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On strong approximation with consequences of almost everywhere type

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1. Let $f=f(x)$ be a continuous 2π-periodic function, i.e. $f\in C_{2\pi}$, and let

(1.1) $$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

be its Fourier series. Denote $s_n=s_n(x)=s_n(f;x)$ the n-th partial sum.

Let $E_n=E_n(f)$ denote the best approximation of f by trigonometric polynomials of order at most n in the space $C_{2\pi}$, and let $\|\cdot\|$ denote the usual supremum norm.

The first result, which deduces structural properties of the function being approximated at a given order by certain strong means, was proved by G. FREUD [2]. He verified that if $p>1$ and

(1.2) $$\left\|\left\{\frac{1}{n+1}\sum_{k=0}^{n}|s_k-f|^p\right\}^{1/p}\right\|=O(n^{-1/p}),$$

then $f\in\mathrm{Lip}\,1/p$ and f satisfies

(1.3) $$\lim_{h\to+0}\big(f(x+h)-f(x)\big)h^{-1/p}=0$$

almost everywhere.

Answering a question of Freud we ([3]) verified that (1.3) cannot be extended to every point, which shows that Freud's results are best possible.

These results have been generalized into several directions, and a theorem like this one has been called converse type.

It is obvious that assumption (1.2) is equivalent to

(1.4) $$\left\|\sum_{k=0}^{\infty}|s_k-f|^p\right\|<\infty,\quad p>1;$$

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this condition seems to be more lucidly arranged than (1.2), therefore the assumptions of converse type theorems have been given in this form instead of (1.2).

Here we recall only two general converse theorems:

Theorem A. ([5]). *Let* $0<\alpha<1$, $p>0$, *and* r *be a nonnegative integer. Then*

$$\left\|\sum_{n=1}^{\infty} n^{(r+\alpha)p-1}|s_n-f|^p\right\|<\infty \tag{1.5}$$

implies $f^{(r)}\in\mathrm{Lip}\,\alpha$; *furthermore*

$$|f^{(r)}(x+h)-f^{(r)}(x)|+|\tilde{f}^{(r)}(x+h)-\tilde{f}^{(r)}(x)| = o_x(h^\alpha) \tag{1.6}$$

holds almost everywhere, where $\tilde{f}$ *denotes the conjugate function of* f.

These statements are best possible, i.e. (1.6) *cannot be extended to every point.*

The following theorem of V. Totik [8] is one of the most general converse theorems, but it does not say anything about consequences of "almost everywhere type".

Theorem B. *Let* Ω *be a convex or concave function with the properties*

$$\Omega(x)>0 \quad (x>0), \quad \lim_{x\to+0}\Omega(x)=\Omega(0)=0; \tag{1.7}$$

furthermore let $\{\lambda_n\}$, $\{\mu_n\}$ *be positive nondecreasing sequences. If*

$$\left\|\sum_{n=0}^{\infty}\lambda_n\Omega(\mu_n|s_n-f|)\right\|<\infty \tag{1.8}$$

then

$$\omega\left(f,\frac{1}{n}\right)=O\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\lambda_k}\right)\right), \tag{1.9}$$

where $\bar{\Omega}$ *and* $\omega(f,\delta)$ *denote the inverse of* Ω *and the modulus of continuity of* f, *respectively.*

Estimation (1.9) *is, in general, the best possible.*

As far as we know nobody has investigated the "almost everywhere" consequences of (1.8).

Our first aim was to consider this problem. In the course of the investigations it turned out that, under the same restrictions which we require in the proof of the consequences of almost everywhere type, the same estimation can be given for the modulus of continuity $\omega(\tilde{f}, 1/n)$ as for $\omega(f, 1/n)$ in (1.9). Our theorem presents these results.

Let $\gamma(x)$ be a monotone function, linear between n and $n+1$, furthermore

$$\gamma(n):=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\lambda_k}\right) \quad \text{and} \quad \gamma(0):=\gamma(1).$$

Theorem. *Suppose that* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *are positive nondecreasing sequences and* Ω *is a convex function with the properties* (1.7) *and its inverse satisfies the condition*

$$\overline{\lim_{x\to 0+0}}\, \frac{\bar{\Omega}(\delta x)}{\bar{\Omega}(x)} < 1 \tag{1.10}$$

for a certain positive $\delta < 1$. *Moreover suppose that the sequences* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *and the function* $\bar{\Omega}(x)$ *jointly satisfy the following additional condition given by means of the function* $\gamma(x)$ *as follows; for any positive* ε *there exist numbers* $N_0 = N(\varepsilon)$ *and* $x_0 = x(\varepsilon)$ *such that for any* $N \geqq N_0$ *and* $x > x_0$

$$\gamma(x) \leqq N\varepsilon\gamma(Nx) \tag{1.11}$$

holds. Then condition (1.8) *implies*

$$\lim_{h\to 0} \gamma^{-1}\left(\frac{1}{|h|}\right)\{|f(x+h)-f(x)|+|\tilde{f}(x+h)-\tilde{f}(x)|\} = 0 \tag{1.12}$$

almost everywhere; furthermore that $\omega(\tilde{f}, 1/n) = O(\gamma(n))$.

Remarks. It is very easy to verify that if Ω is concave then (1.8) implies that

$$\left\|\sum_{n=0}^{\infty} |s_n - f|\right\| < \infty \tag{1.13}$$

also holds, whence, by a joint theorem of the author and E. M. Nikisin ([7]),

$$|f(x+h)-f(x)| = O_x(h) \tag{1.14}$$

follows almost everywhere. On the other hand, (1.14) is the best possible result, since the case $f(x) = \sin x$ shows that in (1.14) the estimation $O_x(h)$ cannot be replaced by $o_x(h)$ almost everywhere. Consequently the discussion of the case of concave Ω looses its interest. Indeed, (1.8) includes (1.5), in the special case $r=0$, only if $p>1$, i.e. if $\Omega(x) = x^p$ is convex.

2. To prove Theorem we require the following lemmas:

Lemma 1. *If* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *are nondecreasing positive sequences then* (1.8) *implies*

$$E_{4n}(f) = O\left(\mu_n^{-1}\bar{\Omega}\left(\frac{1}{n\lambda_n}\right)\right) \tag{2.1}$$

for any convex Ω.

This statement was proved by V. Totik [8].

Lemma 2. *If* Ω *is convex, and* $\{\lambda_n\}$, $\{\mu_n\}$ *are nondecreasing positive sequences then* (1.8) *implies that for any positive* α *and* β *there exists a perfect set* $H_\alpha \subset [0, 2\pi]$ *such* *that* $\operatorname{mes}(H_\alpha)>2\pi-\alpha$, *and on the set* H_α $(t\in H_\alpha)$

$$V_n(t):=\frac{1}{n}\sum_{k=n+1}^{2n}|s_k(t)-f(t)|\leqq\frac{1}{\mu_n}\bar{\Omega}\left(\frac{\beta}{n\lambda_n}\right) \tag{2.2}$$

holds if $n>n_0(\beta)$.

Proof. By Egorov's theorem and (1.8) there exists a perfect set $H_\alpha\subset[0,2\pi]$ such that $\operatorname{mes}(H_\alpha)>2\pi-\alpha$ and on the set H_α the series

$$\sum_{n=0}^{\infty}\lambda_n\Omega\big(\mu_n|s_n(x)-f(x)|\big) \tag{2.3}$$

converges uniformly. Consequently for any given $\beta>0$ there exists an integer $n_0=n_0(\beta)$ such that for all $t\in H_\alpha$

$$\sum_{n=n_0}^{\infty}\lambda_n\Omega\big(\mu_n|s_n(t)-f(t)|\big)\leqq\beta. \tag{2.4}$$

Hence and from the following obvious inequality

$$\mu_nV_n(t)\leqq\frac{1}{n}\sum_{k=n+1}^{2n}\mu_k|s_k(t)-f(t)|,$$

we obtain for any $n>n_0$

$$\Omega\big(\mu_nV_n(t)\big)\leqq\frac{1}{n}\sum_{k=n+1}^{2n}\Omega\big(\mu_k|s_k(t)-f(t)|\big)\leqq\frac{1}{n\lambda_n}\sum_{k=n+1}^{2n}\lambda_k\Omega\big(\mu_k|s_k(t)-f(t)|\big)\leqq\frac{\beta}{n\lambda_n},$$

whence (2.2) clearly follows.

Lemma 3. *Assumption* (1.10) *implies that for any positive* ε *there exists a number* $N(\varepsilon)$ *such that if* $N\geqq N(\varepsilon)$ *then*

$$\gamma(Nx)\leqq\varepsilon\gamma(x) \tag{2.5}$$

holds for any $x>x^*=x^*(\varepsilon)$.

Proof. By (1.10), using iteration, it is quite obvious that there exists a monotone increasing function $\varrho(\varepsilon)$ defined on $(0,1)$ such that $\varrho(\varepsilon)\to0$ as $\varepsilon\to0$ and

$$\bar{\Omega}(\varepsilon x)\leqq\varrho(\varepsilon)\bar{\Omega}(x) \tag{2.6}$$

holds for any $0<x<x_0=x_0(\varrho(\varepsilon))$. Choosing a natural number M $(\geqq2)$ such that $M^{-1}\leqq\bar{\varrho}(\varepsilon/2)$ holds, where $\bar{\varrho}$ denotes the inverse of ϱ, then (2.6) implies

$$\bar{\Omega}(M^{-1}x)\leqq\bar{\Omega}\big(\bar{\varrho}(\varepsilon/2)x\big)\leqq(\varepsilon/2)\bar{\Omega}(x)\quad\text{if}\quad 0<x<x_0. \tag{2.7}$$

Hence the following elementary calculation gives (2.5).

Denote by n the integral part of x, i.e. let $n \leqq x < n+1$. First we prove that $\gamma(Nn) \leqq (\varepsilon/2)\gamma(n)$ holds, which obviously verifies (2.5); here we assume that $n > 1/x_0(\varepsilon/2)$, where x_0 is given by (2.6).

Let $N(\varepsilon) := 4M^2\varepsilon^{-1}$ and $\varepsilon \leqq 1$. Then, if $N \geqq N(\varepsilon)$, we get

$$(2.8)\qquad \gamma(Nn) = \frac{1}{Nn}\sum_{k=1}^{Nn}\frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\lambda_k}\right) = \frac{1}{Nn}\left\{\sum_{k=1}^{Mn} + \sum_{k=Mn+1}^{M^2n} + \sum_{k=M^2n+1}^{Nn}\right\}.$$

Now we estimate these sums using the notation $c_k := \frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\lambda_k}\right)$. Since $\gamma(n)$ is decreasing we have

$$(2.9)\qquad \frac{1}{Nn}\sum_{k=1}^{Mn} c_k = \frac{M}{NnM}\sum_{k=1}^{Mn} c_k = \frac{M}{N}\gamma(Mn) \leqq \frac{\varepsilon}{4M}\gamma(Mn) \leqq \frac{\varepsilon}{8}\gamma(Mn) \leqq \frac{\varepsilon}{8}\gamma(n).$$

At the estimation of the second sum we use (2.7) as follows:

$$(2.10)\qquad \frac{1}{Nn}\sum_{k=Mn+1}^{M^2n} c_k = \frac{1}{Nn}\sum_{\nu=n}^{Mn-1}\sum_{k=M\nu+1}^{M(\nu+1)}\frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\lambda_k}\right) \leqq$$

$$\leqq \frac{1}{Nn}\sum_{\nu=n}^{Mn-1} M\frac{1}{\mu_\nu}\bar{\Omega}\left(\frac{1}{M\nu\lambda_\nu}\right) \leqq \frac{M\varepsilon}{2Nn}\sum_{\nu=n}^{Mn}\frac{1}{\mu_\nu}\bar{\Omega}\left(\frac{1}{\nu\lambda_\nu}\right) \leqq$$

$$\leqq \frac{M^2\varepsilon}{2N}\gamma(Mn) \leqq \frac{\varepsilon}{8}\gamma(n).$$

Finally

$$(2.11)\qquad \frac{1}{Nn}\sum_{k=M^2n+1}^{Nn}\frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\lambda_k}\right) \leqq \frac{Nn}{Nn}\frac{1}{\mu_n}\bar{\Omega}\left(\frac{1}{M^2n\lambda_n}\right) \leqq$$

$$\leqq \frac{\varepsilon^2}{4\mu_n}\bar{\Omega}\left(\frac{1}{n\lambda_n}\right) \leqq \frac{\varepsilon^2}{4n}\sum_{k=1}^{n}\frac{1}{\mu_k}\bar{\Omega}\left(\frac{1}{k\lambda_k}\right) \leqq \frac{\varepsilon}{4}\gamma(n).$$

Summing up, (2.8), (2.9), (2.10) and (2.11) give that

$$\gamma(Nn) \leqq (\varepsilon/2)\gamma(n)$$

holds, whence (2.5) obviously follows for any $x > x^*(\varepsilon) := (x_0(\varepsilon/2))^{-1}+1$.

As Lemma 4 we recall an interesting theorem of L. D. Gogoladze [3] which extends certain structural properties of the function f to its conjugate function $\tilde{f}$.

Lemma 4. *Let φ be a continuous nondecreasing function with the properties $\varphi(t) > 0$ $(t \in (0, \pi])$ and $\lim_{t\to 0}\varphi(t) = \varphi(0) = 0$, moreover*

$$(2.12)\qquad \int_0^\pi \frac{\varphi(t)}{t}\,dt < \infty.$$

Define

$$\psi(s):=\left\{\int_0^s \frac{\varphi(t)}{t}\,dt+s\int_s^\pi \frac{\varphi(t)}{t^2}\,dt\right\},\quad 0<s<\pi. \tag{2.13}$$

If $\omega(f,t)=O(\varphi(t))$ *and on a set* $E\subset[-\pi,\pi]$ *of positive measure*

$$\lim_{h\to 0}\frac{1}{\varphi(|h|)}|f(x+h)-f(x)|=0,$$

then

$$\lim_{h\to 0}\frac{1}{\psi(|h|)}|\tilde f(x+h)-\tilde f(x)|=0$$

also holds on E almost everywhere.

Lemma 5. ([6], Lemma 2.6.). *For any nonnegative sequence* $\{a_k\}$ *the inequality*

$$\sum_{k=1}^m a_k\leqq Ka_m\quad (m=1,2,\ldots,\ K>0)^{*)} \tag{2.14}$$

holds if and only if there exist a positive number c and a natural number μ *such that for any k*

$$a_{k+1}\geqq ca_k \tag{2.15}$$

and

$$a_{k+\mu}\geqq 2a_k \tag{2.16}$$

are valid.

Lemma 6. ([9], (13.30) Theorem). *For any* $f\in C_{2\pi}$

$$\omega(\tilde f,h)\leqq K\left\{\int_0^h t^{-1}\omega(f,t)\,dt+h\int_h^\pi t^{-2}\omega(f,t)\,dt\right\}\quad \left(h\leqq\frac{\pi}{2}\right) \tag{2.17}$$

holds.

3. Proof of Theorem. Let η and ε be arbitrary positive numbers. Let $N=N(\varepsilon)$ denote the least number satisfying (1.11) with this ε, and the inequality $N^{-1}\leqq\varepsilon$.

By Lemma 2 with $\alpha=\eta$ and $\beta=\bar\varrho(N^{-1})$ we get a perfect set $H_\eta\subset[0,2\pi]$ such that $\mathrm{mes}\,(H_\eta)>2\pi-\eta$ and for any $t\in H_\eta$ and $n\geqq n_0(\beta)=n_0(\bar\varrho(N^{-1}))$ (2.2) holds, i.e.

$$V_n(t)\leqq\frac{1}{\mu_n}\,\bar\Omega\left(\frac{\bar\varrho(N^{-1})}{n\lambda_n}\right) \tag{3.1}$$

if $t\in H_\eta$ and $n\geqq n_0$.

*) $K, K_1, \ldots$ will denote positive constants not necessarily the same at each occurrence.

By a known theorem of Lebesgue there exists a subset H_η^* of H_η such that $\text{mes}\,(H_\eta^*)=\text{mes}\,(H_\eta)$ and the points of H_η^* are of density 1.

Let x be an arbitrary fixed point of H_η^*. Setting $\varepsilon_1:=N^{-1}(\leqq\varepsilon)$, by $x\in H_\eta^*$ there exists a positive $\delta=\delta(\varepsilon_1)$ such that if $0<h\leqq\delta$ then

$$\text{mes}\,([x-h,x]\cap H_\eta^*)>(1-\varepsilon_1)h \quad\text{and}\quad \text{mes}\,([x,x+h]\cap H_\eta^*)>(1-\varepsilon_1)h.$$

Now let us choose y such that $|x-y|<\min\left(\delta,\frac{\varepsilon_1}{n_0(\beta)},\frac{1}{x_0},\frac{1}{x^*}\right)$, where x_0 and x^* are the numbers given at inequalities (1.11) and (2.5), respectively. Let $\nu=\nu(y)$ be the smallest natural number with $\varepsilon_1\leqq 2^\nu|x-y|<2\varepsilon_1$. It is clear that $2^\nu>n_0(\beta)$. Since $|x-y|<\delta$ there exists a point $y_1\in H_\eta$ lying between x and y such that $|y-y_1|\leqq\varepsilon_1|x-y|$ and $y_1\neq y$.

By the obvious inequality

(3.2)
$$|f(y)-f(x)|\leqq|f(y)-f(y_1)|+|f(y_1)-V_{2^\nu}^*(y_1)|+|V_{2^\nu}^*(y_1)-V_{2^\nu}^*(x)|+|V_{2^\nu}^*(x)-f(x)|,$$

where $V_n^*(x):=n^{-1}\sum_{k=n+1}^{2n}s_k(x)$, we can prove that

$$\lim_{h\to 0}\gamma^{-1}\left(\frac{1}{|h|}\right)(f(x+h)-f(x))=0 \tag{3.3}$$

holds almost everywhere.

In (3.2) the first term on the right-hand side can be estimated easily by (1.9) and (2.5).

$$|f(y)-f(y_1)|\leqq K\gamma(|y-y_1|^{-1})\leqq K\gamma((\varepsilon_1|x-y|)^{-1})=K\gamma(N/|x-y|)\leqq K\varepsilon\gamma(1/|x-y|).$$

Since x and y_1 belong to H_η, the second and fourth terms, using (3.1), can be estimated jointly:

$$\Sigma_1:=|f(y_1)-V_{2^\nu}^*(y_1)|+|V_{2^\nu}^*(x)-f(x)|\leqq K\left(\frac{1}{\mu_{2^\nu}}\bar\Omega\left(\frac{\bar\varrho(\varepsilon_1)}{2^\nu\lambda_{2^\nu}}\right)\right),$$

whence, using the monotonicity of $\mu_k^{-1}\bar\Omega(1/k\lambda_k)$, (2.6) and (1.11), we obtain

$$\Sigma_1\leqq K\frac{\varepsilon_1}{\mu_{2^\nu}}\bar\Omega(1/2^\nu\lambda_{2^\nu})\leqq K\varepsilon_1\gamma(2^\nu)\leqq K_1\varepsilon_1\gamma(\varepsilon_1/|x-y|)\leqq \tag{3.4}$$
$$\leqq K_2\varepsilon_1 N\varepsilon\gamma(1/|x-y|)=K_2\varepsilon\gamma(1/|x-y|).$$

In order to estimate the third term in (3.2) we set

$$V_{2^\nu}^{*\prime}(x)=\sum_{n=0}^{\nu}(V_{2^n}^{*\prime}(x)-V_{2^{n-1}}^{*\prime}(x))\quad (V_{2^{-1}}^*(x)\equiv 0). \tag{3.5}$$

Using Bernstein's inequality for the functions

$$U_n(x) := V_{2^n}^*(x) - V_{2^{n-1}}^*(x),$$

we get

$$\|U_n'\| \leqq 2^{n+1}\|U_n\| \leqq 2^{n+1}(\|V_{2^n}^* - f\| + \|f - V_{2^{n-1}}^*\|) \leqq K2^n E_{2^{n-1}}(f),$$

whence by (3.5)

$$\|V_{2^\nu}^{*\prime}\| \leqq K \sum_{n=0}^{\nu} 2^n E_{2^n} \leqq K_1 \sum_{n=0}^{2^\nu} E_n$$

follows. Hence, by Lemma 1, we obtain

$$\|V_{2^\nu}^{*\prime}\| \leqq K_2 \sum_{n=1}^{2^\nu} \frac{1}{\mu_n} \bar{\Omega}\left(\frac{1}{n\lambda_n}\right) = K_2 2^\nu \gamma(2^\nu),$$

and so

$$|y_1 - x|\|V_{2^\nu}^{*\prime}\| \leqq K_2|y-x|2^\nu\gamma(2^\nu) < 2K_2\varepsilon_1\gamma(2^\nu).$$

Using this, as in the proof of (3.4), we get

$$|V_{2^\nu}^*(y_1) - V_{2^\nu}^*(x)| \leqq |y_1 - x|\|V_{2^\nu}^{*\prime}\| \leqq K_3\varepsilon\gamma(1/|x-y|).$$

Summing up these estimations, by (3.2), we have

$$|f(x) - f(y)| \leqq K\varepsilon\gamma(1/|x-y|). \tag{3.6}$$

Since ε has been arbitrary, (3.6) implies (3.3) for all $x \in H_\eta^*$. Let $G(f)$ denote the subset of $[0, 2\pi]$ where (3.3) does not hold. It is clear that $G(f) \subset [0, 2\pi] \setminus H_\eta^*$, thus the exterior measure of $G(f)$ is less than η. Since η was also arbitrary, so the measure of $G(f)$ is zero, that is, (3.3) is proved for almost all x.

Now, if we can show that with $\varphi(t) := \gamma(1/t)$ if $t \in (0, \pi]$ and $\varphi(0) = 0$ the assumptions of Lemma 4 are satisfied and that $\psi(s) \leqq K\gamma(1/s)$, then, by Lemma 4, (3.3) implies (1.12).

Thus the rest of the proof of (1.12) is to verify that with $\varphi(t) = \gamma(1/t)$ each of the assumptions of Lemma 4 holds. It is clear that $\lim_{x\to 0} \gamma(x) = 0$ and so this φ is a continuous nondecreasing function on $[0, \pi]$ and positive on $(0, \pi]$.

It is also clear that

$$\psi(s) \leqq K\gamma(1/s) \quad s \in (0, \pi) \tag{3.7}$$

will imply (2.12). So we have to prove (3.7).

Putting $u = 1/t$ we get

$$\int_0^s \frac{1}{t}\gamma\left(\frac{1}{t}\right) dt = \int_{1/s}^{\infty} \frac{1}{u}\gamma(u)\, du \tag{3.8}$$

and since by a theorem of N. K. BARI and S. B. STECKIN [1]

$$\int_{1/s}^{\infty} \frac{1}{u}\gamma(u)\,du = O\left(\gamma\left(\frac{1}{s}\right)\right) \quad (s \to 0) \tag{3.9}$$

holds if and only if there exists a constant $C>1$ such that

$$\varlimsup_{x\to\infty} \frac{\gamma(Cx)}{\gamma(x)} < 1, \tag{3.10}$$

we have to verify this.

By Lemma 3, (1.10) obviously implies (3.10), consequently (3.9) also holds.

Next we estimate the second term in (2.13):

$$\int_{s}^{\pi} \frac{1}{t^2}\gamma\left(\frac{1}{t}\right) dt = \int_{1/\pi}^{1/s} \gamma(u)\,du \leqq K \sum_{n=1}^{1/s} \gamma(n) \leqq K_1 \sum_{k=1}^{n} 2^k\gamma(2^k), \tag{3.11}$$

where $n=\overset{2}{\log}\, 1/s$.

By Lemma 5, with $a_k=2^k\gamma(2^k)$, the estimation

$$\sum_{k=1}^{n} 2^k\gamma(2^k) \leqq K2^n\gamma(2^n) \tag{3.12}$$

holds if

$$2^{k+1}\gamma(2^{k+1}) \geqq c2^k\gamma(2^k) \tag{3.13}$$

and

$$2^{k+\mu}\gamma(2^{k+\mu}) \geqq 2\cdot 2^k\gamma(2^k) \tag{3.14}$$

hold for a positive c and a natural number μ.

By the definition of $\gamma(n)$ (3.13) holds with $c=1$; furthermore (3.14) follows from (1.11) putting $\varepsilon=1/2$, $N=2^\mu$ and $x=2^k$.

Collecting the estimations (3.8), (3.9), (3.11) and (3.12), by (3.13), we have proved (3.7); and this completes the proof of (1.12).

Finally, by Lemma 6 and (1.9), we have

$$\omega(\tilde{f}, h) \leqq K_1\left\{\int_0^h t^{-1}\gamma\left(\frac{1}{t}\right) dt + h\int_h^{\pi} t^{-2}\gamma\left(\frac{1}{t}\right) dt\right\},$$

whence, by (3.8), (3.9) and (3.11), (3.12), as above

$$\omega(\tilde{f}, h) \leqq K_2\gamma(1/h)$$

also follows, i.e.

$$\omega(\tilde{f}, 1/n) = O(\gamma(n))$$

is proved.

This completes the proof of Theorem.

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