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# Some operational formulas of O. V. Viskov involving Laguerre polynomials

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**1. Introduction.** For the classical Laguerre polynomials

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = \binom{n+\alpha}{n} {}_1F_1(-n;\ \alpha+1;\ x), \tag{1}$$

a standard operational representation is provided by the Rodrigues formula:

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!} D^n\{x^{n+\alpha}e^{-x}\}, \quad D \equiv \frac{d}{dx}. \tag{2}$$

Among various other known operational representations for the Laguerre polynomials, we first recall the following formula of O. V. VISKOV [5]:

$$L_n^{(\alpha)}(x) = \frac{(-1)^n e^x}{n!} [xD^2+(\alpha+1)D]^n\{e^{-x}\}. \tag{3}$$

For $\alpha=0$, Viskov's formula (3) reduces to an operational representation for the simple Laguerre polynomials, which was given earlier by L. B. RÉDEI [2]. In fact, as already observed by us elsewhere [3], Rédei's formula is a rather natural consequence of (2) with $\alpha=0$, since

$$(xD^2+D)^n = (DxD)^n = D^n x^n D^n \quad (n = 0, 1, 2, \ldots).$$

It may be of interest to remark in passing that, since

$$D^n\{e^{-x}\} = (-1)^n e^{-x} \quad (n = 0, 1, 2, \ldots), \tag{4}$$

the general result (3) can be proven directly (and simply) by comparing the first part

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of the definition (1) with the operator identity (cf., e.g., [1]):

$$(5)\qquad (xD^2+\lambda D)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda+k)} x^k D^{n+k}$$

for $\lambda=\alpha+1$.

More recently, Viskov [6] gave another interesting operational representation for the Laguerre polynomials:

$$(6)\qquad L_n^{(\alpha)}(x) = \frac{x^{-n}e^x}{n!}[x^2D+(\alpha+1)x]^n\{e^{-x}\}.$$

The object of this note is to present two independent proofs of Viskov's formula (6). Each of our proofs of (6) is markedly different from the proof given by Viskov [6].

**2. First proof.** Many recent developments in the theory of special functions are based upon some remarkable applications of the differential operator (see, for example, [4, p. 368]):

$$(7)\qquad T_\lambda = x(\lambda+xD),\quad \lambda \text{ a constant},$$

which evidently has the property that [loc. cit.]

$$(8)\qquad T_\lambda^n\{x^\mu\} = \frac{\Gamma(\lambda+\mu+n)}{\Gamma(\lambda+\mu)} x^{\mu+n} \quad (n = 0, 1, 2, \dots)$$

for arbitrary parameters $\lambda$ and $\mu$.

Observe first that the operator involved in (6) is precisely the differential operator $T_{\alpha+1}$. Denoting, for convenience, the right-hand side of (6) by $\Omega$, and expanding $e^{-x}$ in powers of $x$, we find from (8) that

$$(9)\qquad \Omega = \frac{x^{-n}e^x}{n!}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}T_{\alpha+1}^n\{x^k\} = e^x\binom{n+\alpha}{n}{}_1F_1(n+\alpha+1;\ \alpha+1;\ -x).$$

Now the confluent hypergeometric ${}_1F_1$ function, occurring in (9), can be transformed by appealing to Kummer's theorem [4, p. 37]

$$(10)\qquad {}_1F_1(a;\ c;\ z) = e^z{}_1F_1(c-a;\ c;\ -z),$$

and we thus have

$$(11)\qquad \Omega = \binom{n+\alpha}{n}{}_1F_1(-n;\ \alpha+1;\ x) = L_n^{(\alpha)}(x),$$

by virtue of the second part of the definition (1).

**3. Second proof.** Yet another interesting proof of Viskov's formula (6) would make use of the operator identity (cf., e.g., [1])

(12) $$(x^2D+\lambda x)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\lambda+n)}{\Gamma(\lambda+k)} x^{n+k} D^k$$

for $\lambda=\alpha+1$. Indeed, in view of (4), the right-hand side of (6) becomes

$$\Omega = \frac{e^x}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+k+1)} x^k D^k \{e^{-x}\} = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!} = L_n^{(\alpha)}(x),$$

where we have employed the first part of the definition (1).

We should like to conclude by observing that, since

(13) $$T_\lambda^n \equiv x^n \prod_{j=1}^{n} (xD+\lambda+j-1),$$

which is easily verified by induction on $n$, Viskov's formula (6) may be rewritten in its equivalent form:

(14) $$L_n^{(\alpha)}(x) = \frac{e^x}{n!} \prod_{j=1}^{n} (xD+\alpha+j)\{e^{-x}\}.$$

On the other hand, Viskov's formula (3) can immediately be put in the alternative form:

(15) $$L_n^{(\alpha)}(x) = \frac{(-1)^n e^x}{n!} (x^{-\alpha} D x^{\alpha+1} D)^n \{e^{-x}\},$$

which incidentally can be proven fairly easily by induction on $n$, using certain well-known derivative and recursion formulas for Laguerre polynomials.

It may be of interest to remark that, in view of the easily verifiable identity [cf. equation (7)]:

$$T_1^n\{x^\alpha f(x)\} = x^\alpha T_{\alpha+1}^n\{f(x)\},$$

the operational formula (6) can also be deduced *directly* from the familiar result:

$$L_n^{(\alpha)}(x) = \frac{x^{-n-\alpha} e^x}{n!} (x^2D+x)^n \{x^\alpha e^{-x}\}.$$

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