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On the strong convergence of orthogonal series with almost everywhere uniformly bounded orthonormal systems

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Strong convergence with index λ was introduced by HYSLOP [2] as a limit case of strong Cesaro summability for positive orders. More exactly, the series $\sum a_k$ with partial sums s_n and Cesaro means $\sigma_n^{(\alpha)}$ is said to be strongly Cesaro summable with parameter $\alpha>0$ and index $\lambda>0$ — or summable $[C, \alpha]_\lambda$ — to the sum s if

$$\frac{1}{n+1}\sum_{k=0}^{n}|\sigma_k^{(\alpha-1)}-s|^\lambda = o(1)$$

and it is said to be strongly convergent with index λ — or $[C, 0]_\lambda$ convergent — to the sum s if $s_n-s=o(1)$ and

(1) $$\frac{1}{n+1}\sum_{k=0}^{n}k^\lambda|a_k|^\lambda = o(1).$$

It is known that if $\beta\geqq\alpha\geqq 0$, $\lambda\geqq 1$ and $\lambda\geqq\mu>0$ then summability $[C, \alpha]_\lambda$ implies summability $[C, \beta]_\mu$ (see [2], Th. 4).

Recently, TANOVIC-MILLER [4] introduced a new definition for convergence $[C, 0]_\lambda$, namely the series $\sum a_k$ is called $[C, 0]_\lambda$ convergent to the sum s if

(2) $$\frac{1}{n+1}\sum_{k=0}^{n}|(k+1)(s_k-s)-k(s_{k-1}-s)|^\lambda = o(1), \quad (s_{-1}=0).$$

Definitions of Hyslop and Tanovic-Miller are equivalent for indices $\lambda\geqq 1$. Furthermore, Tanovic-Miller showed that the series $\sum a_k$ is summable $[C, 0]_\lambda$ $(\lambda\geqq 1)$ if and only if $\sigma_n^{(1)}-s=o(1)$ and (1) holds. (See [5], (iii), 4., p. 128.)

Considering (2), the case of series having terms $a_k=1$ for $k=2^\nu$, $(\nu=0, 1, \ldots)$ and $a_k=0$ otherwise, shows that if $0<\lambda<1$ then the strong convergence does not imply the ordinary convergence, despite of the series $\sum a_k$ is $[C, 0]_\lambda$ convergent to the sum 0, so in next we assume that $\lambda\geqq 1$.

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We say that an orthonormal system $\{\varphi_k(x)\}_{k=0}^{\infty}$ on the interval (0, 1), is almost everywhere uniformly bounded if there exists a constant M, such that for any k, $|\varphi_k(x)|\leqq M$, almost everywhere on (0, 1). Shortly, we write that $\{\varphi_k(x)\}_{k=0}^{\infty}$ is ONSUBae.

In this note we discuss strong convergence of orthogonal series

$$(3) \qquad \sum_{k=0}^{\infty} c_k\varphi_k(x),$$

where the coefficients $\{c_k\}_{k=0}^{\infty}$ are real numbers and $\{\varphi_k(x)\}_{k=0}^{\infty}$ is ONSUBae. The Hyslop's definition says that the series (3) is $[C, 0]_\lambda$, $(\lambda\geqq 1)$ convergent almost everywhere on (0, 1) if it converges almost everywhere and

$$(4) \qquad \frac{1}{n+1}\sum_{k=0}^{n} k^\lambda |c_k\varphi_k(x)|^\lambda = o_x(1), \quad \text{almost everywhere.}$$

Using the result of Tanovic-Miller, mentioned above, we have that the series (3) is $[C, 0]_\lambda$, $(\lambda\geqq 1)$ convergent almost everywhere if and only if it is summable $(C, 1)$ almost everywhere and (4) holds.

Theorem 1. *Let* $\lambda\geqq 1$. *If the series* (3) *is* $[C, 0]_\lambda$ *convergent almost everywhere then*

$$(5) \qquad \frac{1}{n+1}\sum_{k=0}^{n} k^\lambda |c_k|^\lambda = o(1)$$

holds.

Proof. Being the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ ONSUBae, with the bound M, we may assume that $M>1/\sqrt{2}$.

Denoting by $|H|$ the Lebesgue measure of a set H, considering the sets

$$E_k^* = \{x\colon x\in[0, 1] \text{ and } |\varphi_k(x)| \geqq 1/2\}, \quad (k = 0, 1, \ldots)$$

by

$$\int_0^1 \varphi_k^2(x)\,dx = 1,$$

we have that $|E_k^*|\geqq 1/2M^2$, $(k=0, 1, \ldots)$.

Considering now an arbitrary measurable subset E of the interval [0, 1], with $|E|>1-1/4M^2$, we can see that $|E\cap E_k^*|\geqq 1/4M^2$ which yields

$$(6) \qquad \int_E |\varphi_k(x)|^\lambda\,dx \geqq (1/2)^\lambda(1/4M^2), \quad (k = 0, 1, \ldots).$$

On the other hand by (4) we have that sequence of measurable functions

$$\left\{\frac{1}{n+1}\sum_{k=0}^{n} k^\lambda |c_k\varphi_k(x)|^\lambda\right\}_{n=0}^{\infty}$$

converges to the function $f(x)=0$, almost everywhere on $[0, 1]$. According to the well-known Egoroff theorem (see e.g. [3], p. 97) there exists a subset E of the interval $[0, 1]$ with Lebesgue measure greater than $1-1/4M^2$, such that

$$\frac{1}{n+1}\sum_{k=0}^{n} k^{\lambda}|c_k\varphi_k(x)|^{\lambda}=o(1),$$

uniformly on E. This means that for any positive ε there exists a natural number N, such that if $n>N$ then for any $x\in E$

$$\frac{1}{n+1}\sum_{k=0}^{n} k^{\lambda}|c_k\varphi_k(x)|^{\lambda}<\varepsilon. \tag{7}$$

Integrating the left hand side of (7) on E and using (6) we obtain (5).

Remark 1. In Theorem 1 the assumption of the boundedness of the system $\{\varphi_k(x)\}_{k=0}^{\infty}$ is essential. For, let us consider the system of the functions

$$\psi_k(x)=\begin{cases}\sqrt{k(k+1)}, & \text{for } x\in(1/k+1, 1/k)=I_k\\ 0, & \text{otherwise}\end{cases}$$

and the function

$$\psi(x)=\begin{cases}\sqrt{k(k+1)}, & \text{if } x \text{ belongs to an } I_k, \quad (k=1,2,\dots),\\ 0, & \text{otherwise.}\end{cases}$$

It is obvious that the series $\sum_{k=1}^{\infty}\psi_k(x)$ is $[C,0]_{\lambda}$ convergent to $\psi(x)$ everywhere, but it does not fulfil (5) with $c_k\equiv 1$.

Theorem 1 yields the following corollaries immediately.

Corollary 1. *Let* $\lambda\geqq 1$. *The series* (3) *is* $[C,0]_{\lambda}$ *convergent to the function* $f(x)$ *almost everywhere, if and only if it converges to* $f(x)$ *almost everywhere and* (5) *is satisfied.*

Corollary 2. *Let* $\lambda\geqq 1$. *The series* (3) *is* $[C,0]_{\lambda}$ *convergent to the function* $f(x)$ *almost everywhere, if and only if it is* $(C,1)$ *summable to* $f(x)$ *almost everywhere and* (5) *is satisfied.*

The following theorems and remarks show an essential difference between the cases of indices $\lambda>1$ and $\lambda=1$.

Theorem 2. *Let* $\lambda>1$. *If the series* (3) *is* $[C,0]_{\lambda}$ *convergent to* $f(x)$ *almost everywhere then* $f\in\bigcap_{1\leqq p<\infty}L^p$ *and*

$$c_k=\int_0^1 f(x)\varphi_k(x)\,dx, \quad (k=0,1,2,\dots). \tag{8}$$

Proof. By Theorem 1, we have the condition (5) with $\lambda>1$. First we show that this implies

(9) $$\sum_{k=2^n+1}^{2^{n+1}} |c_k|^\mu = o(2^{n(1-\mu)}), \quad (n\to\infty),$$

for any $1<\mu<\lambda$.

Using the Hölder's inequality with indices λ/μ and $\lambda/(\lambda-\mu)$ we obtain

$$\frac{1}{n+1}\sum_{k=0}^{n}|kc_k|^\mu \leqq \left(\frac{1}{n+1}\sum_{k=0}^{n}|kc_k|^\lambda\right)^{\lambda/\mu} = o(1).$$

Hence the inequality

$$\sum_{k=2^n+1}^{2^{n+1}} |c_k|^\mu \leqq \frac{2}{2^{n(\mu-1)}}\frac{1}{2^{n+1}}\sum_{k=0}^{2^{n+1}}|kc_k|^\mu$$

proves (9).

For any $a>1$ we denote by a' the number for which $\frac{1}{a}+\frac{1}{a'}=1$. Considering an arbitrary $p\geqq 1$, we may choose the number $\mu>1$ such that

(10) $$\max(p, 2, \lambda') < \mu',$$

which guarantees $1<\mu<\lambda$ and by (9) we have

$$\sum_{k=2}^{\infty}|c_k|^\mu < \infty.$$

The condition (10) shows that $1<\mu<2$, so we may apply a theorem of F. Riesz (see e.g. [6] vol. II, p. 102, Th. 2.8, point (ii)), which says that there is a function $\Phi\in L^{\mu'}$, such that

(11) $$c_k = \int_0^1 \Phi(x)\varphi_k(x)\,dx, \quad (k=0,1,2,\ldots),$$

holds.

Using again (10), we get that $L^{\mu'}\subset L^p$, so $\Phi\in \bigcap_{1\leqq p<\infty} L^p$ is obtained. By (11), the series (3) is the orthogonal expansion of Φ, so the sequence of its partial sums $s_n(x)$, converges to Φ in L^2-sense. Hence we get a subsequence $\{s_{n_k}(x)\}_{k=0}^{\infty}$ which converges to $\Phi(x)$ almost everywhere. On the other hand Corollary 1 shows that this subsequence is also convergent to $f(x)$, almost everywhere. Thus $f\in \bigcap_{1\leqq p<\infty} L^p$ and by (11) we have (8).

Remark 2. Theorem 2 is false for $\lambda=1$. Really, TANOVIC-MILLER ([5], Th. 3) showed an almost everywhere $[C,0]_1$ convergent trigonometric series with sum $f(x)$ being not in L^p if $p>\frac{3}{2}$.

Theorem 3. *Let* $\lambda>1$. *The series* (3) *is* $[C,0]_\lambda$ *convergent almost everywhere, if and only if the condition* (5) *is fulfilled.*

Proof. Choosing the number μ such that $1<\mu<\min(\lambda,2)$, by (9), we get

$$\sum_{k=3}^{\infty}|c_k|^{\mu}(\log\log k)^2 \leqq \text{const} \sum_{n=1}^{\infty}(\log(n+1))^2/2^{n(\mu-1)}<\infty.$$

Observing that (5) implies $c_k=o(1)$, so assuming that k is large enough we have $|c_k|^{2-\mu}\leqq 1$. Thus we obtain that

$$\sum_{k=3}^{\infty} c_k^2(\log\log k)^2<\infty, \tag{12}$$

and applying the Menchoff—Kaczmarz theorem (see e.g. [1], p. 125, Th. 2.8.1), we have that the series (3) is $(C, 1)$-summable almost everywhere. Hence Corollary 2 completes our proof.

Remark 3. Theorem 3 is false for $\lambda=1$. Really, considering the Rademacher series

$$\sum_{k=0}^{\infty} c_k r_k(x), \quad (r_k(x)=\text{sign}\sin 2^k\pi x), \tag{13}$$

with coefficients $c_k=\nu^{-1/2}$ if $k=2^\nu$, $(\nu=1,2,\ldots)$ and $c_k=0$ otherwise, then for (5) we have

$$\frac{1}{n+1}\sum_{k=0}^{n} k|c_k| = O\left(\left\{\frac{1}{\log(n+2)}\right\}^{1/2}\right)=o(1),$$

but by $\sum_{k=0}^{\infty} c_k^2=\infty$, the series (13) diverges almost everywhere (see e.g. [1], p. 54. Th. 1.7.4). Now the Corollary 1, shows that the series (13) is not $[C,0]_1$-summable almost everywhere.

From the Menchoff—Kaczmarz theorem and Corollary 2 we may reduce a sufficient coefficient test for $[C,0]_1$-convergence of orthogonal series with system ONSUBae as follows.

Theorem 4. *The series* (3) *is* $[C,0]_1$*-convergent almost everywhere, if the condition* (12) *is fulfilled and the condition* (5) *is satisfied, with* $\lambda=1$.

Observing that if the sequence $\{c_k\}_{k=0}^{\infty}$ is a positive monotone decreasing sequence then (5) with $\lambda=1$ is equivalent to $c_k=o(1/k)$, hence Theorem 4 yields.

Corollary 3. *If* $\{c_k\}_{k=0}^{\infty}$ *is a positive monotone decreasing sequence then the* *series* (3) *is* $[C, 0]_1$*-convergent almost everywhere if and only if*

$$c_k = o(1/k)$$

holds.

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