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Separation of the radical in ring varieties. II

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Solving a problem posed in [1] the second author described in [2] all associative ring varieties $\mathscr{W}$ satisfying the following property: the Jacobson radical of every ring in $\mathscr{W}$ is a direct summand. He has proved in particular that this property is local, i.e. if each finitely generated ring of a variety has the property then all rings of the variety have it. It turns out that there are rather few varieties having this property: they are exactly the unions of a nilvariety and a variety generated by a finite (possibly empty) set of finite fields.

In [2] the following question was posed: what are the ring varieties such that the Jacobson radical of 1) every 2) every finitely generated member is a semidirect summand? (We recall that a ring R is a semidirect sum of an ideal J and a subring S if $S+J=R$, $S\cap J=0$. In this situation we write $R=J\rtimes S$). It is the latter version of the question that is most interesting, because by the classical Wedderburn theorem all locally finite varieties of prime characteristic satisfy this property.

In this paper we solve problem 2) by giving a complete description for the varieties $\mathscr{W}$ such that the Jacobson radical of every finitely generated ring in $\mathscr{W}$ is a semidirect summand, i.e. for the varieties in which an analogue of Wedderburn's theorem is valid.

Theorem. *In a variety $\mathscr{W}$ of associative rings the Jacobson radical of every finitely generated member is a semidirect summand if and only if the identities $x^n=x^{2n}$ and $mx^n=0$ hold in $\mathscr{W}$ for some natural numbers n and m, m square free.*

Proof. Necessity. By $\mathscr{A}_p$ we denote the variety of associative rings defined by the identities $xy=yx$, $px=0$, where p is prime. Let $\mathbf{Z}_n$ be the ring of integers modulo n.

Suppose that $\mathscr{W}$ contains $\mathscr{A}_p$ for some p. Consider the ring R_p with generators a, b and relations $pa=pb=a^2b=b^2a=ab-ba=0$. The Jacobson radical J of R_p coincides with the principal ideal (ab) generated by ab. Indeed, $(ab)^2=0$, hence

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$(ab) \subseteq J$. On the other hand, the quotient ring $R_p/(ab)$ is isomorphic to the direct product of the polynomial rings $\mathbf{Z}_p[a]$ and $\mathbf{Z}_p[b]$ without constant term. Thus, $R_p/(ab)$ is semisimple and $J \subseteq (ab)$.

By definition, R_p belongs to $\mathscr{A}_p$ and, therefore, to $\mathscr{W}$. So the radical of the finitely generated ring R_p is a semidirect summand, $R_p = J \rtimes S$ for some S. Hence there exist x, y in S and i, j in J such that $a = x + i$, $b = y + j$. Multiplying these equalities we obtain $ab = xy$, because $JR_p = R_p J = 0$. However $ab \in J$, $xy \in S$, giving a contradiction. We have proved that $\mathscr{W}$ does not contain any variety $\mathscr{A}_p$.

By the main theorem of [3] the identity $x^k = x^l$ holds in $\mathscr{W}$ for some $k > l > 0$. Let $n = k(k-l)$, then $x^{2n} = x^{2n-k} x^k = x^{2n-k} x^l = \ldots = x^n$. Thus, the identity $x^{2n} = x^n$ holds in $\mathscr{W}$.

Now we consider the free ring F of rank one with free generator c in the variety $\mathscr{W}$. The element c^n is idempotent. Since the ring of integers does not satisfy $x^n = x^{2n}$, we obtain that the subring S generated by c^n is finite. Let it consist of m elements, then $mc^n = 0$. As F is free, the identity $mx^n = 0$ holds in $\mathscr{W}$. We have to show only that m is square free. Assuming the contrary we get $m = p^2 l$ for some prime p, $S = \mathbf{Z}_m$ implying that $\mathbf{Z}_{p^2} = \mathbf{Z}_m/(l)$ belongs to $\mathscr{W}$. However, the radical $p\mathbf{Z}_{p^2}$ of $\mathbf{Z}_{p^2}$ is not a semidirect summand, a contradiction.

Sufficiency. Suppose that $\mathscr{W}$ satisfies the identities $x^n = x^{2n}$, $mx^n = 0$, and let R be a finitely generated ring belonging to $\mathscr{W}$. Denote by J the radical of R and by G the set $\{r \in R \mid mr = 0\}$. By the main theorem of [3] the additive group of G is finitely generated, since it is a subgroup of the finitely generated abelian group of R. So G is a finite dimensional $\mathbf{Z}_m$-modul. The number m is a product of pairwise distinct primes $p_1, \ldots, p_k$. Let $G_i = \{x \in G \mid p_i x = 0\}$, $i = 1, \ldots, k$. It is easy to verify that $G = G_1 + \ldots + G_k$.

By Wedderburn's classical theorem the radical J_i of the finite dimensional $\mathbf{Z}_p$-algebra G_i is a semidirect summand. So $G = J(G) \rtimes S$ for some S. We shall show that $R = J \rtimes S$.

It is well known that the artinian semisimple ring R/J has an identity element. We lift it to an idempotent e of R. Then, for each x in R, $x - ex$ belongs to J and ex belongs to G, because $mex = me^n x = 0$. Thus $R = J + G$. The nilpotent ideal $J(G)$ is contained in J, implying $R = J + S$.

It remains to note that $J \cap S = \{0\}$, because J is a nilpotent ideal and the semisimple ring S does not contain nonzero nilpotent ideals. The theorem is proved.

Now we construct an example showing that the property of the Jacobson radical to be a semidirect summand is not local.

Example. Let A be a $\mathbf{Z}_p$-algebra presented by the variables x_r, where r belongs to the set $\mathbf{R}$ of real numbers, and by the relations $x_r^2 = x_r$ for each $r \in \mathbf{R}$, $x_r x_s = 0$ when $r > s$, $x_r x_s x_t = 0$ if $r < s < t$. Suppose $\mathscr{W}$ is the variety generated by A. In all finitely generated algebras in $\mathscr{W}$ the Jacobson radicals are semidirect summands, but in A the Jacobson radical is not a semidirect summand.

Proof. Let J be the ideal of A consisting of all sums of products $x_r x_s$, $r<s$. Since $J^2=0$ and A/J is semisimple, J is the radical of A.

Now take some z in A. Write

$$z = \sum_{i=1}^{n} k_i x_{r_i} + y \quad \text{with} \quad k_i \in \mathbf{Z}_p,\ y \in J.$$

Then

$$z^p = \sum_{i=1}^{n} k_i^p x_{r_i} + y_1 = \sum_{i=1}^{n} k_i x_{r_i} + y_1$$

for some $y_1 \in J$, so $z - z^p$ belongs to J. Since J is a zero-ring, we have $(x-x^p)(y-y^p)=$ $=0$ for arbitrary x, y in A. By the main theorem of [3] the identity $x^n = x^{2n}$ holds in A for some n. By the theorem proved above, in all finitely generated algebras in $\mathscr{W}$ the Jacobson radicals are semidirect summands.

Further, we prove that the Jacobson radical J of A is not a semidirect summand. Suppose the contrary, let $A = J \rtimes S$ for a subring S. Each element x_r, $r \in \mathbf{R}$, may be written as $x_r = a_r + b_r$, where $a_r \in J$, $b_r \in S$. If b_r is of the form

$$b_r = \sum_{i=1}^{m} k_i x_{r_i} + \sum_{j=1}^{n} h_j x_{s_j} x_{t_j} \quad \text{with} \quad k_i, h_j \in \mathbf{Z}_p,\ s_j < t_j,$$

then we denote by $w(r)$ the set $\{r_1, \ldots, r_m, s_1, \ldots, s_n, t_1, \ldots, t_n\}$. Considering the countable set $W = \bigcup_{t \in \mathbf{Z}} w(t)$, we choose $u \in \mathbf{R} \setminus W$ and $v \in W \setminus w(u)$. Clearly we may assume that $u < v$.

The product $b_v b_u$ is not zero because it has a summand $x_v x_u$ and all the other summands belong to the set $\{x_\alpha x_\beta \mid \alpha \in w(v),\ \beta \in w(u)\}$ not containing $x_v x_u$. Also, $b_v b_u$ belongs to J, contradicting $J \cap S = \{0\}$. The proof is complete.

The ring A constructed above is essentially uncountable. Indeed, we are able to prove that, for each variety $\mathscr{W}$ satisfying the conditions of the theorem, if all rings in $\mathscr{W}$ are commutative modulo the Jacobson radical then the radical of an arbitrary countably generated ring is a semidirect summand.

The problem of describing varieties such that the radical of each of their member is a semidirect summand still remains open. One can easily verify that, for instance, the varieties defined by the identities $xy - yx = x^n - x^{2n} = px$ have this property.

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