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On $*$-bases in lattices of finite length

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The notion of $*$-independence was introduced by G. CZÉDLI in [2] as an analogue of weak independence (see [1]). A subset H of a lattice L is called $*$-independent if for all $h, h_1, \ldots, h_n \in H$ satisfying $h = h_1 \vee \ldots \vee h_n$ there is an i $(1 \leqq i \leqq n)$ such that $h = h_i$. A maximal $*$-independent subset is called a $*$-basis of L. Let L be a lattice of finite length. Then two basic examples of $*$-bases are maximal chains and the set of all join-irreducible elements of L.

In this note we extend the following result of [2]:

Theorem A. *Every $*$-basis of a finite distributive lattice L has at least $|J_0(L)|$ elements ($J_0(L)$ is the set of all join-irreducibles of L).*

We also determine the class of lattices of finite length which have the property that any two $*$-bases have the same number of elements. This class turns out to be exactly the class of planar distributive lattices.

We will need the following well-known

Lemma B (see [3]). *Let D be a finite distributive lattice. If for the elements $j, x_1, \ldots, x_n \in D$ we have $j \in J_0(D)$ and $j \leqq x_1 \vee \ldots \vee x_n$ then $j \leqq x_i$ for some i, $1 \leqq i \leqq n$.*

Let L be a lattice of finite length. For any interval $[a, b]$ of length two in L let $N_{a,b}$ be a (possibly empty) set of new elements such that $N_{a,b} \cap N_{c,d} = \emptyset$ if $a \neq c$ or $b \neq d$. We define a new lattice $\tilde{L}$ containing L as a sublattice on the base set $L \cup \bigcup_{l([a,b])=2} N_{a,b}$ by adding to the Hasse diagram of L the covering relations $a \prec u$ and $u \prec b$ for all $N_{a,b}$ and for all $u \in N_{a,b}$. Then we say that $\tilde{L}$ can be obtained by inserting new elements into L.

Observe that for any join $x_1 \vee \ldots \vee x_n$ in $\tilde{L}$ we have either

$$x_1 \vee \ldots \vee x_n = x_i$$

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for some $1 \leqq i \leqq n$ or

$$x_1 \vee \ldots \vee x_n = \bar{x}_1 \vee \ldots \vee \bar{x}_n$$

where $x \to \bar{x}$ denotes the mapping of $\tilde{L}$ to L defined by

$$\bar{x} = \begin{cases} b & \text{if } x \in N_{a,b} \\ x & \text{if } x \in L. \end{cases}$$

Let $\mathscr{L}_0$ be the class of lattices which can be obtained by inserting new elements into some finite distributive lattice. The following result extends Theorem A:

Theorem 1. *If* $L \in \mathscr{L}_0$ *then for any* $*$*-basis of* L *we have* $|H| \geqq l(L)+1$.

Proof. Suppose that L can be obtained by inserting new elements into the distributive lattice D. Let $N = L \setminus D$. We define a mapping $\Phi\colon J_0(D) \to \mathscr{P}(H)$ by

$$\Phi(j) = \begin{cases} \{j\} \text{ if } j \in H \text{ (case } \alpha) \\ \{h \in N \cap H \mid j \text{ covers } h \text{ and } j = h \vee h_1 \vee \ldots \vee h_n \text{ for some } h_1, \ldots, h_n \in H\} \text{ if} \\ j \notin H \text{ and this set is nonempty (case } \beta) \\ \{h \in H \mid h = j \vee h_1 \vee \ldots \vee h_n \text{ for some } h_1, \ldots, h_n < h;\ h_1, \ldots, h_n \in H\} \\ \text{otherwise (case } \gamma). \end{cases}$$

Since H is a $*$-basis $\Phi(j)$ is always nonempty.

Now assume that for some $j \neq k$ we have $h \in \Phi(j) \cap \Phi(k)$. It is easy to check that this can happen only in two cases:

— $\Phi(j)$ was defined according to case α and $\Phi(k)$ was defined according to case γ;

— both $\Phi(j)$ and $\Phi(k)$ were defined according to case γ.

Suppose the latter. Then

$$h = j \vee h_1 \vee \ldots \vee h_n = k \vee g_1 \vee \ldots \vee g_m$$

for some $h_1, \ldots, h_n, g_1, \ldots, g_m \in H$ with $h_1, \ldots, h_n, g_1, \ldots, g_m < h$. This implies

$$j \vee \bar{h}_1 \vee \ldots \vee \bar{h}_n = k \vee \bar{g}_1 \vee \ldots \vee \bar{g}_m$$

and by distributivity

$$j = (j \wedge k) \vee (j \wedge \bar{g}_1) \vee \ldots \vee (j \wedge \bar{g}_m).$$

Since j is join-irreducible in D, we have $j = j \wedge \bar{g}_i$ for some $1 \leqq i \leqq m$. Then

$$\bar{g}_i \vee h_1 \vee \ldots \vee h_n = h$$

and since H is a $*$-basis we have

$$g_i \vee h_1 \vee \ldots \vee h_n < h.$$

As $g_i \vee h_1 \vee \ldots \vee h_n = h_t$ cannot hold ($1 \leqq t \leqq n$) we have $g_i \vee h_1 \vee \ldots \vee h_n = g_i$ thus $g_i \in N$ and $h = g_i$, whence $j \vee g_i = h$. Furthermore $k \vee g_i = h$, as otherwise $k \vee g_i = g_i$ which would imply $g_1 \vee \ldots \vee g_m = h$, a contradiction.

We obtained that if $h \in \Phi(j) \cap \Phi(k)$ for some $j \neq k$ then $h = \bar{h}'$ and $j \vee h' = k \vee h' = h$ for some $h' \in N \cap H$. The same conclusion can be derived in a similar way if $\Phi(j) = \{j\}$.

It is easy to see that $h' \notin \Phi(l)$ for any $l \in J_0(D)$. Indeed, since $h' \in N$ and $\bar{h}' = h$, the only possibility for $h' \in \Phi(l)$ is $l = h$ but then $h' \notin \{h\} = \Phi(l)$.

Now suppose that there is a third element $l \in J_0(D)$ with $h \in \Phi(l)$. Then by the above observations for some $h'' \in N \cap H$ we have $h = \bar{h}''$ and $j \vee h'' = l \vee h'' = h$. As $h' \vee h'' = h$ cannot hold we must have $h' = h''$ and $j \vee h' = k \vee h' = l \vee h' = h$. Assume that h' was inserted into the interval $[a, h]$. It is obvious that $j \vee a$, $k \vee a$, $l \vee a > a$. Further, they are pairwise distinct by Lemma B. If, say $j \vee a = h$, then $j \vee a = k \vee l \vee a$. Then again by Lemma B we have $j \geqq k, l$ and either $j \leqq k$ or $j \leqq l$, i.e. $j = k$ or $j = l$. Hence the elements $a < j \vee a$, $k \vee a$, $l \vee a < h$ form a sublattice isomorphic to M_3, the five element non-distributive modular lattice, which is a contradiction.

Finally define

$$H = \{h \in H \,|\, |\{j | h \in \Phi(j)\}| = i\}$$

for $i = 0, 1, 2$. Summarizing the above observations we can write

$$|H| = |H_0| + |H_1| + |H_2| \geqq |H_1| + 2|H_2| =$$
$$= \sum_{j \in J_0(D)} |\Phi(j)| \geqq |J_0(D)| = l(D) + 1 = l(L) + 1.$$

A finite lattice is said to be planar if its Hasse diagram can be described in the plane by using non-intersecting straight line segments.

Theorem 2. *Let L be a lattice of finite length. Then any two $*$-bases of L have the same number of elements iff L is a planar distributive lattice.*

Proof. Let L be a planar distributive lattice and let $H \subseteq L$ be a $*$-basis. Now for any join $x_1 \vee \ldots \vee x_n$ in L there are $1 \leqq i, j \leqq n$ such that $x_1 \vee \ldots \vee x_n = x \vee x_j$, i.e. $b(L)$ (=the breadth of L) $\leqq 2$ (see [5]). Consider the following map $\Phi \colon J_0(L) \to \mathscr{P}(H)$ which was introduced in [2]:

$$\Phi(j) = \begin{cases} \{j\} & \text{if } j \in H \\ \{h \in H \,|\, h = j \vee h_1 \vee \ldots \vee h_n \text{ for some } h_1, \ldots, h_n < h;\ h_1, \ldots, h_n \in H\} \\ \quad \text{if } j \notin H. \end{cases}$$

In view of Theorem A it is enough to show that for any $j \in J_0(L)$ we have $|\Phi(j)| = 1$ and for any $h \in H$ there is a $j \in J_0(L)$ with $h \in \Phi(j)$.

Suppose that $h \neq h'$ and $h, h' \in \Phi(j)$, i.e. $h = j \vee h_1 \vee \ldots \vee h_m$ and $h' = j \vee h_1' \vee \ldots \vee h_n'$ for some $h_1, \ldots, h_m < h$, $h_1, \ldots, h_m \in H$ and for some $h_1', \ldots, h_n' < h'$, $h_1', \ldots, h_n' \in H$. By the starting note on joins $h = j \vee h_s$ and $h' = j \vee h_t'$ for some $1 \leqq s \leqq m$ and for some $1 \leqq t \leqq n$. Then $h \vee h' = j \vee h_s \vee h_t'$ and by $b(L) \leqq 2$ again, we have three possibilities:

$$h \vee h' = \begin{cases} j \vee h_s & \text{(a)} \\ j \vee h_t' & \text{(b)} \\ h_s \vee h_t' & \text{(c).} \end{cases}$$

In case (a) $h \vee h' = h$, whence $h' \vee h_s = h$. Since H is a basis, we must have $h' = h$. Case (b) is similar to case (a) and in case (c) we have $j \leqq h \vee h' = h_s \vee h_t'$ where h_s, $h_t' \not\geqq j$. By Lemma B this is a contradiction.

Now le $h_0 \in H$. If $h_0 \in J_0(L)$ then $h_0 \in \Phi(h_0)$. Let $h_0 \notin J_0(L)$. Then (again by $b(L) \leqq 2$) there are two incomparable elements $j, j' \in J_0(L)$ such that $j \vee j' = h_0$. If $j \in H$ then $j' \in H$ cannot hold, thus $h_0 \in \Phi(j')$. Suppose that $j \notin H$. Then $h = j \vee h_1$ for some $h_1 < h$; h_1, $h \in H$. We have three possibilities:

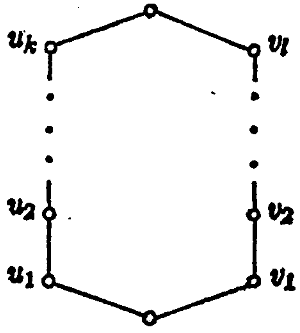
$$j \vee j' \vee h_1 = \begin{cases} j \vee j' & \text{(a)} \\ j \vee h_1 & \text{(b)} \\ j' \vee h_1 & \text{(c).} \end{cases}$$

Case (a). Then $h \leqq h_0$. If $h = h_0$ then $h_0 \in \Phi(j)$. If $h < h_0$ then $j' \vee h = h_0$ yealds $h_0 \in \Phi(j')$.

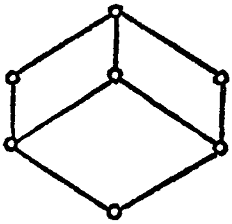
Case (b). Then $h_0 \leqq h$. If $h_0 = h$ then $h_0 \in \Phi(j)$. If $h_0 < h$ then $h_0 \vee h_1 = h$ contradicts the fact that H is a $*$-independent subset.

Case (c). Then $j \leqq j' \vee h_1$ and $j \not\geqq j'$, $j \not\geqq h_1$, a contradiction.

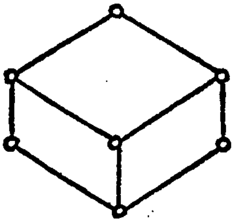
Now suppose that L is not a planar distributive lattice. If L is not modular then as it was shown in [4], L contains a c-sublattice L_0 isomorphic to $L_{k,l}$ ($k \geqq 2$, $l \geqq 1$), B or B' (see the diagram).



$L_{k,l}$



B

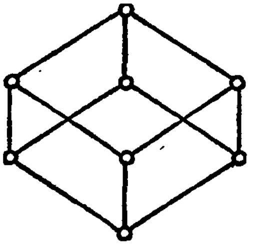


B'

(A sublattice L_0 of a lattice L is a c-sublattice if $a \prec b$ in L whenever $a \prec b$ in L_0.)

It is straightforward to check that in each case there is a $*$-basis $H_0 \subseteqq L_0$ and there is a maximal chain $C_0 \subseteqq L_0$ with $|H_0|>|C_0|$. Let $a=x_0 \prec \ldots \prec x_m = a_0$ and $b_0 = y_0 \prec \ldots \prec y_n = b$ be two maximal chains between a and a_0 and between b_0 and b, respectively, where $a = \bigwedge L$, $b = \bigvee L$ and $a_0 = \bigwedge L_0$, $b_0 = \bigvee L_0$. Then $H = H_0 \cup \{x_0, \ldots, x_m\} \cup \{y_1, \ldots, y_n\}$ is a $*$-independent subset (which can be extended to a $*$-basis of L) and $C = C_0 \cup \{x_0, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ is a maximal chain of L (which is itself a $*$-basis of L) such that $|H|>|C|$.

If L is modular but not distributive then L contains M_3 as a c-sublattice (see [3]).



Since M_3 has a $*$-basis with 4 elements, we can argue as above, in the non-modular case. If L is distributive and L is not planar then L contains a cube (see the diagram) as a c-sublattice (cf. [6]) and again, we can argue as in the non-modular case.

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