dots, \mathbf{A}_k$, and such that $\mathbf{A}_1$ appears precisely over $\mathbf{B}^*$. By 3.3 we have $\mathbf{C} \underset{a}{=} \mathrm{PSh}(\mathbf{C})$; and as $\mathbf{C} \cong \mathbf{A}[\mathbf{B}]^*$ it follows that $\mathbf{C} \underset{a}{=} \mathrm{PSh}(\mathbf{A}[\mathbf{B}]^*)$. Thus we can recapture $\mathbf{B}$ from $\mathrm{PSh}(\mathbf{A}[\mathbf{B}]^*)$ as the Boolean algebra of all clopen sets in the base space $X(\mathrm{PSh}(\mathbf{A}[\mathbf{B}]^*))$ which have all stalks above them isomorphic to $\mathbf{A}_1$. Thus $\mathbf{A}$ is B-separating.

For (b) we only need to show that any finite algebra not isomorphic to an algebra of the form described is not B-separating. But this is true in general, and the argument is in APPS [1].

Thus we have a description of the finite B-separating algebras in any congruence distributive variety and in any variety with a semilattice operation.

References

[1] A. B. APPS, Boolean powers of groups, *Math. Proc. Camb. Phil. Soc.*, **19** (1982), 375—395.

[2] D. BIGELOW, *B-Separating Algebras*, M. Math. Thesis, University of Waterloo, 1986.

[3] A. P. BIRYUKOV, Varieties of idempotent semigroups, *Alg. i. Logika*, **9** (1970), 255—273.

[4] S. BULMAN-FLEMING and H. WERNER, Equational compactness in quasi-primal varieties, *Algebra Universalis*, **7** (1977), 33—46.

[5] W. D. BURGESS and W. STEPHENSON, Pierce sheaves of non-commutative rings, *Comm. Algebra*, **4** (1976), 51—75.

[6] S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra.* Graduate Texts in Math. 78, Springer-Verlag (New York, Heidelberg, Berlin, 1981).
[7] S. Burris and H. Werner, Sheaf constructions and their elementary properties, *Trans. Amer. Math. Soc.*, **248** (1979), 269—309.
[8] C. C. Chang, B. Jónsson and A. Tarski, Refinement properties for relational structures, *Fund. Math.*, **50** (1964), 249—281.
[9] C. C. Chen, On the unique factorization property of algebras, *Nanta Math.*, **10** (1977), 149—157.
[10] S. Comer, Representations by algebras of sections over Boolean spaces. *Pacific J. Math.*, **38** (1971), 29—38.
[11] C. Fennemore, All varieties of bands, *Semigroup Forum*, **1** (1970), 172—179.
[12] G. A. Fraser and A. Horn, Congruence relations in direct products., *Proc. Amer. Math. Soc.*, **26** (1970), 390—394.
[13] J. A. Gerhard, The lattice of equational classes of idempotent semigroups, *J. Algebra*, **15** (1970), 195—224.
[14] T.-K. Hu, On equational classes of algebras in which congruences of finite direct products are induced by congruences on their factors, Mimeographed, Southern Illinois University (1968?).
[15] J. Kalicki and D. Scott, Equational completeness of abstract algebras. *Indag. Math.*, **17** (1955), 650—659.
[16] K. Keimel and H. Werner, Stone duality for varieties generated by quasi-primal algebras, *Mem. Amer. Math. Soc.*, **148** (1974), 59—85.
[17] R. S. Pierce, Modules over commutative regular rings, *Mem. Amer. Math. Soc.*, **70** (1967).
[18] U. M. Swamy and G. S. Murti, Boolean centre of a universal algebra, *Algebra Universalis*, **13** (1981), 202—205.
[19] W. Taylor, Characterizing Mal'cev conditions, *Algebra Universalis*, **3** (1973), 351—397.

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