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A maximal partial clone and a Slupecki-type criterion

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0. Introduction

In this paper we study certain aspects of partial algebras, composition of partial operations and partial clones. Partial algebras are known to be more flexible than the full ones but also to present conceptual difficulties (in the sense that certain notions which are obvious for full algebras, may be extended in several ways in partial algebras) and often require more complex treatment. We approach the definition of the composition of partial operations via the onepoint extension (completion), an idea which has been around for a long time (cf. e.g. [8]) whithout being systematically exploited. It embeds partial algebras on A into full algebras on $B:=A\cup\{\infty\}$ (Where $\infty\notin A$). The images of partial operations are in the clone $\mathbf{R}_B$ on B consisting of all n-ary operations f on B such that 1) $f(\infty, ..., \infty)=\infty$ and 2) for each $1\leqq i\leqq n$ such that f depends on its i-th variable we have

$$f(x_1, ..., x_{i-1}, \infty, x_{i+1}, ..., x_n) = \infty$$

for all $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n\in B$ $(n=1, 2, ...)$. Partial clones are then defined as natural restrictions of subclones of $\mathbf{R}_B$ (for more details see [16]).

In [7], [14], [15] partial clones are understood in a more restrictive sense. They are not only closed under composition but also under taking suboperations (i.e. with each f such a clone also contains every partial operation obtained by restricting the domain). We call such clones strong. This definition is compatible with the relational (i.e. SP) definition and has been introduced for $|A|=2$ in [2], [3], for finite universes or operations with finite domains in [14], subalgebras of direct powers were considered in [7] for the finite case and the Galois connection for A infinite is in [15].

For finite full algebras one of the immediate questions arising naturally e.g. in propositional calculi of logics or in switching theory is wether they are complete (or primal). A general completeness criterion may be based on the knowledge of all

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maximal clones (i.e. the dual atoms or co-atoms of the lattice of clones). For partial operations on A finite the question, albeit not so immediate, leads first to the study of maximal clones. These were determined for $|A|=2$ in [2], [3] and for $|A|=3$ in [20] (it seems that the three maximal clones of partial $\leqq$ isotone operation for $\leqq$ a chain were inadvertedly omitted); it is reported in [22] that a solution is in [25]. For $|A|>2$ some maximal strong partial clones are described in [14] and very recently the two first authors characterized them combinatorially [21]; another approach is in [22]—[24] which were not available during the redaction of this paper. Not surprisingly, the list of maximal partial clones is much more extensive and more complex to state than the list of maximal full clones. However, this does not tell the whole story if non-strong clones are considered as well.

In this paper we address this problem. The answer is quite simple: there is a unique maximal partial non-strong clone. It consists of all full operations and all partial operations with the empty domain (the latter seem to be somewhat paradoxal but still useful here and elsewhere). For $|A|=2$ this clone is in [2], [3], for $|A|=3$ it is mentioned in [20] (without proof). For A finite we deduce that the partial clone of all partial operations is generated by two operations and obtain an analogue of Słupecki criterion [18]. The proofs are quite straightforward. Since the topic is off the beaten path and not quite well understood we have tried to make the paper self-contained and present certain details in a more leasurely way.

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1. Preliminaries

1.1. Let B be a fixed nonempty universe. For a positive integer n let $\mathbf{O}_B^{(n)}$ denote the set of *n-ary-operations on* B (i.e. maps $B^n \to B$) and let $\mathbf{O}_B := \bigcup_{n=1}^{\infty} \mathbf{O}_B^{(n)}$ (the use of 0-ary operations, although possible, entails unnecessary formal complications and so we prefer to replace them by unary constant operations). The composition of operations may be formally defined in several (equivalent) ways. In the literature often this is glossed over but none of the more precise definitions seem to be short. The following formal definition [8] in terms of a monoid on $\mathbf{O}_B$ together with three unary operations neatly avoids the explicit use of arities. We start with the following monoid $\langle \mathbf{O}_B; *, \mathrm{id}_B \rangle$. Let $f \in \mathbf{O}_B^{(m)}$ and $g \in \mathbf{O}_B^{(n)}$. Put $p := m+n-1$ and let $h := f * g \in \mathbf{O}_B^{(p)}$ be defined by setting

$$h(b_1, \dots, b_p) := f(g(b_1, \dots, b_n), b_{n+1}, \dots, b_p) \quad \text{for all} \quad b_1, \dots, b_p \in B.$$

Thus $f*g$ is the result of replacing the first variable of f by g while keeping all variables distinct. In order to be able to permute variables, we introduce the following unary operations ζ and τ on $\mathbf{O}_B$. Let $f\in\mathbf{O}_B^{(n)}$. For $n>1$ we define $\tau(f), \zeta(f)\in\mathbf{O}_B^{(n)}$ by setting

$$\tau(f)(x_1, \dots, x_n):=f(x_2, x_1, x_3, \dots, x_n),$$
$$\zeta(f)(x_1, \dots, x_n):=f(x_2, \dots, x_n, x_1)$$

for all $x_1, \dots, x_n\in B$. For $n=1$ we put $\tau(f):=\zeta(f):=f$. Thus $\tau(f)$ is the result of the exchange of the two first variables in f while $\zeta(f)$ is obtained from f by a cyclic shuffle of variables. Since a transposition and a cyclic permutation generate the symmetric group S_n, we can get all permutations of variables via repeated applications of ζ and τ.

The following unary operation Δ on $\mathbf{O}_B$ designed for the fusion (identification) of variables. Let $f\in\mathbf{O}_B^{(n)}$. For $n>1$ the operation $\Delta(f)\in\mathbf{O}_B^{(n-1)}$ is defined by setting $\Delta(f)(x_1, \dots, x_{n-1}):=f(x_1, x_1, x_2, \dots, x_{n-1})$ for all $x_1, \dots, x_{n-1}\in B$ while for $n=1$ we put $\Delta(f):=f$. Finally for $n\geqq i\geqq 1$ let $e_i^n(x_1, \dots, x_n):=x_i$ for all $x_1, \dots, x_n\in B$. The universal algebra (more precisely a monoid with three unary operations and a constant)

$$O_B:=\langle \mathbf{O}_B;\ *, \zeta, \tau, \Delta, e_1^2\rangle$$

may be called the *full post-iterative algebra* on B. A subuniverse (i.e. the carrier of a subalgebra) of $\mathbf{O}_B$ is called a *clone* on B. In simpler terms, a clone on B is a composition closed set of operations on B containing all projections. Thus a clone is a multivariable analogue of a transformation monoid on B or a permutation group on B (whereby projections play the role of the identity). The set $\bar{F}$ of all term operations (or polynomials) of a universal algebra $\langle B; F\rangle$ is a clone (it is the least clone containing F). The clones on B, ordered by $\subseteqq$, form an algebraic lattice L_B which, apart from the case $|B|=2$ ([13], cf. [11]) is largely unknown. For $F\subset\mathbf{O}_B$ and a positive integer n put $F^{(n)}:=F\cap\mathbf{O}_B^{(n)}$.

We say that $f\in\mathbf{O}_B^{(n)}$ depends on its i-th variable if there are $c_1, \dots, c_{i-1}, c_{i+1}, \dots \dots, c_n\in B$ such that $h\in\mathbf{O}_B^{(1)}$ defined by setting $h(x):=f(c_1, \dots, c_{i-1}, x, c_{i+1}, \dots, c_n)$ for all $x\in B$ is non-constant.

1.2. Let A be a fixed universe with $|A|>1$. For a positive integer n a *partial n-ary operation* is a map $f\colon D_f\to A$ where $D_f\subseteqq A^n$. Let $\mathbf{P}_A^{(n)}$ denote the set of partial n-ary operations and let $\mathbf{P}_A:=\bigcup_{n=1}^{\infty}\mathbf{P}_A^{(n)}$. For our purposes we may treat $\mathbf{P}_A$ via the following one point extension. Let ∞ denote an element outside A and let $B:=A\cup\{\infty\}$. For a positive integer n let $\mathbf{R}_B^{(n)}$ denote the set of $f\in\mathbf{O}_B^{(n)}$ such that 1) $f(\infty, \dots, \infty)=\infty$ and 2) for every $1\leqq i\leqq n$ such that f depends on its i-th variable we have

$$f(x_1, \dots, x_{i-1}, \infty, x_{i+1}, \dots, x_n)=\infty$$

for all $x_1, \ldots, x_{i+1}, \ldots, x_n \in B$. To $f \in \mathbf{R}_B^{(n)}$ we may assign the partial operation $f^- \in \mathbf{P}_A^{(n)}$ defined by $D_{f^-} := \{\tilde{x} \in A^n : f(\tilde{x}) \in A\}$ and $f^-(\tilde{x}) := f(\tilde{x})$ for all $\tilde{x} \in D_{f^-}$. In this sense $f \in \mathbf{R}_B^{(n)}$ is obtained from f^- by setting $f(\tilde{x}) = \infty$ for all $\tilde{x} \in A^n \setminus D_{f^-}$ (i.e. there where f^- is not defined; note that if f^- is given by table usually a symbol like $^-$ or * is put in such places) and then we extend f^- to B^n. One is tempted to putt $f(\tilde{x}) := \infty$ everywhere no $B^n \setminus A^n$, however with this definition the new operations depend on all their variables while the clone on B generated by them will contain operations with dummy variables — for example projections on B must be added — and for this reason we have adopted the above more flexible definition. It is not difficult to see that $\mathbf{R}_B$ is a clone on B [16]. Now a *partial clone* on A is the image $C^- := \{f^- : f \in C\}$ of a subclone C of $\mathbf{R}_B$. Note that the map $C \to C^-$ (from the lattice of subclones of $\mathbf{R}_B$ onto the lattice of partial clones) is not injective. The advantage of this definition is twofold. First, we don't need to develop the theory of partial clones separately but may view it as (a slight modification of) the theory of subclones of $\mathbf{R}_B$ and secondly we are led to an interesting clone $\mathbf{R}_B$. Note that for $1 < |A| < \aleph_0$ the clone $\mathbf{R}_B$ sits at the bottom of a descending infinite chain of clones [15] and that $\mathbf{R}_B$ (and $\mathbf{P}_A$) has $2^{\aleph_0}$ subclones [4]. Perhaps it is worth mentioning that each post-iterative algebra C on B containing $\mathbf{R}_B$ has only one non-trivial congruence (with blocks $C \cap \mathbf{O}_B^{(n)}$, $n = 1, 2, \ldots$) and only automorphisms induced by permutations of B [8].

In [27] our definition of a partial clone is compared with another definition attributed to A. I. MAL'CEV [28, p. 12]. For f an n-ary and g an m-ary partial operation on A the composition $h = f *_A g$ is an r-ary $(r = m + n - 1)$ partial operation on A with the domain

$$D_h := \{(a_1, \ldots, a_r) : (a_1, \ldots, a_m) \in D_g,\ (g(a_1, \ldots, a_m), a_{m+1}, \ldots, a_r) \in D_f\}$$

and value $f(g(a_1, \ldots, a_m), a_{m+1}, \ldots, a_r)$ for all $(a_1, \ldots, a_r) \in D_h$. It is shown in [27] that if $f \in \mathbf{R}_B$ depends on its first variable, then $f^- *_A g^- = (f * g)^-$ but this needs not hold if the first variable of f is fictitious. Partial clones based on $*_A$, called M-clones in [27], are a special case of our clones but not vice versa. According to BÖRNER [27] our results hold for M-clones as well. Some people may prefer M-clones because they may be handled in a categorical way [29]. We are grateful to R. PÖSCHEL [29] ad H. J. HOEHNKE [30] for pointing out the connections and for interesting comments. We mention one possible application. For a binary relation p on A (i.e. a subset of A^2) consider $e_p \in \mathbf{R}_B^{(2)}$ defined by

$$e_p(x, y) = x \quad \text{if} \quad (x, y) \in p \quad \text{and} \quad e_p(x, y) = \infty$$

otherwise. The algebra $\langle B; e_p \rangle$ is called the *graph algebra* of p and its variety has been studied in the context of finitely axiomatizable varieties (cf. [10, 12, 19] for references).

1.3. Let $f, g\in \mathbf{P}_A^{(n)}$. We say that f is a *suboperation* of g, in symbols $f\leqq g$, if $D_f\subseteqq D_g$ and $f(x)=g(x)$ for all $x\in D_f$. A partial clone C is *strong* if $g\in C$ whenever $g\leqq f\in C$. Let $\mathbf{O}_A^{*(n)}$ denote the set of $f\in \mathbf{P}_A^{(n)}$ with $D_f=A^n$ and let $\mathbf{O}_A^*=\bigcup_{n=1}^{\infty}\mathbf{O}_A^{*(n)}$. We have the following:

1.4. Lemma. *Let C be a partial clone on A. Then*

$$C':=\{f\in \mathbf{P}_A\colon f\leqq g \text{ for some } g\in C\}$$

is a strong partial clone such that

$$C'\cap \mathbf{O}_A^* = C\cap \mathbf{O}_A^*.$$

Proof. Let C_1 be a subclone of $\mathbf{R}_B$ such that $C_1^-=C$. Let C_2 consist of all operations in $\mathbf{R}_B$ obtained from an operation belonging to C_1 by replacing some of its values by ∞. Clearly $C_2^-=C'$, hence for the first statement it suffices to show that C_2 is subclone of $\mathbf{R}_B$. Let $f\in C_2^{(m)}$ and $g\in C_2^{(n)}$ be obtained from $f'\in C_1$ and $g'\in C_1$ by converting some of their values from A to ∞.

1) Put $h':=f'*g'$ and $h:=f*g$. Let $p:=m+n-1$ and $\tilde{a}=(a_1, \ldots, a_p)\in B^p$ be such that $h(a)\in A$. We show that $h'(\tilde{a})\in A$. Put

$$(1)\qquad \begin{aligned} a_0' &:= g'(a_1, \ldots, a_n),\\ a_0 &:= g(a_1, \ldots, a_n). \end{aligned}$$

Note that

$$(2)\qquad \begin{aligned} h'(\tilde{a}) &= f'(a_0', a_{n+1}, \ldots, a_p),\\ h(\tilde{a}) &= f(a_0, a_{n+1}, \ldots, a_p). \end{aligned}$$

Suppose $a_0=\infty$. In view of $f\in \mathbf{R}_B$ and $h(\tilde{a})=f(a_0, a_{n+1}, \ldots, a_p)\in A$, the operation f does not depend on its first variable, thus $h(\tilde{a})=f(a_0, a_{n+1}, \ldots, a_p)\in A$ and *a fortiori* $h'(\tilde{a})\in A$ and we are done. Thus let $a_0\in A$. Now it follows from (1) that $a_{0'}\in A$ and $a_0=a_0'$, hence (2) shows $h'(\tilde{a})\in A$.

2) We show that $\tau(f)\in C_2$. This is evident if $n<2$ and so let $n\geqq 2$. Suppose $\tilde{a}=(a_1, \ldots, a_m)\in B^m$ is such that $\tau(f)(\tilde{a})\in A$. Then $f(a_2, a_1, a_3, \ldots, a_m)\in A$ which implies $f'(a_2, a_1, a_3, \ldots, a_m)\in A$ and $\tau(f')(\tilde{a})\in A$. Thus $\tau(f)$ is obtained from $\tau(f')$ by replacing some values in A by ∞. As $\tau(f')\in C_1$ we have $\tau(f)\in C_2$.

3) The proof $\zeta(f)\in C_2$ and $\Delta(f)\in C_2$ is quite analogous. Thus C_2 is a clone and consequently a subclone of $\mathbf{R}_B$.

For the equality $C'\cap \mathbf{O}_A^*=C\cap \mathbf{O}_A^*$ the inclusion $\supseteqq$ is obvious from $C'\supseteqq C$. If $f\in C\cap \mathbf{O}_A$, then $f\leqq g$ for some $g\in C$. Here $g\in \mathbf{O}_A^*$, hence $f=g\in C\cap \mathbf{O}_A^*$, proving $\subseteqq$ and completing the proof.

1.5. It follows from Lemma 1.4 that each partial clone C such that $C\cap \mathbf{O}_A^*\subset \mathbf{O}_A^*$ (i.e. $\neq \mathbf{O}_A^*$) extends to a strong partial clone distinct from $\mathbf{P}_A$. Thus for the study of large partial clones we may study separately strong partial clones and partial clones containing $\mathbf{O}_A^*$. This paper studies the second type. The study of maximal strong partial clones has started in [2, 3] for $|A|=2$, [20] for $|A|=3$ and is continued in [6, 26], see also [5].

2. Finite partial clones containing all operations

2.1. In the sequel we study partial clones containing the set $\mathbf{O}_A^*$ of all operations. Put

$$M_1 := \{f \in \mathbf{R}_B^{(n)} \colon n < \omega,\ f(A^n) \subseteq A\}$$
$$M_2 := \{f \in \mathbf{R}_B^{(n)} \colon n < \omega,\ f(A^n) = \{\infty\}\}$$

We need the following:

2.2. Lemma. *If $f \in \mathbf{R}_B \setminus (M_1 \cup M_2)$, then $\mathbf{R}_B = \overline{M_1 \cup \{f\}}$ (i.e. $M_1 \cup \{f\}$ generates $\mathbf{R}_B$.)*

Proof. Let f be n-ary. Put $Q := \overline{M_1 \cup \{f\}}$ and $D_f := \{\tilde{x} \in A^n \colon f(\tilde{x}) \in A\}$. We have $\emptyset \neq D_f \subset A^n$ and thus there are

$$\tilde{a} = (a_1, \ldots, a_n) \in D_f \quad \text{and} \quad \tilde{b} = (b_1, \ldots, b_n) \in A^n \setminus Df.$$

Fix $0 \in A$ and put $A^* := A \setminus \{0\}$. The proof is done in four steps.

Fact 1. *Let $I \colon B \to B$ be defined by setting $I(0) = I(\infty) := \infty$ and $I(x) = 0$ otherwise. Then $I \in Q$.*

Proof. Let $g_i \colon B \to B$ be defined by

$$g_i(0) := b_i, \quad g_i(\infty) := \infty \quad \text{and} \quad g_i(x) := a_i$$

otherwise ($i = 1, \ldots, n$). Clearly $g_i \in M_1$. Define $h \colon B \to B$ by setting

$$h(x) := f(g_1(x), \ldots, g_n(x)) \quad \text{for all} \quad x \in B.$$

Clearly $h \in Q$ and $h(0) = f(\tilde{b}) = \infty = h(\infty)$ and $h(x) = f(\tilde{a}) \in A$ for all $x \in A^*$. Now M_1 contains $g \colon B \to B$ defined by $g(\infty) := \infty$ and $g(x) := 0$ otherwise. Clearly $I = g \circ h \in Q$.

Fact 2. *Let $p \colon B \to B$ be defined by setting $p(0) = p(\infty) := \infty$ and $p(x) := x$ for all $x \in A^*$. Then $p \in Q$.*

Proof. Let $e \in \mathbf{O}_B^{(2)}$ be defined by setting $e(x, y) = y$ for all $x, y \in A$ and $e(x, y) = \infty$ otherwise. Clearly $e \in M_1$ and $p(x) = e(I(x), x)$ for all $x \in B$. Thus $p \in Q$.

Let 1 be a fixed element of A^*.

Fact 3. *There is a binary operation* $r\in Q$ *such that* $r(0,0)=\infty$, $r(1,0)=0$ *and* $r(x,x)=x$ *for all* $x\in B\setminus\{0\}$.

Proof. Let $q\in \mathbf{O}_B^{(2)}$ be defined by setting $q(\infty, x):=\infty$ for all $x\in B$ and $q(x,y):=y$ otherwise. Clearly $q(A^2)=A$ and so $q\in M_1\subseteq Q$. Put $r(x,y):=$ $:=q(p(x),y)$ for all $x,y\in B$. We have $r(0,0)=q(\infty,0)=\infty$,

$$r(1,0)=q(1,0)=0$$

and

$$r(x,x)=q(p(x),x)=q(x,x)=x \quad \text{for all} \quad x\in B\setminus\{0\}.$$

Now we can finish the proof. Let $q\in \mathbf{R}_B^{(m)}$ depend on all its variables. For $x\in B$ put $C_x:=g^{-1}(x)\cap A^m$. Define $c_1, c_2\in \mathbf{O}_B^{(m)}$ by setting

$$\begin{aligned} c_1(\tilde{y}) &= c_2(\tilde{y}) = 0 && \text{if} \quad \tilde{y}\in C_\infty, \\ c_1(\tilde{y}) &= 1, \quad c_2(\tilde{y}) = 0 && \text{if} \quad \tilde{y}\in C_0, \\ c_1(\tilde{y}) &= c_2(\tilde{y}) = x && \text{if} \quad x\in A\setminus O, \quad \tilde{y}\in C_x \\ c_1(\tilde{y}) &= c_2(\tilde{y}) = \infty && \text{otherwise} \quad (\text{i.e. for } \tilde{y}\in B^m\setminus A^m). \end{aligned}$$

Clearly $c_1, c_2\in M_1^{(m)}$. Indeed, using Fact 3 we can verify that $g(\tilde{y})=r(c_1(\tilde{y}), c_2(\tilde{y}))$ for all $\tilde{y}\in B^m$. It follows that $g\in Q$ i.e. the subclone Q of $\mathbf{R}_B$ contains all operations from $\mathbf{R}_B$ depending on all variables. However, Q being a clone, this shows the required $Q=\mathbf{R}_B$.

For $n<\omega$ let p_n denote the partial n-ary operation on A with empty domain (i.e. $D_{p_n}=\emptyset$). We have our main result.

2.3. Theorem. *The set*
$M:=\mathbf{O}_A^*\cup\{p_n\colon 0<n<\omega)$ *is*
(i) *a maximal partial clone, and*
(ii) *a unique clone properly between* $\mathbf{O}_A^*$ *and* $\mathbf{P}_A$.

Proof. Put $M_3:=M_1\cup M_2$ (see 2.1). Clearly $M_3^-=M$ and thus it suffices to show that M_3 is a unique maximal subclone of $\mathbf{R}_B$ containing M_1.

(i) We prove that M_3 is a clone. Indeed, let $f\in M_3^{(n)}$ and $g\in M_3^{(m)}$. 1) Consider $h:=f*g$ and let $p:=m+n-1$. If $f\in M_2$ then clearly $h\in M_2$. Thus assume that $f\in M_1$. If $g\in M_1$, then clearly $h\in M_1$. Finally let $g\in M_2$. If f depends on its first variable, then due to $f\in \mathbf{R}_B$, we have $h\in M_2$, hence it remains to consider the case of f not depending on its first variable. Let $a_1, \ldots, a_p\in A$. Then

$$h(a_1, \ldots, a_p) = f(\infty, a_{n+1}, \ldots, a_p) = f(a_1, a_{n+1}, \ldots, a_p)\in A$$

shows $h\in M_1$. 2) The verification of $\alpha(M_3)\subseteq M_3$ for $\alpha\in\{\zeta, \tau, \Delta\}$ and $e_1^2\in M_3$ is direct and immediate. Thus M_3 is a subclone of $\mathbf{R}_B$. The fact $M_3^-=M$ is also obvious.

(ii) We show that M_3 is a maximal subclone of $\mathbf{R}_B$. Clearly $M_3 \subset \mathbf{R}_B$. Let $f \in \mathbf{R}_B \setminus M_3$. According to Lemma 2.2 we have

$$\mathbf{R}_B \supseteq \overline{M_3 \cup \{f\}} \supseteq \overline{M_1 \cup \{f\}} = \mathbf{R}_B,$$

hence M_3 is a maximal subclone of $\mathbf{R}_B$,

(iii) Finally let S be a clone on B such that $M_1 \subset S \subset \mathbf{R}_B$. By Lemma 2.2 we have $S \subseteq M_3$. Let $g \in S \cap M_2$ be n-ary. Then S contains the unary operation $h_1 := \Delta^{n-1} g$. Clearly $h_1 \in M_2$. Setting $h_n := h_1 * e_1^n$ we obtain $p_n = h_n^-$ for all $0 < n < \omega$. Thus $S^- \supseteq \{p_n : 0 < n < \omega\}$ and $S^- = M$.

3. The finite case: Slupecki type criterion

3.1. We look at the case of A finite. As usual, an n-ary partial operation f on A is *essential* if it is surjective (i.e. takes on all values from A) and depends on at least two variables. A subset F of $\mathbf{P}_A$ is *complete* (or primal) if $\bar{F} = \mathbf{P}_A$, i.e. $\mathbf{P}_A$ is the least partial clone containing F. We have:

3.2. Theorem. *Let A be finite. A subset F of P_A is complete if and only if*

(i) *F contains an essential operation,*

(ii) *F generates all unary operations and*

(iii) *For some $0 < n < \omega$ the set F contains an n-ary partial operation with proper domain (i.e. whose domain is a proper subset of A^n).*

Proof. ($\Rightarrow$) Since $\bar{F} \supseteq \mathbf{O}_A^*$ evidently (i) and (ii) hold. Moreover $F \not\subseteq M$ (see 2.3) proving (iii).

($\Leftarrow$) By Slupecki's criterion ([18] cf. [11]) from (i) and (ii) we obtain $\mathbf{O}_A^* \subseteq \bar{F}$.

3.3. Corollary. *Let A be finite and $f \in \mathbf{P}_A$. Then $\{f\}$ is complete in $\mathbf{P}_A$ if and only if f has proper domain and $\overline{\{f\}}$ contains all unary operations and at least one essential operation.*

We show that the lattice of subclones of $\mathbf{P}_A$ is dually atomic and has a finite number of dual atoms (coatoms):

3.4. Corollary. *If A is finite, then*

(i) *each proper subclone of $\mathbf{P}_A$ extends to a maximal subclone of $\mathbf{P}_A$, and*

(ii) *the set of maximal subclones of $\mathbf{P}_A$ consists of finitely many strong partial clones and the clone M (defined in 2.3).*

A similar statement holds for the clone $\mathbf{R}_B$ on $B = A \cup \{\infty\}$.

Proof. By definition, the subclones of $\mathbf{R}_B$ are the subuniverses of $\mathbf{R}_B :=$ $:= \langle \mathbf{R}_B;\ *, \zeta, \tau, \Delta, e_1^2 \rangle$. By Corollary 3.3 the algebra $\mathbf{R}_B$ is finitely generated. It is well known that this implies that each proper subalgebra of R_B extends to a maximal one proving (i). For (ii) cf. [9, Thm. 7.2] or [21, 26].

5. Remark

The number of maximal partial clones on a finite universe A exceeds largely the number of maximal clones on A. Some relations determine both a maximal clone and a maximal partial clone but there are relations determining only one of them. The combinatorial description of the relations determining maximal strong partial clones is cryptic in the sense that for some it is an NP-complete problem to decide wether a given relation is such.

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