Acta Sci. Math., 54 (1990), 37—52

Endomorphism monoids in minimal quasi primal varieties

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In this paper we address the following question: Given a finitely generated variety of algebras, how closely is an arbitrary algebra B determined by its endomorphism monoid, End (B)? A survey of the literature reveals a spectrum of possible answers which focus at two extremes. We begin by citing some of these examples.

Independently, K. D. MAGILL [20], C. J. MAXSON [21] and B. M. SCHEIN [29] have shown that nontrivial Boolean algebras are determined up to isomorphism by their endomorphism monoids. T. K. HU [15] proved that the variety generated by a primal algebra is equivalent, as a category, to the variety of Boolean algebras. It follows that nontrivial algebras in any primal variety are also determined up to isomorphism by their endomorphism monoids. The same is true of median algebras (H. J. BANDELT [3]), distributive lattices with 0 and distributive lattices with 1 (B. M. SCHEIN [29]). The conclusion is slightly weaker for distributive lattices (without bounds): SCHEIN [29] showed that if $\mathrm{End}\,(L_1)\cong\mathrm{End}\,(L_2)$, then L_1 is isomorphic either to L_2 or to the lattice obtained by inverting the order in L_2. The same result for bounded distributive lattices was proven by R. MCKENZIE and C. TSINAKIS [22].

At the other extreme, a number of authors have found finitely generated varieties for which there are monoids that are isomorphic to the endomorphism monoid of a proper class of nonisomorphic algebras (c.f. A. PULTR and J. SICHLER [26], V. KOUBEK and J. SICHLER [18], P. GORALČÍK, V. KOUBEK and P. PRÖHLE [11], or the text A. PULTR and V. TRNKOVÁ [27]). A striking case of this dichotomy was discovered in [1] and [2]: By K. B. LEE [19] the varieties of pseudocomplemented distributive lattices form an $\omega+1$ chain

$$\mathscr{H}_{-1}\subseteqq\mathscr{H}_0\subseteqq\mathscr{H}_1\subseteqq\mathscr{H}_2\subseteqq\ldots\subseteqq\mathscr{H}_n\subseteqq\ldots\subseteqq\mathscr{H}_\omega$$

where $\mathscr{H}_{-1}$, $\mathscr{H}_0$, $\mathscr{H}_1$ are the varieties of trivial, Boolean and Stone algebras, respectively. In [1] and [2] it is shown that nontrivial algebras in $\mathscr{H}_1$ (like $\mathscr{H}_{-1}$ and $\mathscr{H}_0$) are determined up to isomorphism, in $\mathscr{H}_2$ (like distributive lattices) they are determined

Received March 10, 1987 and in revised form May 16, 1989.

to within two, while for $n \geqq 3$ $\mathscr{H}_n$ contains a proper class of nonisomorphic algebras with isomorphic endomorphism monoids.

Where in this spectrum would one expect the variety generated by a randomly chosen finite algebra to fall? In this study we consider varieties generated by a *minimal quasi primal algebra.* In the sense of R. O. DAVIES [10] and V. L. MURSKIĬ [23] (see also R. W. QUACKENBUSH [28]), almost all finite algebras are minimal quasi primal and, consequently, are addressed by the results presented here. We will show that in such a variety an algebra is almost always uniquely determined by its endomorphism monoid.

To make this assertion precise, we refer again to the case of distributive lattices. How can we construct, from a distributive lattice L, the lattice obtained by inverting the order in L? One way to do it begins by representing L as a sublattice of a power of $A=(\{0, 1\}, \wedge, \vee)$. Now apply pointwise the permutation $\beta=(0, 1)$ to each member of L to obtain a new sublattice of the same power of A. This approach can be extended to an arbitrary algebra using *weak automorphisms.* If β is a weak automorphism of A and B is a subalgebra of a power of A, then applying β^{-1} pointwise to the members of B will yield a new subalgebra B^β of the same power of A. For any algebra A, the automorphism group Aut (A) forms a normal subgroup of the group of weak automorphisms WAut (A). Weak automorphisms β and δ of A which are in the same coset of Aut (A) produce isomorphic algebras B^β and B^δ. In general B^β and B^δ are not isomorphic, but we will show that their endomorphism monoids always are.

We now state our main result which is established in Section 3.

Theorem 1. *Let Q be a minimal quasi primal algebra. For nontrivial algebras B and C in the variety generated by Q,* End $(B) \cong$ End (C) *if and only if there is a weak automorphism β of Q such that $C \cong B^\beta$.*

Thus, from Theorem 1 and the comments preceding it, it follows that for the variety generated by a minimal quasi primal algebra Q the number of nonisomorphic algebras in the variety with a given endomorphism monoid is bounded by the (necessarily finite) index of Aut (Q) in WAut (Q).

Unlike most work on quasi primal varieties which relies on sheaf representations, our proof of Theorem 1 is based on topological dualities. Starting with the topological duality of M. H. STONE [32] for Boolean algebras, topological dualities for quasi primal varieties have been developed in several steps by K. KEIMEL and H. WERNER [16], then B. A. DAVEY and H. WERNER [9] and simultaneously [7]. Our main tool will be a result from the latter two papers which states that the variety generated by a minimal quasi primal algebra is equivalent, as a category, to a category of structured Boolean spaces.

In most familar cases Aut (A) has a small index in WAut (A). For example, if A is a lattice the index is one unless $A=(A, \wedge, \vee) \cong (A, \vee, \wedge)$, in which case it is two.

If A is an abelian group it is always one. Nevertheless, we can find minimal quasi primal algebras A for which the index is arbitrarily large. In the Section 4 we prove

Theorem 2. *For $0<n<\omega$, there exists a minimal quasi primal algebra A, an algebra B in the variety generated by A and a weak automorphism β of A such that $B^{\beta^r}\not\cong B^{\beta^s}$ for $0\leqq r<s<n$.*

As will be seen, the algebra B in the statement of Theorem 2 is, by necessity, infinite (see Proposition 3.3).

We conclude in Section 5 with some examples.

We would like to acknowledge the very helpful discussions with A. Higgins and J. Sichler concerning this topic.

1. Preliminaries

We begin by fixing some notation and terminology, and reviewing some recent results that will be the basis of our work. For more extensive background we refer the reader to standard texts such as S. Burris and H. P. Sankappanavar [6] or G. Grätzer [12].

We use an arbitrary set Op of finitary *operation symbols* to determine a *similarity type*. Relative to Op we construct the set Tm of *terms* in a fixed denumerable sequence of variables. $B=(B,f^B)_{f\in \mathrm{Op}}$ is an *algebra* of type Op if B is a nonempty set (the *carrier* of B) and for each n-ary operation symbol $f\in\mathrm{Op}$, $f^B\colon B^n\to B$. In this case each n-ary term $t\in\mathrm{Tm}$ also defines an n-ary *term function* $t^B\colon B^n\to B$ on B. The *clone* of B is the set of all t^B where $t\in\mathrm{Tm}$. A nonempty subset C of B *determines a subalgebra* of B if it is closed under the operations of B. Here we denote by C the subalgebra determined by C. B is a *trivial* algebra if it has only one element. It is *minimal* if it is finite, nontrivial and every proper subalebra of B is trivial. Given a class $\mathcal{M}$ of (possibly topological) algebras of type Op, we denote by $\mathbf{H}\mathcal{M}$, $\mathbf{I}\mathcal{M}$, $\mathbf{S}\mathcal{M}$ and $\mathbf{P}\mathcal{M}$ the classes of (continuous) homomorphic images, (homeomorphic) isomorphic images, (closed) subalgebras (with the relative topology) and products (with the product topology) over nonempty index sets of members of $\mathcal{M}$, respectively. $\mathbf{V}\mathcal{M}=\mathbf{HSP}\mathcal{M}$ is the *variety* or *equational class* generated by a class $\mathcal{M}$.

A finite nontrivial algebra Q is *quasi primal* if every operation on the set Q which preserves the subalgebras and isomorphisms between subalgebras of Q is a term function of Q. This notion, introduced by A. F. Pixley [24], [25], is equivalent to the assertion that the ternary discriminator, $t(x,y,z)=z$ if $x=y$ else x, is a term function of Q. In particular a quasi primal algebra has only simple subalgebras.

A slight reformulation of the definition of minimal quasi primal algebra will more directly suit our needs. Given a finite nontrivial algebra Q, let G be the set of automorphisms of Q and let $E(Q)$ be the set of $e \in Q$ which determine trivial subalgebras of Q. Then

$$\mathbf{Q} = (Q, \mu, e)_{\mu \in G, e \in E(Q)}$$

is an algebra with $|G|$ unary operations, $|E(Q)|$ nullary operations (constants) and the same carrier as Q. If Q is a minimal algebra, then the homomorphisms

$$f\colon \mathbf{Q}^n \to \mathbf{Q}$$

are exactly the n-ary operations on Q which preserve all subalgebras and isomorphisms between subalgebras of Q. Thus we have

Lemma 1.1. *Let Q be a minimal algebra, $\mathbf{Q}$ as above. Then Q is quasi primal if and only if the homomorphisms $f\colon \mathbf{Q}^n \to \mathbf{Q}$ are exactly the n-ary term functions of Q for each $n < \omega$.*

The key tool in our investigation will come from the topological duality theory developed in [7] and, independently, in B. A. DAVEY and H. WERNER [9]. We review here the necessary theorem. For a minimal quasi primal algebra Q, $\mathbf{ISP}Q$ is identical with $\mathbf{V}Q$ if Q has a one element subalgebra and otherwise consists of the nontrivial members of $\mathbf{V}Q$. Let $\mathbf{Q}$ be defined as above, but augment it with the *discrete topology*. The category $\mathbf{ISPQ}$ contains structured Boolean spaces $\mathbf{X}$ having the same type as $\mathbf{Q}$. It is easy to check that, for each such $\mathbf{X}$ in $\mathbf{ISPQ}$, the hom set

$$\mathrm{Hom}(\mathbf{X}, \mathbf{Q}) \subseteq Q^X$$

of continuous homomorphisms from $\mathbf{X}$ into $\mathbf{Q}$ determines a subalgebra of Q^X. We denote this algebra by $\Phi(\mathbf{X})$.

Proposition 1.2. ([7], [9]). *If Q is a minimal quasi primal algebra, then*

$$\Phi\colon \mathbf{ISPQ} \to \mathbf{ISP}Q$$

is a dual category equivalence (i.e., a full and faithful contravariant functor such that every algebra in $\mathbf{ISP}Q$ *is isomorphic to the image of some algebra in* $\mathbf{ISPQ}$*).*

2. Weak automorphisms

The primary object of this section is to establish the more direct part of Theorem 1. We will show how weak automorphisms can generate nonisomorphic algebras with the same endomorphisms (Corollary 2.4), and we will give a bound on the number of algebras that can so arise (Lemma 2.6). In this section $A = (A, f^A)_{f \in \mathrm{Op}}$ will denote a fixed but arbitrary algebra. For each permutation β of its carrier A, let

$$\beta A = (A, f^{\beta A})_{f \in \mathrm{Op}}$$

be the unique algebra such that β is an isomorphism from A onto βA. More explicitly, $f^{\beta A}=\beta f^A(\beta^{-1})_n$ where $(\beta^{-1})_n$ is the pointwise application of β^{-1} to A^n. β is an *automorphism* of A exactly when $A=\beta A$. More generally, β is a *weak automorphism* if A and βA have the same *clone*. J. R. SENFT [30] showed that the weak automorphisms of A form a group WAut (A) which contains the automorphism group Aut (A) as a normal subgroup. (See also J. SICHLER [31]). Given a set I and a nonempty subset $B\subseteq A^I$, let

$$B^\beta = \{\beta^{-1}x \mid x\in B\} \subseteq A^I.$$

Lemma 2.1. *If B determines a subalgebra of A^I and β is a weak automorphism, then B^β also determines a subalgebra B^β of A^I.*

Proof. Suppose $B\subseteq A^I$, $f\in\mathrm{Op}$ is n-ary and $\beta^{-1}x_0, \beta^{-1}x_1, \dots, \beta^{-1}x_{n-1}\in B^\beta$. Since $f^{\beta A}$ is in the clone of A, there is a term $t\in\mathrm{Tm}$ such that $t^A=f^{\beta A}$, Then for each $i\in I$,

$$f^{A^I}(\beta^{-1}x_0, \dots, \beta^{-1}x_{n-1})(i) = f^A(\beta^{-1}x_0(i), \dots, \beta^{-1}x_{n-1}(i)) =$$
$$= \beta^{-1}\beta f^A(\beta^{-1}x_0(i), \dots, \beta^{-1}x_{n-1}(i)) = \beta^{-1}f^{\beta A}(x_0(i), \dots, x_{n-1}(i)) =$$
$$= \beta^{-1}t^A(x_0(i), \dots, x_{n-1}(i)) = \beta^{-1}t^{A^I}(x_0, \dots, x_{n-1})(i).$$

Thus,

$$f^{A^I}(\beta^{-1}x_0, \dots, \beta^{-1}x_{n-1})\in\beta^{-1}B = B^\beta.$$

B^β is not, in general, isomorphic to B. For example, if $A=(\{0,1\}, \wedge^A, \vee^A)$ is the two element lattice, then $\beta=(0,1)$ is a weak automorphism where $\wedge^{\beta A}=\vee^A$ and $\vee^{\beta A}=\wedge^A$. For a lattice $B\subseteq A^I$, B^β is the lattice obtained by reversing the order on B. Although $B\not\cong B^\beta$ in general, in this case it is clear that End $(B)\cong$End (B^β). To see that End (B^β) is always isomorphic to End (B), we will give an alternative construction ${}^\beta B$ for B^β.

Lemma 2.2. *Let* $\beta\in\mathrm{WAut}(A)$, $B\subseteq A^I$.

(i) *B determines a subalgebra A^I if and only if it determines a subalgebra of $(\beta A)^I$ (which will be denoted by ${}^\beta B$).*

Moreover, in this case

(ii) *B and ${}^\beta B$ have the same clone and (therefore) the same endomorphism monoid.*

Lemma 2.3. *If $\beta\in\mathrm{WAut}(A)$ and $B\subseteq A^I$, then $B^\beta\cong{}^\beta B$.*

Proof. The pointwise extension $\boldsymbol{\beta}$ of β to A^I is clearly a bijection from B^β onto ${}^\beta B$. Moreover, if $f\in\mathrm{Op}$ and $y_k=\beta^{-1}x_k$ where $x_k\in B$ for $k<n$, we have

$$\boldsymbol{\beta} f^{A^I}(y_0, \dots, y_{n-1})(i) = \beta f^A(\beta^{-1}x_0(i), \dots, \beta^{-1}x_{n-1}(i)) =$$
$$= f^{\beta A}(x_0(i), \dots, x_{n-1}(i)) = f^{\beta A^I}(x_0, \dots, x_{n-1})(i) =$$
$$= f^{\beta A^I}(\boldsymbol{\beta} y_0, \dots, \boldsymbol{\beta} y_{n-1})(i).$$

Starting with B we can change the carrier but retain the operations to obtain B^β or, alternately, change the operations but retain the carrier to obtain ${}^\beta B$. For example, in case

$$B = (B, \wedge^B, \vee^B) \subseteqq (\{0, 1\}, \wedge^A, \vee^A)^I,$$

we obtain B^β by applying $\beta=(0, 1)$ pointwise to members of B while ${}^\beta B=(B, \vee^B, \wedge^B)$; these are clearly isomorphic.

Algebras C and D will be called *clone equivalent* if there is a $B \in \mathbf{SP}A$ and a weak automorphism β of A such that $C \cong B$ and $D \cong B^\beta$. From Lemmas 2.2 and 2.3 we conclude

Corollary 2.4. *Clone equivalent algebras have isomorphic endomorphism monoids.*

As will be seen, weak automorphisms induce clone equivalences which impose an absolute limit on our ability to retrieve an algebra from its endomorphism monoid. But, as remarked prior to Theorem 1 and established below (Lemma 2.6), this limit is ammeliorated by the automorphisms of A.

Lemma 2.5. *If β and δ are permutations of A, then $(\beta\delta)A=\beta(\delta A)$.*

Proof. Let $f \in \mathrm{Op}$ be n-ary. Then

$$f^{(\beta\delta)A} = (\beta\delta)f^A((\beta\delta)^{-1})_n = \beta(\delta f^A(\delta^{-1})_n)(\beta^{-1})_n = f^{\beta(\delta A)}.$$

Lemma 2.6. *Let A be an algebra. For $\beta, \delta \in \mathrm{WAut}(A)$, the following are equivalent:*

(i) ${}^\beta B = {}^\delta B$ *for every* $B \in \mathbf{SP}A$.

(ii) $\beta A = \delta A$.

(iii) β *and* δ *are in the same coset of* $\mathrm{Aut}(A)$.

Proof. For (i)→(ii), take $B=A$. (ii)→(i) is trivial. To prove (ii)→(iii), we have

$$(\beta^{-1}\delta)A = \beta^{-1}(\delta A) = \beta^{-1}(\beta A) = (\beta^{-1}\beta)A = A$$

so that $\beta^{-1}\delta \in \mathrm{Aut}(A)$. For (iii)→(ii), let $f \in \mathrm{Op}$ be n-ary and assume (iii). Then $\beta^{-1}\delta A = A$ and therefore

$$f^{\delta A} = f^{(\beta\beta^{-1})\delta A} = f^{\beta(\beta^{-1}\delta A)} = f^{\beta A}.$$

Corollary 2.7. *If $\beta, \delta \in \mathrm{WAut}(A)$ are in the same coset of* $\mathrm{Aut}(A)$ *and $B \subseteqq A^I$, then $B^\beta \cong B^\delta$.*

3. Proof of Theorem 1

Throughout this section Q will denote a fixed minimal quasi primal algebra and $B \in \mathbf{ISP}Q$ will represent a fixed algebra whose endomorphism monoid is given abstractly. Our goal is to retrieve from End(B) an algebra that is clone equivalent to B, demonstrating that this is the only possibility for an algebra with endomorphism monoid isomorphic to End(B).

Let G be the set of automorphisms of Q, $E(Q)$ the set of elements of Q which form a one element subalgebra, and

$$\mathbf{Q} = (Q, \mu, e)_{\mu \in G, e \in E(Q)}$$

the dual topological structure. As given in Proposition 1.2, there exists $\mathbf{X} \in \mathbf{ISP}Q$ such that $B \cong \Phi(\mathbf{X})$ and End(B) is anti isomorphic to End$(\mathbf{X})$=Hom$(\mathbf{X}, \mathbf{X})$.

The topological duality [7] and B. A. Davey and H. Werner [9] will play an integral and essential role in our argument. This contrasts with conventional applications of duality in which one transfers a problem to a dual category, solves it there, and then transfers back. It is also the first instance we are aware of in which the duality representation, rather than the Boolean sheaf representation of S. Bulman-Fleming, K. Keimel and H. Werner [5], [16], has been used to solve a nontrivial algebraic problem for quasi primal varieties.

We begin by examining the structure of the members of $\mathbf{ISPQ}$. Let $\mathbf{E}(Q)$ be the subalgebra of $\mathbf{Q}$ determined by $E(Q)$, and let

$$\mathbf{G}' = (G, \mu)_{\mu \in G}$$

be the unary algebra determined by G (where, for $\mu, \gamma \in G$, $\mu(\gamma)$ is the product $\mu\gamma$ in G).

Lemma 3.1. (i) *Each* $\mathbf{X} \in \mathbf{ISPQ}$ *is the disjoint union of an isomorphic copy of* $\mathbf{E}(Q)$ *determined by its constants and a collection of isomorphic copies of* $\mathbf{G}'$. *The union of its copy of* $\mathbf{E}(Q)$ *and any set of these copies of* $\mathbf{G}'$ *is a subalgebra of* $\mathbf{X}$, *and these are its only subalgebras.*

(ii) *If* $\mathbf{P} \in \mathbf{ISPQ}$ *is finite,* $T \subseteq P$ *contains exactly one member of each copy of* G', *then* $\mathbf{P}$ *is* $\mathbf{ISPQ}$*-freely generated by* T.

Proof. (i) If $\mu \in G$ is not the identity, then the set of fixed points of μ is either empty or is a proper subalgebra of Q and therefore consists of exactly one element in $E(Q)$; in particular, μ has at most one fixed point. It follows that, for $\sigma, \tau \in G$ and $x \in X$ not a constant element,

$$\sigma x = \tau x \quad \text{implies} \quad \sigma = \tau.$$

Thus $\{\sigma x \mid \sigma \in G\}$ determines an isomorphic copy of $\mathbf{G}$. The remainder follows easily.

(ii) Next, let

$$f\colon T \to Y \quad \text{where} \quad \mathbf{Y}\in \mathbf{ISPQ}.$$

If $x\in T$ and $\sigma x=\tau x$, then $\sigma=\tau$ so that $\sigma fx=\tau fx$. This shows that f extends to a homomorphism.

Lemma 3.2. *$\mathbf{X}$ is finite if and only if* End ($\mathbf{X}$) *is finite.*

Proof. Suppose $\mathbf{X}\subseteq \mathbf{Q}^I$ is infinite. For $i\in I$, let $\pi_i\colon \mathbf{X}\to\mathbf{Q}$ be the projection with kernel θ_i. Then $\{\theta_i \mid i\in I\}$ must be infinite. By Lemma 3.1 (ii) there is an embedding $f\colon \mathbf{Q}\to\mathbf{X}$. Whence θ_i is the kernel of $f\pi_i$ so $\{f\pi_i \mid i\in I\}$ is an infinite subset of End ($\mathbf{X}$).

Proposition 3.3. *If B is finite, then it is determined up to isomorphism by* $|\mathrm{End}\,(B)|$, *the cardinality of* End (B).

Proof. If B is finite, then so is $\mathbf{X}$ and therefore End ($\mathbf{X}$) and End (B). Suppose $\mathbf{X}$ consists of a copy of $\mathbf{E}(Q)$ augmented by m copies of $\mathbf{G}$ as in Lemma 3.1(i). Then $\mathbf{X}$ is **IPSQ**-free on m generators so that

$$|\mathrm{End}(B)| = |\mathrm{End}(\mathbf{X})| = |X|^m = \big(|E(Q)|+m\,|G|\big)^m.$$

This number determines m, $\mathbf{X}$ and therefore B.

In the remainder of this section we assume that B, and therefore $\mathbf{X}$, is infinite.

Our next objective is to identify a subset X^* of End ($\mathbf{X}$) that corresponds to a copy of X and a subset A^* of X^* that corresponds to a copy of $\mathbf{Q}$. That is to say, we will establish the existence of a one-to-one function $*\colon X\to X^*\subseteq \mathrm{End}\,(\mathbf{X})$ and identify, in Lemma 3.6, A^* and X^* as subsets of End ($\mathbf{X}$). We remark that it is appropriate that we can only identify sets corresponding to $\mathbf{Q}$ and $\mathbf{X}$. Indeed, were we able to determine the accompanying algebraic and topological structures, B would be determined up to isomorphism. As shown in the next section, this need not be the case.

According to Lemma 3.1 (i) there is an $n<\omega$ such that $\mathbf{Q}$ is isomorphic to $\mathbf{E}(Q)$ augmented by n copies of $\mathbf{G}'$, and is free on n generators. Since $\mathbf{X}$ is infinite, it is the disjoint union of a copy $\mathbf{E}(X)$ of $\mathbf{E}(Q)$ and infinitely many isomorphic copies of $\mathbf{Q}$.

Lemma 3.4. *Every finite* $\mathbf{P}\subseteq\mathbf{X}$ *is a retract of $\mathbf{X}$, i.e., the image of $\mathbf{X}$ under an idempotent surjection.*

Proof. Choose a finite subset $J\subseteq I$ so that the projection $\pi_J\colon \mathbf{X}\to\mathbf{Q}^J$ is one-to-one on $\mathbf{P}$. Since $\pi_J\mathbf{X}$ is finite, Lemma 3.1 shows that there is a retraction g from $\pi_J\mathbf{X}$ onto $\pi_J\mathbf{P}$. Let h be the isomorphism from $\pi_J\mathbf{P}$ onto $\mathbf{P}$ such that $h\pi_J y=y$ for $y\in P$. Then $f=hg\pi_J$ retracts $\mathbf{X}$ to $\mathbf{P}$.

We can identify such a retract from within End $(\mathbf{X})$:

Lemma 3.5. *Let* $f \in M = \text{End}(\mathbf{X})$, $m < \omega$. *Then* f *is a retraction of* $\mathbf{X}$ *onto the union of* $\mathbf{E}(X)$ *and m copies of* $\mathbf{G}'$ *if and only if* $f^2 = f$ *and* $|fMf| = (|E(Q)| + m|G|)^m$.

Proof. If $f^2 = f$, $|fMf|$ is the number of endomorphisms of $f(\mathbf{X})$. If $f(\mathbf{X})$ is infinite, then by Lemma 3.2, End $(f(\mathbf{X}))$ is also infinite. Otherwise, $|\text{End}(f(\mathbf{X}))| = |f(\mathbf{X})|^r$ where $f(\mathbf{X})$ is free on r generators (Lemma 3.1(ii)).

Using Lemmas 3.1(i), 3.4 and 3.5 we now choose a fixed retraction f of $\mathbf{X}$ onto a copy $\mathbf{A}$ of $\mathbf{Q}$:

$$f\colon \mathbf{X} \to \mathbf{A} \subseteqq \mathbf{X}, \quad f^2 = f.$$

Let π denote an isomorphism from $\mathbf{Q}$ onto $\mathbf{A}$ which is fixed for the remainder of this section: $\pi\colon \mathbf{Q} \to \mathbf{A}$. Again using Lemma 3.5 we can choose a fixed retraction g of $\mathbf{X}$ onto a copy $\mathbf{G}$ of $\mathbf{E}(Q)$ augmented by one copy of $\mathbf{G}'$:

$$g\colon \mathbf{X} \to \mathbf{G} \subseteqq \mathbf{X}, \quad g^2 = g.$$

$\mathbf{G}$ is free on one generator by Lemma 3.1 (ii). Let x_0 be any free generator of G. For each $x \in X$ let x^* be the unique hg in End $(\mathbf{X})$ where $h(x_0) = x$. Then $*\colon X \to X^*$ is a bijection which takes A onto a subset A^* of X^*. Both X^* and A^* can be identified inside $M = \text{End}(\mathbf{X})$ as *sets:*

Lemma 3.6. $X^* = Mg$ *and* $A^* = fMg$.

Now there do exist unique algebraic and topological structures $\mathbf{X}^*$ and $\mathbf{A}^*$ on X^* and A^* such that $*\colon \mathbf{X} \to \mathbf{X}^*$ and $*\colon \mathbf{A} \to \mathbf{A}^*$ are isomorphisms. Although we do not have access to these structures, the next lemma shows that we are able to determine Hom $(\mathbf{X}^*, \mathbf{A}^*)$ as a subset of $(A^*)^{X^*}$. For each endomorphism $k\colon \mathbf{X} \to \mathbf{X}$ we define $k^*\colon X^* \to X^*$ by

$$k^*(x^*) = (k(x))^*$$

to obtain the commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{*} & X^* \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k^*} \\ X & \xrightarrow{*} & X^* \end{array}$$

Next we observe that each k^* and Hom $(\mathbf{X}^*, \mathbf{A}^*)$ can be defined within $M = \text{End}(\mathbf{X})$:

Lemma 3.7. (i) Hom $(\mathbf{X}, \mathbf{A}) = fM$.

(ii) If $k \in M$ *and* $hg \in X^* = Mg$, *then* $k^*(hg) = khg$.

(iii) $*\colon \text{End}(\mathbf{X}) \to \text{End}(\mathbf{X}^*)$ *is an isomorphism, and takes* Hom $(\mathbf{X}, \mathbf{A})$ *onto* Hom $(\mathbf{X}^*, \mathbf{A}^*)$.

Proof. (i) is easy. For (ii), let $x=h(x_0)$ and, thus, $x^*=hg$. Then $k^*(hg)= =k*(x^*)=(k(x))*=(khg(x_0))*=khg$.

Finally, the first part of (iii) follows from (ii) and the fact that $k*=*k*^{-1}$. The second part is seen from the following list of equivalent statements: $k(x)\in A$; $fk(x)=k(x)$; $fkhg(x_0)=khg(x_0)$; $fkhg=khg$; $khg\in A^*$; $k^*(x^*)\in A^*$.

Now let A and A^* be the unique algebras such that the maps

$$Q \xrightarrow{\pi} A \xrightarrow{*} A^*$$

are both isomorphisms. Then, in the category **ISP**Q of algebras we have

$$B \cong \Phi(\mathbf{X}) = \mathrm{Hom}(\mathbf{X}, \mathbf{Q}) \cong \mathrm{Hom}(\mathbf{X}, \mathbf{A}) \cong \mathrm{Hom}(\mathbf{X}^*, \mathbf{A}^*) \subseteq (A^*)^{X^*}$$

where the second isomorphism is induced by π and the third is the restriction of $*$. If we could, at this point, discover the correct structure on the set A^*, then Hom $(\mathbf{X}^*, \mathbf{A}^*)$, would determine a subalgebra of $(A^*)^{X^*}$ isomorphic to B. As already noted, this need not be possible. Suppose, however, that we could identify which maps $f\colon (A^*)^n \to A^*$ are in the *clone* of A^*. Since Q is given, we could then impose any algebraic structure on A^* that made it isomorphic to Q (and therefore to A^*) and gave it the same clone as A^* to obtain a weak isomorph of A^*. We would then be finished, since this algebraic structure would lift pointwise to Hom $(\mathbf{X}^*, \mathbf{A}^*)$ to determine an algebra clone equivalent to B.

Thus our final objective in this section is to identify the maps from $(A^*)^n$ into A^* which are in the clone of A^*. Since A^* is minimal quasi primal, an n-ary operation is a *term function* if and only if it is a *morphism* from $(\mathbf{A}^*)^n$ into $\mathbf{A}^*$ (Lemma 1.2). To identify such morphisms, we will pick out a copy $\mathbf{P}^*$ of $(\mathbf{A}^*)^n$ inside $\mathbf{X}^*$ and use our access to Hom $(\mathbf{X}^*, \mathbf{A}^*)$ to identify the morphisms from $\mathbf{P}^*$ into $\mathbf{A}^*$. Finally, we will use the category-theoretic definition of product to back these morphisms up to the morphisms from $(\mathbf{A}^*)^n$ into $\mathbf{A}^*$.

Lemma 3.8. *The homomorphisms from a finite substructure* $\mathbf{P}_1^*\subseteq\mathbf{X}^*$ *into a finite substructure* $\mathbf{P}_2^*\subseteq\mathbf{X}^*$ *are exactly the restrictions to* $\mathbf{P}_1^*$ *of members of* End $(\mathbf{X}^*)$ *which take* $\mathbf{P}_1^*$ *into* $\mathbf{P}_2^*$.

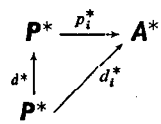
Proof. Use Lemma 3.4 and the fact that $*\colon \mathbf{X}\to\mathbf{X}^*$ is an isomorphism.

Let $0<n<\omega$. By Lemma 3.4 the finite substructures of $\mathbf{X}$ are determined by the finite subsets of the form kX where $k^2=k$. It follows that the finite substructures of $\mathbf{X}^*$ can be identified as being determined by the finite subsets of the form

$$k^*X^* = k^*(Mg) = kMg$$

where $k^2=k$. By Lemma 3.1 (i) there is a copy of $(\mathbf{A}^*)^n$ contained in $\mathbf{X}^*$. We can identify such a structure $\mathbf{P}^*$ as determined by any set $P^*=kMg$ where $k^2=k$ and

$|P^*|=|(A^*)^n|$. Thus $\mathbf{P}^*\cong(\mathbf{A}^*)^n$. The projections $\{\pi_i \mid i<n\}$, $\pi_i\colon (\mathbf{A})^n\to\mathbf{A}$, have the property that for any $\mathbf{Y}$ and $\delta_i\colon \mathbf{Y}\to\mathbf{A}$, $i<n$, there is a unique $\delta\colon\mathbf{Y}\to(\mathbf{A})^n$ such that each $\delta_i=\pi_i\delta$. Since $\mathbf{P}^*\cong(\mathbf{A})^n$, $\mathbf{P}^*$ must have the same property. Taking $\mathbf{Y}=\mathbf{P}^*$ and using Lemma 3.8, we can find a set of n "projection homomorphisms", $\{p_i^* \mid i<n\}$, whose restrictions to $\mathbf{P}^*$ take $\mathbf{P}^*$ onto $\mathbf{A}^*$ in such a way that for any choice of n homomorphisms $d_i^*\colon\mathbf{P}^*\to\mathbf{A}^*$, $i<n$, there is a unique homomorphism $d^*\colon\mathbf{P}^*\to\mathbf{P}^*$ such that for each $i<n$, d_i^* and $p_i^* d^*$ agree on $\mathbf{P}^*$:

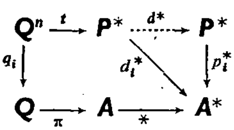


Now define $h\colon\mathbf{P}^*\to(\mathbf{A}^*)^n$ by

$$h(x^*)=(p_0^*x^*, p_1^*x^*, \dots, p_{n-1}^*x^*).$$

Lemma 3.9. *h is an isomorphism from $\mathbf{P}^*$ onto $(\mathbf{A}^*)^n$.*

Proof. Since $|P^*|=|(A^*)^n|=|Q^n|$, by Lemma 3.1 there is an isomorphism t from $\mathbf{Q}^n$ onto $\mathbf{P}^*$. For each $i<n$ let q_i be the i-th projection from $\mathbf{Q}^n$ onto $\mathbf{Q}$ and let $d_i^*=*\pi q_i t^{-1}$. Then there is a unique map d^* which makes the diagram



commute for each $i<n$.

We first claim that d^* is one-to-one. Suppose $x, y\in Q^n$ where $d^*t(x)=d^*t(y)$. Then $d_i^*t(x)=d_i^*t(y)$ for each $i<n$. It follows that $x=y$ and $t(x)=t(y)$.

Now we show that h is one-to-one. Let $x^*, y^*\in P^*$ where $h(x^*)=h(y^*)$. Since d^* is one-to-one, it is also onto so that there are $u, v\in Q^n$ such that $d^*t(u)=x^*$ and $d^*t(v)=y^*$. From the diagram we see that for each $i<n$, $q_i(u)=q_i(v)$. Thus $u=v$ and $x^*=y^*$.

Since $|P^*|=|(A^*)^n|$, h is an isomorphism onto.

Finally, because h is concretely given, we can give a criterion to test if k is in the clone of A^* which can be checked from within End $(\mathbf{X})$.

Lemma 3.10. *For a map $k\colon(A^*)^n\to A^*$ the following are equivalent:*

(i) *k is in the clone of A^*.*

(ii) *$kh\colon\mathbf{P}^*\to\mathbf{A}^*$ is a homomorphism.*

Proof. (i) holds if and only if $k\colon (\mathbf{A}^*)^n \to \mathbf{A}^*$ is a homomorphism, which is equivalent to (ii) by Lemma 3.9.

Finally, given End (B), and therefore End $(\mathbf{X})$, we construct X^*, $A^* \subseteqq$ End $(\mathbf{X})$ (Lemmas 3.5, 3.6) as well as the set Hom $(\mathbf{X}^*, \mathbf{A}^*)$ (Lemma 3.7). Next we use Lemmas 3.8 and 3.10 to determine the (set of maps in) the clone of A^*. Now we choose any algebraic structure A_0^* on the set A^* such that $A_0^* \cong Q \cong A^*$ and A_0^* has the same clone as A^*. Then the isomorphism from A_0^* onto A^* is a weak automorphism of A^*, and Hom $(\mathbf{X}^*, \mathbf{A}^*)$ determines a subalgebra of $(A_0^*)^{X^*}$ (Lemma 2.2) which is clone equivalent to the subalgebra it determines of $(A^*)^{X^*}$. But this subalgebra of $(A^*)^{X^*}$ is isomorphic to $\Phi(\mathbf{X})$ and therefore to B. This completes the proof of Theorem 1.

4. Proof of Theorem 2

For a fixed positive integer n we take $A = \{0, 1\}^n$. Let m and $'$ be the pointwise extension of the median $([x, y, z] = (x \vee y) \wedge (z \vee y) \wedge (y \vee z))$ and complementation operations on A respectively, and let t be the ternary discriminator operation on A. For $a, b \in A$ define

$$a * b = \begin{cases} a, & \text{if } a = b \\ a', & \text{if } a \neq b \end{cases}$$

For each element $e \in A$ we define a binary operation $[e]$ on A by

$$a[e]b = m(a, e, b).$$

Finally, let

$$A = (\{0, 1\}^n, t; *, [e])_{e \in A}.$$

Lemma 4.1. (i) *A is a minimal quasi primal algebra.*

(ii) *Each element of A determines a trivial subalgebra.*

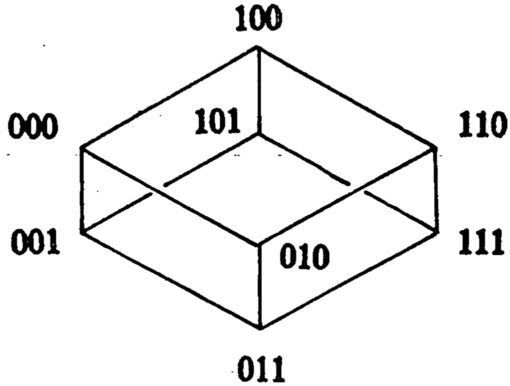
(iii) *For each $e \in A$, $(A, [e])$ is a meet semilattice with e as zero.*

Proof. (i) If $a \neq b$, $a, b \in A$, then for any $e \in A$ we obtain

$$a[e](a * b) = a[e]a' = e$$

is in the subalgebra generated by $\{a, b\}$. (ii) and (iii) are straight forward to verify.

By way of example, we illustrate the meet semilattice $(\{0, 1\}^3, [011])$.



Next we define an action of each β in the permutation group S_n on A by

$$\beta(e_0, e_1, \ldots, e_{n-1}) =: (e_{\beta 0}, e_{\beta 1}, \ldots, e_{\beta(n-1)})$$

for each $e=(e_0, e_1, \ldots, e_{n-1})\in A$. Then β permutes the elements of equal height in the meet semilattice $(A, [0, 0, \ldots, 0])$. Moreover, each $\beta\in S_n$ is a weak automorphism of A as we see by checking that

$$t^\beta = t, \quad *^\beta = *, \quad [e]^\beta = [\beta e]$$

for each $e\in A$.

For each $c\in A$, we construct a direct sum of ω copies of A relative to c. Let

$$B_c = \{a\in A^\omega \mid a^{-1}(c) \text{ is cofinite in } \omega\}.$$

By Lemma 4.1 (ii) B_c determines a subalgebra $\mathbf{B}_c$ of $\mathbf{A}^\omega$.

Lemma 4.2. *For each $d\in A$, the meet semilattice $(B_c, [d])$ has a zero if and only if $d=c$.*

Proof. If $d=c$, the constant $(c, c, c, \ldots)$ is a zero. Suppose $d\neq c$ and choose any element $z\in B_c$. Let $x\in B_c$ be obtained from z by replacing one occurrence of c by d. Then $x[d]z=x\neq z$ so that z is not a zero.

Lemma 4.3. ${}^\beta\mathbf{B}_c\cong\mathbf{B}_{\beta^{-1}c}$ *for each* $\beta\in S_n$.

Proof. Let $\boldsymbol{\beta}^{-1}$ be the pointwise extension of β^{-1} to B_c. Clearly $\boldsymbol{\beta}^{-1}$ is a bijection from B_c onto $B_{\beta^{-1}c}$ which preserves both t and $*$. To see that it preserves $[e]$, let $x, y\in B_c$. Then

$$\begin{aligned}\boldsymbol{\beta}^{-1}(x[e]^\beta y) &= \boldsymbol{\beta}^{-1}((x_0, x_1, \ldots)[\beta e](y_0, y_1, \ldots)) = \\ &= \boldsymbol{\beta}^{-1}(m(x_0, \beta e, y_0), m(x_1, \beta e, y_1), \ldots) = \\ &= (\beta^{-1}m(x_0, \beta e, y_0), \beta^{-1}m(x_1, \beta e, y_1), \ldots) = \\ &= (m(\beta^{-1}x_0, e, \beta^{-1}y_0), m(\beta^{-1}x_1, e, \beta^{-1}y_1), \ldots) = \\ &= (\beta^{-1}x_0[e]\beta^{-1}y_0, \beta^{-1}x_1[e]\beta^{-1}y_1, \ldots) = \boldsymbol{\beta}^{-1}x[e]\boldsymbol{\beta}^{-1}y.\end{aligned}$$

Let $c=(1,0,0,0,\ldots,0)\in A$ and let β be the n-cycle $(0,1,2,3,\ldots,n-1)$. Suppose ${}^{\beta^r}B_c\cong{}^{\beta^s}B_c$. By Lemma 4.3, $B_{\beta^{-r}c}\cong B_{\beta^{-s}c}$. From Lemma 4.2 it follows that $\beta^{-r}c=\beta^{-s}c$ so that r is congruent to s modulo n. This completes the proof of Theorem 2.

5. Two examples

Our first example is somewhat of a prototype. A *relatively complemented distributive lattice* is a distributive lattice $L=(L,\wedge^L,\vee^L,r^L)$ augmented by a ternary operation r^L satisfying

$$(*)\qquad \begin{cases}[(y\wedge(x\vee z))\vee(x\wedge z)]\wedge r(x,y,z)=x\wedge z\\ [(y\wedge(x\vee z))\vee(x\wedge z)]\vee r(x,y,z)=x\vee z\end{cases}$$

Here $r^L(x,y,x)$ is the complement of y projected into the interval $[x\wedge^L z,\ x\vee^L z]$. H. Werner [34] noticed that the two element lattice $R=(\{0,1\},\wedge,\vee,t)$, augmented by the ternary discriminator t, generates the variety of relatively complemented distributive lattices. R has exactly one weak automorphism, $\beta=(0,1)$, which is not an automorphism. Thus, for L in the variety generated by R, there is at most one nonisomorphic algebra L^β for which $\mathrm{End}\,(L)\cong\mathrm{End}\,(L^\beta)$ and it is obtained by inverting the order in L. In case L is finite, $L^\beta\cong L$ is a Boolean lattice.

Let $L=(\{0,1\},\wedge,\vee,-)$ be the two element Boolean algebra. As stated earlier, Boolean algebras are uniquely determined by their endomorphism monoids ([20], [21], [29]). However, $\beta=(0,1)$ is a weak automorphism of L which is not an automorphism. Our second example (Proposition 5.2) shows how algebras can be uniquely determined by their endomorphism monoids even in the presence of proper weak automorphisms.

Lemma 5.1. *If every weak automorphism of A is a term function of A, then $B\cong{}^\beta B$ for every $B\in\mathbf{SPA}$, $\beta\in\mathrm{WAut}\,(A)$.*

Proof. Let $\beta\in\mathrm{WAut}\,(A)$, $B\subseteq A^I$, $\boldsymbol{\beta}$ and $\boldsymbol{\beta}^{-1}$ the pointwise extensions to A^I. We first observe that (in general) $\boldsymbol{\beta}\colon B\to A^I$ is an embedding of B into $(\beta A)^I$. Let $g\in\mathrm{Op}$ be n-ary, $x\in B^n$, and $i\in I$. Then

$$\boldsymbol{\beta}g^B(x_0,\ldots,x_{n-1})(i)=\beta g^A(x_0(i),\ldots,x_{n-1}(i))=$$
$$=g^{\beta A}(\beta x_0(i),\ldots,\beta x_{n-1}(i))=g^{(\beta A)^I}(\boldsymbol{\beta}x_0,\ldots,\boldsymbol{\beta}x_{n-1})(i).$$

Now suppose each weak automorphism is a term function. Since B is a subalgebra of A^I, $\boldsymbol{\beta}B\subseteq B$, and since $\beta^{-1}\in\mathrm{WAut}\,(A)$, $\boldsymbol{\beta}^{-1}B\subseteq B$. Thus $B=\boldsymbol{\beta}B$ and $B\cong\boldsymbol{\beta}B={}^\beta B$.

A quasi primal algebra is *semi primal* if the only isomorphisms between its subalgebras are identity maps. A semi primal algebra (like L above) is *primal* if it has no proper subalgebras.

Proposition 5.2. *If A is a minimal semi primal algebra and $B, C \in \mathbf{ISP}A$ such that* $\mathrm{End}(B) \cong \mathrm{End}(C)$, *then* $B \cong C$.

Proof. By Theorem 1, there is a weak automorphism β of A such that $C \cong^{\beta} B$. A has at most one proper subalgebra which must be trivial, and therefore is preserved by each weak automorphism. Thus each weak automorphism is a term function and we use Lemma 5.1.

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