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An interpretation of Cellina's example: negligibility via the failure of Peano's theorem

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1. Introduction

Let $(X, \|\cdot\|)$ be an arbitrary infinite-dimensional Banach space. The unit sphere and the closed unit ball of X are defined by $S=\{x\in X \,|\, \|x\|=1\}$ and $B=\{x\in X \,|\, \|x\|\leqq 1\}$, respectively. The closed unit interval $[0, 1]\subset \mathbf{R}$ is denoted by I.

The following theorems well-known from infinite-dimensional topology [2] are fundamental — none of these results remains valid if X is allowed to be finite dimensional.

Theorem A.

(a) *S is a retract of X. In other words, there exists a continuous mapping $r_1\colon X\to S$ such that $r_1(x)=x$ whenever $x\in S$.*

(b) *Moreover, S is a deformation retract of X. In other words, r_1 can be chosen so that there exists a continuous mapping $r\colon I\times X\to X$ with the properties that $r(0, x)=x$, $r(1, x)=r_1(x)$ for all $x\in X$ and $r(t, x)=x$ for all $t\in I$, $x\in S$.*

Theorem B.

(a) *0_x, the origin of X is negligible in B. In other words, there exists a homeomorphism $h_1\colon X\setminus\{0_x\}\to X$ which is limited by B, i.e. $h_1(x)=x$ whenever $x\in S\cup(X\setminus B)$.*

(b) *Moreover, 0_x can be pushed off B by an invertible isotopy. In other words, h_1 can be chosen so that there exists a homeomorphism $h\colon (I\times X)\setminus(\{1\}\times\{0_x\})\to I\times X$ preserving the first coordinate (i.e. if $(t, y)=h(s, x)$ then $t=s$) with the properties that $h(0, x)=x$ and $h(1, x)=h_1(x)$ for all $x\in X$ and, in addition, which is limited by B, i.e. $h(t, x)=(t, x)$ whenever $t\in I$, $x\in S\cup(X\setminus B)$.*

Theorem B (b) is a special case of [7, Corollary 1]. (Cutler's original result is valid in a Frèchet space setting.) Theorem A (b) is an immediate corollary of Theorem

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B (a). In fact, the desired deformation retraction $r: I \times X \to X$ can be defined by

$$r(t, x) = h((1-t)h_1^{-1}(x) + th_1^{-1}(x))/\|h_1^{-1}(x)\|.$$

It is quite natural that the geometric properties of infinite-dimensional Banach spaces incorporated in Theorems A and B have some consequences in the geometric theory (stability, attraction, isolated blocks, boundedness, parallelizability etc.) of infinite-dimensional dynamical systems. In fact, Theorems A (b) and B (b) have found several applications [11], [12] in topological dynamics. More precisely, the mapping r and h were represented as translation operators along the trajectories of dynamical systems constructed by using Theorems A and B.

The present paper is devoted to the converse problem.

In case of $X=l_1$, the Banach space of absolutely summable real sequences, we present examples of differential systems such that the translation operator (along the solutions of these systems) exhibits properties like those of r and h. In particular, we give a new (cf. the references in [2]) proof for the $X=l_1$ case of Theorems A and B. All of our considerations will be based on classical methods of the theory of ordinary differential equations [6]. The idea of applying ordinary differential equations in proving various homeomorphism results is widely used in nonlinear analysis; see e.g. Chapters 8 and 10 in [3].

The examples we give below are closely related to the failure of Peano's existence theorem. Especially, it will turn out that, at least in case of $X=l_1$, Theorems A and B provide a simple geometric explanation for the failure of Peano's existence theorem in infinite dimensions.

It seems plausible that Theorems A and B can be proved via differential equations. However, the technical details seem to be rather difficult. One is tempted to start from the Godunov example [14], a differential equation constructed for pointing out the failure of Peano's theorem in general infinite-dimensional Banach spaces. Unfortunately, as an easy consequence of Formula 3 in [14], the solutions of the Godunov example do, in general not depend continuously on initial data and this difficulty can not be easily overcome.

Concluding this paper, we present a new example pointing out that uniqueness does not imply continuous dependence on initial data.

2. Peano's theorem in infinite-dimensional Banach spaces

For completeness, we give now a brief review on existence theorems for ordinary differential equations in Banach spaces focusing to the failure of Peano's theorem in infinite dimensions.

The Peano theorem [6, Theorem I.1.2] holds only in finite-dimensional Banach spaces [14] (as well as in some locally convex topological vector spaces [1]). Under some additional compactness and/or monotonicity assumptions, the Peano theorem remains valid in infinite-dimensional Banach spaces as well [9], [15], [16].

It is well known that the Picard—Lindelöf theorem [6, Theorem I.3.1] on successive approximations is true in arbitrary Banach spaces. The conditions of this theorem ensure continuous dependence of the solutions upon initial data. No alterations are needed: the proof of the $X=\mathbf{R}^n$ case [6, Theorem I.7.1] can be repeated in the infinite-dimensional setting as well. On global versions of the Picard—Lindelöf theorem in Banach spaces see the nonlinear semigroup results in [9], [15], [16].

The first example pointing out the failure of Peano's theorem was given by DIEUDONNE [10]: the initial value problem $x_n(0)=0$ for the infinite system of ordinary differential equations $\dot{x}_n=|x_n|^{1/2}+n^{-1}$, $n=1, 2, \ldots$, defined in c_0, the Banach space of all real sequences $x=(x_1, x_2, \ldots)$ with limit zero and norm $\|x\|=$ $=\max\{|x_n| \mid n=1, 2, \ldots\}$, has no solution. Actually, as it was observed by YORKE [20], Dieudonne's equation has no solutions at all (i.e. no local solutions in c_0). For initial value problems in general infinite-dimensional Banach spaces, the first counterexample was given by GODUNOV [14]. The best result into this direction is due to SAINT-RAYMOND [18]: given an arbitrary infinite-dimensional Banach space X, there exists a continuous function $w\colon \mathbf{R}\times X\to X$ such that the differential equation $\dot{x}=w(t, x)$ has no solution in a neighbourhood of zero for any initial value $x_0\in X$.

Because of their simplicity and clear geometrical background, the counterexamples of CELLINA [5] are highly remarkable. For further references, we describe here briefly one of his examples for pointing out the failure of Peano's theorem in $X=l_1$. For $x=(x_1, x_2, \ldots)\in X=l_1$, $\|x\|=\sum\{|x_n| \mid n=1, 2, \ldots\}$, CELLINA [5] has considered the initial value problem $\dot{x}=C(t, x)$, $x(0)=0_x$ where $c\colon B\to B$, $(x_1, x_2, \ldots, x_n, \ldots)\to$ $\to(1-\|x\|, x_1, \ldots, x_{n-1}, \ldots)$ is a continuous fixed-point-free mapping of B into itself, $\mathscr{C}\colon X\to B$ is an arbitrary continuous extension of c to the whole X, with range B, and $C\colon R\times X\to X$ is the continuous function defined by $C(t, x)=2t\cdot\mathscr{C}(x/t^2)$ if $t\neq 0$ and zero if $t=0$. Since $\|C(t, x)\|\leqq 2|t|$, a possible solution x of the initial value problem $\dot{x}=C(t, x)$, $x(0)=0_x$ has to satisfy the inequality $\|x(t)\|\leqq t^2$ as well as, on some interval $(0, \delta)$, $\delta>0$, the differential system

$$\dot{x}_1 = 2t(1-\|x/t^2\|)$$

$$\dot{x}_{n-1} = 2x_n/t, \quad n = 1, 2, \ldots.$$

But this is impossible [5].

3. Cellina's example revisited

From now on, let $X=l_1$, the Banach space of absolutely summable real sequences $x=(x_1, x_2, \dots)$ with norm

$$\|x\| = \sum \{|x_n| \,|\, n = 1, 2, \dots\}.$$

For $x \in B$, define

$$f(x) = (-1+\|x\|+x_1,\ x_2-x_1,\ x_3-x_2, \dots, x_n-x_{n-1}, \dots).$$

It is easy to see that $f\colon B \to X$ is a continuous function satisfying the inequalities

(1a) $$\|f(x)-x\| = 1 \quad \text{for all} \quad x \in B,$$

and

(2a) $$\|f(x)-f(y)\| \leqq 3\|x-y\| \quad \text{for all} \quad x, y \in B.$$

For $x \in X$, define

$$F(x) = \begin{cases} f(x) & \text{if} \quad x \in B, \\ x & \text{if} \quad \|x\| \geqq 2, \\ f(x/\|x\|)(2-\|x\|)+x(\|x\|-1) & \text{if} \quad 1 \leqq \|x\| \leqq 2. \end{cases}$$

It is clear that $F\colon X \to X$ is well-defined and satisfies the equalities

(1b) $$\|F(x)-x\| = 0 \quad \text{whenever} \quad \|x\| \geqq 2,$$

and

(2b) $$\|F(x)-F(y)\| = \|x-y\| \quad \text{whenever} \quad \|x\|, \|y\| \geqq 2.$$

Applying (1a) and (2a), it is a rather lengthy but straightforward task to prove that

(1c) $$\|F(x)-x\| \leqq 1 \quad \text{whenever} \quad 1 \leqq \|x\| \leqq 2,$$

and

(2c) $$\|F(x)-F(y)\| \leqq 11\|x-y\| \quad \text{whenever} \quad 1 \leqq \|x\|,\ \|y\| \leqq 2.$$

In fact,

$$\|F(x)-x\| = (2-\|x\|)\big\|\big(f(x/\|x\|)-x/\|x\|\big)+x(1-\|x\|)/\|x\|\big\| \leqq$$

$$\leqq (2-\|x\|)\big(1+(\|x\|-1)\big) = 1-(\|x\|-1)^2 \leqq 1$$

whenever $1 \leqq \|x\| \leqq 2$, and

$$\begin{aligned}\|F(x)-F(y)\| &= \|(f(x/\|x\|)-f(y/\|y\|))(2-\|x\|)+\\ &+f(y/\|y\|)(\|y\|-\|x\|)+(x-y)(\|x\|-1)+y(\|x\|-\|y\|)\| \leqq\\ &\leqq 3\|x/\|x\|-y/\|y\|\|+2\|y-x\|+\|x-y\|+2\|x-y\| =\\ &= 3\|(x-y)+y(\|y\|-\|x\|)/\|y\|\|/\|x\|+5\|x-y\| \leqq\\ &\leqq 3(\|x-y\|+\|y-x\|)/\|x\|+5\|x-y\| \leqq 11\|x-y\|\end{aligned}$$

whenever $1 \leqq \|x\|$, $\|y\| \leqq 2$.

Summarizing (1a), (1b) and (1c), we arrive at the inequality

(1) $$\|F(x)-x\| \leqq 1 \quad \text{for all} \quad x \in X.$$

Similarly, in virtue of elementary properties of Lipschitzian functions in normed spaces, (2a), (2b) and (2c) imply that

(2) $$\|F(x)-F(y)\| \leqq 11\|x-y\| \quad \text{for all} \quad x, y \in X.$$

Applying a well-known global version (e.g. [15, Theorem 2.8.1]) of the Picard—Lindelöf theorem, it follows that the solution operator

$$\Phi\colon \mathbf{R}\times\mathbf{R}\times X \to X, \quad (\tau, \tau_0, z_0) \to \Phi(\tau, \tau_0, z_0)$$

of the infinite system of ordinary differential equations $dz/d\tau = F(z)$ is a continuous function uniquely defined for all $(\tau, \tau_0, z_0) \in \mathbf{R}\times\mathbf{R}\times X$. Here, of course, $\Phi(\tau, \tau_0, z_0)$ denotes the value of the solution of the Cauchy problem

(3C) $$dz/d\tau = F(z), \quad z(\tau_0) = z_0$$

at time τ.

Lemma A. *For each $z_0 \in X$, there exists a unique $\omega(z_0) \in \mathbf{R}$ such that $\|\Phi(\omega(z_0), 0, z_0)\| = 2$. $\omega(z_0) \geqq 0$ if and only if $\|z_0\| \leqq 2$. Further, the function ω: $X \to \mathbf{R}$ is continuous.*

The proof of Lemma A is postponed to Section 4.

Proposition A. *S is a deformation retract of X.*

Proof. This is an immediate corollary of Lemma A. In fact, the desired deformation retraction can be defined by

$$r(\tau, z_0) = 2^{-1}\Phi(\tau\omega(2z_0), 0, 2z_0), \quad (\tau, z_0) \in I\times X.$$

For $t \in (0, \infty)$, $x \in X$, define

$$g(t, x) = 2t(x/t^2 - F(x/t^2)).$$

Further, for $t \in \mathbf{R}$, $x \in X$, define

$$G(t, x) = \begin{cases} g(t, x) & \text{if } t > 0, \quad x \in X, \\ 0_x & \text{if } t = 0, \quad x \in X, \\ -g(-t, x) & \text{if } t < 0, \quad x \in X. \end{cases}$$

By definition, $G(t, x) = 0_x$ whenever $\|x\| \geqq 2t^2$. In virtue of (1), we have that $\|G(t, x)\| \leqq 2|t|$ for all $(t, x) \in \mathbf{R} \times X$. Consequently, (2) implies that the function $G \colon \mathbf{R} \times X \to X$ is (everywhere) continuous and — excepting at the point $(0, 0_x) \in \in \mathbf{R} \times X$ — locally Lipschitzian with respect to its second variable.

Consider now the co-ordinate transformation

$$\mathbf{R} \times X \ni (\tau, z) \leftrightarrow (t, x) \in (0, \infty) \times X$$

defined by

$$x = z \exp(-\tau), \quad t = \exp(-\tau/2) \leftrightarrow z = x/t^2, \quad \tau = -2 \ln t.$$

For brevity, we write $(t, x) = J(\tau, z)$. Observe that J is a homeomorphism of $\mathbf{R} \times X$ onto $(0, \infty) \times X$ mapping

$$\{(\tau, z) \in \mathbf{R} \times X \mid \|z\| = 2\} \quad \text{onto} \quad \{(t, x) \in (0, \infty) \times X \mid \|x\| = 2t^2\}.$$

It can be checked directly that our differential equation

(3) $$dz/d\tau = F(z); \quad (\tau, z) \in \mathbf{R} \times X$$

goes over into

(4) $$dx/dt = g(t, x); \quad (t, x) \in (0, \infty) \times X.$$

In fact,

$$dx/dt = [(dz/d\tau)(d\tau/dt) - z(d\tau/dt)] \exp(-\tau) = [F(z)(-2/t) - z(-2/t)] \exp(-\tau) =$$
$$= [F(x/t^2)(-2/t) - (x/t^2)(-2/t)]t^2 = g(x, t).$$

Choose $t_0 \in (0, \infty)$, $x_0 \in X$ arbitrarily and consider the Cauchy problem

(4C) $$dx/dt = g(t, x), \quad x(t_0) = x_0.$$

It follows immediately from the previous considerations that the unique non-continuable solution $\tilde{x}_{t_0, x_0}$ of (4C) can be given by

$$\tilde{x}_{t_0, x_0}(t) = t^2 \Phi(-2 \ln t, -2 \ln t_0, x_0/t_0^2), \quad t \in (0, \infty).$$

Since (3) is an autonomous equation, Lemma A implies that

$$\|\Phi(\omega(x_0/t_0^2) - 2 \ln t_0, -2 \ln t_0, x_0/t_0^2)\| = \|\Phi(\omega(x_0/t_0^2), 0, x_0/t_0^2)\| = 2.$$

Consequently, a direct computation shows that

$$\|\tilde{x}_{t_0, x_0}(\Omega(t_0, x_0))\| = 2\Omega^2(t_0, x_0),$$

where

$$\Omega(t_0, x_0) = t_0 \exp\left(-\omega(x_0/t_0^2)/2\right).$$

Recall that $g(t, x)=G(t, x)=0_x$ whenever $\|x\| \geqq 2t^2$, $t \in (0, \infty)$. Consequently,

$$\tilde{x}_{t_0, x_0}(t) = \tilde{x}_{t_0, x_0}(\Omega(t_0, x_0)) \quad \text{for all} \quad t \in (0, \Omega(t_0, x_0)).$$

Keeping $(t_0, x_0) \in (0, \infty) \times X$ fixed, consider now the Cauchy problem

$$\text{(5C)} \qquad dx/dt = G(t, x), \quad x(t_0) = x_0.$$

Recall that $G: R \times X \to X$ is (everywhere) continuous and — excepting at the point $(0, 0_x) \in \mathbf{R} \times X$ — locally Lipschitzian with respect to its second variable. Since

$$\left\|\tilde{x}_{t_0, x_0}(\Omega(t_0, x_0))\right\| = 2\Omega^2(t_0, x_0) > 0,$$

a simple symmetry argument shows that the unique noncontinuable solution $\tilde{\tilde{x}}_{t_0, x_0}$ of (5C) can be given by

$$\tilde{\tilde{x}}_{t_0, x_0}(t) = \begin{cases} \tilde{x}_{t_0, x_0}(t) & \text{if} \quad t \in (0, \infty), \\ \tilde{x}_{t_0, x_0}(\Omega(t_0, x_0)) & \text{if} \quad t = 0, \\ \tilde{x}_{t_0, x_0}(-t) & \text{if} \quad t \in (-\infty, 0). \end{cases}$$

The same reasoning shows that the differential equation

$$\text{(5)} \qquad dx/dt = G(t, x)$$

has a unique solution through each $(t_0, x_0) \neq (0, 0_x) \in \mathbf{R} \times X$. These solutions are defined for all real t and depend continuously on initial data. (Continuous dependence follows from the continuity of Φ and Ω resp. ω.)

On the other hand, the Cauchy problem

$$dx/dt = G(t, x), \quad x(0) = 0_x$$

has no (one-sided, local) solutions. To the contrary, suppose e.g. that $\eta(t)$, $t \in [0, 2\delta)$, $\delta > 0$ is a right-hand local solution. It is clear that $\eta(t) = \tilde{x}_{\delta, \eta(\delta)}(t)$ for all $t \in (0, \delta)$. If $t \to 0^+$, one arrives at $0_x = \tilde{x}_{\delta, \eta(\delta)}(\Omega(\delta, \eta(\delta)))$, a contradiction.

Thus, we have proved the following

Lemma B. *Let Ψ denote the solution operator of* (5). *The domain of Ψ is the set* $(\mathbf{R} \times \mathbf{R} \times X) \setminus (\mathbf{R} \times \{0\} \times \{0_x\})$. *The solution operator*

$$(\mathbf{R} \times \mathbf{R} \times X) \setminus (\mathbf{R} \times \{0\} \times \{0_x\}) \to X$$

is uniquely defined and continuous. Further, $\Psi(t, t_0, x_0) = x_0$ whenever $t_0 > 0$, $\|x_0\| \geqq 2t_0^2$, $0 \leqq t \leqq t_0$. (Observe that $t_0 > 0$, $\|x_0\| \geqq 2t_0^2$ imply that $t_0 \leqq \Omega(t_0, x_0)$.)

It is easy to check that the properties of Ψ established before yield

Proposition B. *The mapping*

$$h\colon (I\times X)\setminus(\{1\}\times\{0_x\}) \to I\times X$$

defined by

$$h(t, x) = \big(t, \Psi(2^{-1/2}, 2^{-1/2}(1-t), x)\big), \quad (t, x)\in(I\times X)\setminus(\{1\}\times\{0_x\})$$

is an invertible isotopy pushing 0_x *off* **B**.

Concluding this section, we remark that (by the definition of G) (5) goes over into the infinite system of ordinary differential equations

$$\dot{x}_1 = 2t(1-\|x/t^2\|)$$

$$\dot{x}_{n+1} = 2x_n/t, \quad n = 1, 2, \dots,$$

provided that $t>0$, $\|x\|\leqq t^2$. Hence, our differential equation (5) can be considered as a modified version of Cellina's example outlined in the last paragraph of Section 2. As a matter of fact, the construction of (5) was inspired by Cellina's example [5].

4. The proof of Lemma A

The proof of Lemma A is subdivided into the proof of several claims.

Recall that the solution operator $\Phi\colon \mathbf{R}\times\mathbf{R}\times X\to X$ of (3) is a continuous function uniquely defined for all $(\tau, \tau_0, z_0)\in\mathbf{R}\times\mathbf{R}\times X$.

Claim 1. *Let* $z\colon \mathbf{R}\to X$ *be a solution of* (3C). *Assume that* $z(\tau)\in B$ *for all* $\tau\geqq\tau_0$. *Then* $z_n(\tau)\geqq 0$ *for all* $\tau\geqq\tau_0$, $n=1, 2, \dots$.

Proof. If $z(\tau)\in B$ for all $\tau\geqq\tau_0$, then $z(\tau)$, $\tau\geqq\tau_0$ satisfies the following infinite system of differential equations:

$$\dot{z}_1(\tau) = -1+\|z(\tau)\|+z_1(\tau) \tag{6_1}$$

$$\dot{z}_2(\tau) = z_2(\tau)-z_1(\tau) \tag{6_2}$$

$$\vdots$$

$$\dot{z}_{n+1}(\tau) = z_{n+1}(\tau)-z_n(\tau). \tag{6_{n+1}}$$

$$\vdots$$

To the contrary, suppose first that $z_1(\tau^*)<0$ for some $\tau^*\geqq\tau_0$. Then (6_1) implies that $\dot{z}_1(\tau)\leqq z_1(\tau)\leqq z_1(\tau^*)$ for all $\tau\geqq\tau^*$ and $z_1(\tau)\to-\infty$ as $\tau\to\infty$. Consequently, $\|z(\tau)\|\geqq|z_1(\tau)|\to\infty$, a contradiction. We proceed by induction. Suppose now that $z_i(\tau)\geqq 0$, $i=1, 2, \dots, n$ for all $\tau\geqq\tau_0$ but $z_{n+1}(\tau^*)<0$ for some $\tau^*\geqq\tau_0$. Argueing as in the case $n=1$, the contradiction follows from (6_{n+1}).

Claim 2. *Let* $z\colon R\to X$ *be a solution of* (3C). *Assume that* $z(\tau)\in B$ *for all* $\tau\geqq\tau_0$. *Then* $z(\tau)\in S$ *for all* $\tau\geqq\tau_0$.

Proof. For $\tau>\tau_0$, define $V(\tau)=\|z(\tau)\|$. Since

$$\sum_{i=1}^{n}|z_{i+1}(s)-z_i(s)|\leqq 2\|z(s)\|\leqq 2,\quad n=1,2,\ldots,$$

for all $s>\tau_0$, Lebesque's dominated convergence theorem and Claim 1 imply that

$$V(\tau+h)-V(\tau)=\sum_{i=1}^{\infty}\big(z_i(\tau+h)-z_i(\tau)\big)=\int_{\tau}^{\tau+h}\dot z_1(s)\,ds+\sum_{i=2}^{\infty}\int_{\tau}^{\tau+h}\dot z_i(s)\,ds=$$

$$=\int_{\tau}^{\tau+h}\big(-1+\|z(s)\|+z_1(s)\big)\,ds+\int_{\tau}^{\tau+h}\sum_{i=1}^{\infty}\big(z_{i+1}(s)-z_i(s)\big)\,ds=$$

$$=\int_{\tau}^{\tau+h}\big(-1+V(s)+z_1(s)\big)\,ds+\int_{\tau}^{\tau+h}\big(-z_1(s)\big)\,ds=\int_{\tau}^{\tau+h}\big(-1+V(s)\big)\,ds$$

for all $\tau, \tau+h>\tau_0$. Since $V\colon(\tau_0,\infty)\to\mathbf{R}$ is a continuous function, it follows that V is differentiable and satisfies the differential equation $\dot V(\tau)=-1+V(\tau)$, $\tau>\tau_0$. Consequently, $V(\tau)=1+k\cdot\exp(\tau)$, $\tau>\tau_0$ for some constant k. Recall that $z(\tau)\in B$ for all $\tau\geqq\tau_0$. Thus, $k=0$ and $V(\tau)=\|z(\tau)\|=1$, $z(\tau)\in S$ for all $\tau\geqq\tau_0$.

Claim 3. *Let* $z\colon\mathbf{R}\to X$ *be a solution of* (3C). *Assume that* $z(\tau_0)\in B$. *Then there exists a* $\tau^*>\tau_0$ *such that* $z(\tau^*)\notin B$.

Proof. In virtue of Claim 2, the indirect hypothesis can be formulated as $z(\tau)\in S$ for all $\tau\geqq\tau_0$. Thus, z satisfies the infinite system of differential equations

$$\text{(7)}\qquad \begin{cases}\dot z_1(\tau)=z_1(\tau)\\ \dot z_{n+1}(\tau)=z_{n+1}(\tau)-z_n(\tau),\quad n=1,2,\ldots\end{cases}$$

for all $\tau\geqq\tau_0$. It follows immediately by induction that the boundedness of $z(\tau)$ with τ increasing implies $z_1(\tau)=0$, $z_2(\tau)=0$, ..., $z_{n+1}(\tau)=0$, ... for all $\tau\geqq\tau_0$, a contradiction. (As a matter of fact, (7) is linear and can easily be solved explicitely. It is worth to mention that $z(\tau)\in S$ for all $\tau\leqq\tau_0$ provided that $z(\tau_0)\in S$ and $z_1(\tau_0)\geqq0$, $z_2(\tau_0)\geqq0$, ..., $z_{n+1}(\tau_0)\geqq0$, ...)

Claim 4. *Let* $z\colon\mathbf{R}\to X$ *be a solution of* (3). *Assume that* $\|z(\tau_0)\|>1$ *for some* $\tau_0\in R$. *Then* z *satisfies the inequality*

$$\text{(8)}\qquad \|z(\tau)\|\geqq 1+\big(\|z(\tau_0)\|-1\big)\exp(\tau-\tau_0)\quad\textit{for all}\quad \tau\geqq\tau_0.$$

Proof. Observe (it can be checked by a direct computation) that

$$\text{(9a)}\qquad \|x+mf(x)\|\geqq 1\quad\text{whenever}\quad m\geqq0,\ \|x\|=1.$$

Next we show that

$$(9)\qquad \|z+hF(z)\| \geqq \|z\|+h(\|z\|-1) \quad \text{whenever} \quad h \geqq 0, \ \|z\| \geqq 1.$$

We have to distinguish two cases according as $\|z\|\geqq 2$ or $1\leqq\|z\|\leqq 2$. The first case is trivial. In the second case, applying (9a) for $x=z/\|z\|$ and

$$m = h(2-\|z\|)/(\|z\|+h\|z\|(\|z\|-1)),$$

we obtain that

$$\|z+hF(z)\| = \left\|z+h\big(f(z/\|z\|)(2-\|z\|)+z(\|z\|-1)\big)\right\| =$$
$$= \left\|z\big(1+h(\|z\|-1)\big)+h(2-\|z\|)f(z/\|z\|)\right\| \geqq \|z\|\big(1+h(\|z\|-1)\big) \geqq \|z\|+h(\|z\|-1).$$

Now we turn back to inequality (8). For $\tau\in R$, define $u(\tau)=1+\big(\|z(\tau_0)\|-1\big)\cdot$ $\cdot\exp(\tau-\tau_0)$. Observe that u is the unique solution to the initial value problem

$$du/d\tau = u(\tau)-1, \quad u(\tau_0) = \|z(\tau_0)\| > 1.$$

For each $\tau\geqq\tau_0$, $h>0$, inequality (9) implies that

$$\|z(\tau+h)\| = \left\|z(\tau)+hF(z(\tau))+\varepsilon_\tau(h)\right\| \geqq \|z(\tau)\|+h(\|z(\tau)\|-1)-\|\varepsilon_\tau(h)\|,$$

where $\varepsilon_\tau(h)\in X$ and $\varepsilon_\tau(h)/h\to 0_x$ as $h\to 0$.

Consequently,

$$\liminf\left\{(\|z(\tau+h)\|-\|z(\tau)\|)/h \,\middle|\, h\to 0^+\right\} \geqq \|z(\tau)\|-1$$

and (8) follows by an elementary comparison argument.

Now we are in a position to prove Lemma A. In fact, Claims 3 and 4 imply that, given a sphere $S(0_x, r)=\{z\in X \mid \|z\|=r\}$ centered at the origin and of radius $r>1$, then each solution of (3) intersects $S(0_x, r)$ at exactly one point. In particular, for each $z_0\in X$, the solution of (3) through $(0, z_0)\in\mathbf{R}\times X$ intersects $S(0_x, 2)$ at exactly one point. In other words, there exists a unique $\omega=\omega(z_0)\in\mathbf{R}$ such that $\left\|\Phi(\omega(z_0), 0, z_0)\right\|=2$. It is clear that $\omega(z_0)\geqq 0$ if and only if $\|z_0\|\leqq 2$. The continuity of the function $\omega\colon X\to\mathbf{R}$ follows easily from the uniqueness of $\omega=\omega(z_0)$ and of the continuity of Φ.

Remark. For $\mu\in\mathbf{R}$, consider the one-parameter family of infinite systems of differential equations

$$(7_\mu)\qquad \begin{cases} \dot z_1(\tau) = \mu\min\{0, -1+\|z(\tau)\|\}+z_1(\tau) \\ \dot z_{n+1}(\tau) = z_{n+1}(\tau)-z_n(\tau), \quad n=1,2,\ldots. \end{cases}$$

For $\mu=0$, (7_μ) is linear and, as it was observed in proving Claim 3, each nonzero solution of $(7_0)=(7)$ is unbounded. On the other hand, for $\mu\neq 0$, (7_μ) has no bounded solutions at all. Even in case of $\mu<0$, this follows easily from the proof of Claims 1 and 2. We arrived at the conclusion that, in general, the (isolated) equilibrium point of a linear equation (in infinite-dimensional Banach spaces) can be perturbed away (by a perturbation arbitrarily small) without giving rise to bounded trajectories.

5. Uniqueness without continuous dependence

Concluding this paper, we remark that a slight modification of the construction described in Section 3 provides an example in $X=l_1$ for which the assertion "Uniqueness implies continuous dependence on initial data" — valid in finite dimension — fails to hold.

In recent years, several authors have dealt with the problem of constructing ordinary differential equations in infinite-dimensional normed spaces, with continuous right-hand side, such that the initial value problem for each initial time and state has exactly one noncontinuable solution but continuous dependence on initial data fails to hold. For some classical Banach spaces, the first examples of this kind were given by Pasika [17] and independently, by Schäffer [19]. The case of general infinite-dimensional normed spaces was solved in a joint paper by Prof. Schäffer and the present author [13] and independently, by DeBlasi and Pianigiani [8]. For linear scalar equations with infinite delay, the same phenomenon was observed by Burton and Dwiggins [4].

Now we present a further example of a very simple geometric background: the emphasis is lead on the co-ordinate transformation J and on the properties of the linear equation (7), in particular, on the concluding remark in the proof of Claim 3.

For $x\in X$, let

$$P(x)=\begin{cases} p(x) & \text{if } x\in B,\\ x & \text{if } \|x\|\geqq 2,\\ p(x/\|x\|)(2-\|x\|)+x(\|x\|-1) & \text{if } 1\leqq\|x\|\leqq 2,\end{cases}$$

where $p\colon B\to X$ is defined by

$$p(x)=(x_1, x_2-x_1, x_3-x_2, \dots, x_n-x_{n-1}, \dots).$$

For $t\in(0,\infty)$, $x\in X$, let $q(t,x)=2t\big(x/t^2-P(x/t^2)\big)$.

Finally, for $t\in\mathbf{R}$, $x\in X$, define

$$Q(t,x)=\begin{cases} q(t,x) & \text{if } t>0,\ x\in X,\\ 0_x & \text{if } t=0,\ x\in X,\\ -q(-t,x) & \text{if } t<0,\ x\in X,\end{cases}$$

and consider the Cauchy problem

$$dx/dt = Q(t, x), \quad x(t_0) = x_0. \tag{10C}$$

It is not hard to show that the Cauchy problem (10C) for each initial time and state has exactly one noncontinuable solution (the domain of this solution being **R**) but continuous dependence on initial data fails badly at $(t_0, x_0)=(0, 0_x)\in \mathbf{R}\times X$. The details are left to the reader.

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