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Approximation by unitary operators

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Introduction. Several authors have determined the distance $d(T,\mathcal{M}):=$ $:=\inf\{\|T-A\|: A\in\mathcal{M}\}$ between an operator T and a non-empty set $\mathcal{M}$ of operators in a Hilbert space H. In an arbitrary Hilbert space the distances between an operator and the sets of Hermitian, positive and compact operators have been established in, resp., [3], [5] and [6]. In a separable Hilbert space the distance to the set $\mathcal{U}n$ of all unitary operators has been found by D. D. ROGERS [7]. In the present paper we generalize his result to arbitrary Hilbert spaces. It will be proved that (with $N_T=$ $=\operatorname{Ker} T$): $d(T,\mathcal{U}n)=\max(1-m(T), \|T\|-1)$ if $\dim N_T=\dim N_{T^*}$ with $m(T)=$ $=\inf\sigma(|T|)$, and $d(T,\mathcal{U}n)=\max(1+n(T), \|T\|-1)$ if $\dim N_T<\dim N_{T^*}$ with $n(T)=\inf\{x>0: \dim 1_{[0,x]}(|T|)(H)\geqq\max(\aleph_0, \dim N_{T^*})\}$.

Definitions and notations. Let H be a Hilbert space with norm $\|\cdot\|$. The dimension of a closed linear subspace D of H is $\dim D=\operatorname{card} I$ if $(e_i)_{i\in I}$ is an orthonormal basis of D. (See [8], page 45.) The set of unitary operators in $\mathcal{L}(H)$ is denoted by $\mathcal{U}n(H)$ or by $\mathcal{U}n$ if no confusion can arise.

The identity function on $\mathbf{C}$ is denoted by χ and the indicator function of a set $A\subset\mathbf{C}$ by 1_A.

Let $T\in\mathcal{L}(H)$ be an operator. N_T denotes the kernel, $\sigma(T)$ the spectrum of T. The absolute value of T is $|T|=(T^*T)^{1/2}$. Define: $m(T):=\inf\sigma(|T|)$ and

$$n(T):=\inf\{x>0: \dim 1_{[0,x]}(|T|)(H)\geqq\max(\aleph_0, \dim N_{T^*})\}.$$

If L is a linear subspace of H and $T(L)\subset L$, then by $T|_L$ we mean the restriction of T to L, considered as an operator $L\to L$.

The set $\mathcal{C}$ of compact operators is a closed two-sided ideal in $\mathcal{L}(H)$. The *Calkin algebra* is the quotient $\mathcal{E}=\mathcal{L}(H)/\mathcal{C}$. Let π be the natural projection of $\mathcal{L}(H)$ onto $\mathcal{E}$ and $\|\cdot\|$ the quotient norm on $\mathcal{E}$. The Calkin algebra is a Banach algebra with unity. An operator $T\in\mathcal{L}(H)$ is *Fredholm* iff $\pi(T)$ is invertible in $\mathcal{E}$, its *essential*

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norm is $\|T\|_e := \|\pi(T)\|$ and its *essential spectrum* is $\sigma_e(T) := \sigma(\pi(T))$. It is trivial that $\sigma_e(T) \subset \sigma(T)$. Define: $m_e(T) := \inf \sigma_e(|T|)$. Note that $m_e(T) = n(T)$ if $\dim N_{T^*} \leqq \aleph_0 \leqq \dim H$. (See [4], page 185.)

For a separable infinite dimensional Hilbert space, ROGERS [7] has proved that $d(T, \mathcal{U}n) = \max(1 - m(T), \|T\| - 1)$ if $\dim N_T = \dim N_{T^*}$ and that $d(T, \mathcal{U}n) = \max(1 + m_e(T), \|T\| - 1)$ if $\dim N_T < \dim N_{T^*}$.

The index, ind T, of an operator T in $\mathcal{L}(H)$ is very useful for the determination of $d(T, \mathcal{U}n)$ in a separable Hilbert space; here $\operatorname{ind} T = 0$ if $\dim N_T = \dim N_{T^*}$ and $\operatorname{ind} T = \dim N_T - \dim N_{T^*}$ otherwise. The index of an operator T in an arbitrary Hilbert space is defined as follows:

$$\operatorname{ind} T := \begin{cases} 0 & \text{if } \dim N_T = \dim N_{T^*}, \\ \dim N_T - \dim N_{T^*} & \text{if } \dim N_T < \aleph_0 \text{ and } \dim N_{T^*} < \aleph_0, \\ \dim N_T & \text{if } \dim N_T \geqq \aleph_0 \text{ and } \dim N_T > \dim N_{T^*}, \\ -\dim N_{T^*} & \text{if } \dim N_{T^*} \geqq \aleph_0 \text{ and } \dim N_{T^*} > \dim N_T. \end{cases}$$

Negative cardinal numbers are defined in the obvious way.

Theorem 1. *Let* $T \in \mathcal{L}(H)$ *and* $\operatorname{ind} T = 0$. *Then* $d(T, \mathcal{U}n) = \max(1 - m(T), \|T\| - 1)$.

Proof. With the aid of the polar decomposition we find a partial isometry V from $N_T^\perp$ to $N_{T^*}^\perp$ such that $T = V|T|$. Because $\dim N_T = \dim N_{T^*}$, there also exists a unitary $U \in \mathcal{L}(H)$ with $T = U|T|$. From the Spectral Theorem it follows that $d(T, \mathcal{U}n) = d(|T|, \mathcal{U}n) \leqq \||T| - I\| = \max(1 - m(T), \|T\| - 1)$. The triangle inequality gives $d(T, \mathcal{U}n) \geqq \|T\| - 1$. Suppose there exists a V in $\mathcal{U}n$ with $\||T| - V\| < 1 - m(|T|)$. Let $Q := |T| - m(|T|)I$. Then Q is not invertible. But Q has to be invertible since $\|QV^{-1} - I\| < 1$. This proves the theorem.

It takes more trouble to determine $d(T, \mathcal{U}n)$ if $\operatorname{ind} T \neq 0$.

Theorem 2. *Let* $T \in \mathcal{L}(H)$, $\operatorname{ind} T < 0$ *and* $\varepsilon > 0$. *Then there exists a unitary* $U \in \mathcal{L}(H)$ *with* $\|T - U\| \leqq \max(1 + n(T) + \varepsilon, \|T\| - 1)$.

Proof. This proof is analogous to the proof of Theorem 1.3 in [7] with H a separable Hilbert space and $m_e(T) = n(T)$.

Let $E_\varepsilon := 1_{[0, n(T)+\varepsilon]}(|T|)$. Then $\dim E_\varepsilon(H) \geqq \max(\aleph_0, \dim N_{T^*})$. Because $\operatorname{ind} T < 0$ there exists an isometry $S \in \mathcal{L}(H)$ with $T = S|T|$ and $\dim N_{S^*} \leqq \dim N_{T^*}$. Then $\dim N_{S^*} \leqq \dim S^*SE_\varepsilon(H) \leqq \dim SE_\varepsilon(H) = \dim E_\varepsilon(H)$, with the result $\dim (N_{S^*} \oplus SE_\varepsilon(H)) = \dim E_\varepsilon(H)$. Therefore, there exists a unitary $V \colon E_\varepsilon(H) \to N_{S^*} \oplus SE_\varepsilon(H)$. Note that $H = N_{S^*} \oplus [SE_\varepsilon(H) \oplus S(E_\varepsilon(H)^\perp)]$, so that $[N_{S^*} \oplus SE_\varepsilon(H)]^\perp = S(E_\varepsilon(H)^\perp)$.

Define $U := VE_\varepsilon + S(I - E_\varepsilon) \in \mathcal{L}(H)$. Trivially, U is an isometry. Because $U(E_\varepsilon(H)) = N_{S^*} \oplus SE_\varepsilon(H)$ and $U(E_\varepsilon(H)^\perp) = S(E_\varepsilon(H)^\perp) = [N_{S^*} \oplus SE_\varepsilon(H)]^\perp$, U is also onto and U is unitary.

Since $\|T-U\|=\|U^*T-I\|$, consider U^*T. Now the space $E_\varepsilon(H)$ reduces the operator U^*T (and also U^*T-I) because

$$U^*TE_\varepsilon(H)=U^*S\big((\chi 1_{[0,n(T)+\varepsilon]})(|T|)\big)(H)\subset U^{-1}SE_\varepsilon(H)\subset E_\varepsilon(H)$$

and, similarly, $U^*T(E_\varepsilon(H)^\perp)\subset E_\varepsilon(H)^\perp$. Let X and Y be the restrictions of U^*T to $E_\varepsilon(H)$ and $E_\varepsilon(H)^\perp$, respectively. Then

$$\|X\|\leqq \big\|U^*S\big((\chi 1_{[0,n(T)+\varepsilon]})(|T|)\big)\big\|\leqq n(T)+\varepsilon.$$

The operator Y is simple: it is the restriction of $|T|$ to $E_\varepsilon(H)^\perp$. Using these facts we get $\|T-U\|=\|U^*T-I\|\leqq\max(\|X-I\|,\|Y-I\|)\leqq\max\big(1+n(T)+\varepsilon,\||T|-I\|\big)=$ $=\max\big(1+n(T)+\varepsilon,\|T\|-1\big)$.

In order to prove that also $d(T,\mathcal{U}n)\geqq\max\big(1+n(T),\|T\|-1\big)$ in case $\operatorname{ind}T<0$ we need some lemmas.

Lemma 3. *Let* $T_1, T_2\in\mathcal{L}(H)$ *and* $\|T_1-T_2\|_e<m_e(T_1)$. *Then*

$$\operatorname{ind}T_1=\operatorname{ind}T_2 \quad \textit{and}\quad \dim N_{T_2^*}<\aleph_0 \quad \textit{if}\quad \dim N_{T_1^*}<\aleph_0,$$
$$\dim N_{T_2}<\aleph_0 \quad \textit{and}\quad \dim N_{T_2^*}\geqq\aleph_0 \quad \textit{if}\quad \dim N_{T_1^*}\geqq\aleph_0.$$

Proof. This lemma extends Theorem 1.1 of [7] to a non-separable Hilbert space. For its proof, which is similar to Rogers', see the appendix.

Lemma 4. *Let* $T\in\mathcal{L}(H)$ *and let* $L\subset H$ *be a closed linear subspace invariant under* T *and* T^*. *Let* $a>0$ *and* $L\subset 1_{[a,\infty)}(|T|)(H)$. *Then* $\sigma(|T|_L|)\subset[a,\infty)$.

Proof. $\sigma(|T|_L|)=\sigma(|T||_L)=\sigma([|T|+a1_{[0,a)}(|T|)]|_L)\subset\sigma(|T|+a1_{[0,a)}(|T|))\subset[a,\infty)$.

Theorem 5. *Let* $T\in\mathcal{L}(H)$, $\operatorname{ind}T<0$. *Then* $d(T,\mathcal{U}n)\geqq\max\big(1+n(T),$ $\|T\|-1\big)$.

Proof. With the triangle inequality it follows that $d(T,\mathcal{U}n)\geqq\|T\|-1$. The part $d(T,\mathcal{U}n)\geqq 1+n(T)$ will be proved in three cases.

Case 1. Suppose $n(T)=0$. Let $U\in\mathcal{L}(H)$ be unitary. Then $\dim N_{T^*U}=$ $=\dim N_{T^*}>0$, so $\|T-U\|=\|U^*T-I\|=\|T^*U-I\|\geqq 1=1+n(T)$.

Case 2. Suppose $n(T)>0$ and $\dim N_{T^*}\leqq\aleph_0$. The proof now runs similarly to the one of Rogers ([7] Theorem 1.2) for the separable case. (Note: generally it is impossible to restrict the operator T to a separable Hilbert space H_1 such that $T=0$ on $H_1^\perp$.) Let $U\in\mathcal{L}(H)$ be unitary. Then $\{\zeta\in\mathbf{C}\colon|\zeta|<n(T)\}\subset\sigma(U^*T)$. This can be proved as follows: let $\zeta\in\mathbf{C}$, $|\zeta|<n(T)$; let $T_1:=U^*T$ and $T_2:=U^*T-\zeta$. Then $\|T_1-T_2\|_e<n(T)=m_e(T_1)$. Lemma 3 gives now:

— if $\dim N_{T_1^*}<\aleph_0$, then $\operatorname{ind}(U^*T-\zeta)=\operatorname{ind} T_2=\operatorname{ind} T_1=\operatorname{ind} T<0$, so $\dim N_{(U^*T-\zeta)^*}>0$,

— if $\dim N_{T_1^*}\geqq\aleph_0$, then $\dim N_{(U^*T-\zeta)^*}>0$.

In both cases $\bar{\zeta}\in\sigma((U^*T)^*)$ and $\zeta\in\sigma(U^*T)$. It follows that $-(1+n(T))\in\in\sigma(U^*T-I)$ so that $\|T-U\|\geqq 1+n(T)$.

The last inequality holds for all unitary U. Hence, $d(T,\mathscr{U}n)\geqq 1+n(T)$.

Case 3. Suppose $n(T)>0$ and $\dim N_{T^*}>\aleph_0$. Let $U\in\mathscr{L}(H)$ be unitary, $a>0$, $a<n(T)$. It will be proved that $\|T-U\|\geqq 1+a$. Let E be the spectral measure for $|T|$. Note that $N_T\subset E([0,a))(H)$. From the definition of $n(T)$ it follows that $\dim E([0,a))(H)<\dim N_{T^*}$. Let H_1 be the smallest closed subspace of H which contains $E([0,a))(H)$ and is invariant under T, T^*, U and U^*. It is well known that $\dim H_1\leqq\max(\aleph_0,\dim E([0,a))(H))<\dim N_{T^*}$, so $N_{T^*}\not\subset H_1$. Because $N_{T^*}==(N_{T^*}\cap H_1)+(N_{T^*}\cap H_1^\perp)$ we have $N_{T^*}\cap H_1^\perp\neq\{0\}$. Choose $p\in N_{T^*}$, $p\perp H_1$, $p\neq 0$.

Let L be the smallest closed subspace of H which contains p and is invariant under T, T^*, U and U^*. Then $\dim L\leqq\aleph_0$. Let $S:=T|_L$. Then $N_{S^*}\neq\{0\}$ since $S^*p=0$. Because $L\subset H_1^\perp$ and $N_T\subset H_1$, we have $N_S=\{0\}$. Together: $\operatorname{ind} S<0$. Also, $E([0,a))(H)\subset H_1$, so that $L\subset H_1^\perp\subset E([a,\infty))(H)$. Now Lemma 4 can be used and $\sigma_e(|S|)\subset\sigma(|S|)\subset[a,\infty)$. If we apply Case 2 for the separable Hilbert space L and the operator S with the unitary $U|_L$ we get $\|T-U\|\geqq\|S-U|_L\|\geqq d(S,\mathscr{U}n(L))\geqq 1+n(S)=1+m_e(S)=1+\min\sigma_e(|S|)\geqq 1+a$.

Our results on the distance of an operator to the set of unitary operators can now be summarized as follows:

Theorem 6. *Let* $T\in\mathscr{L}(H)$. *Then*

$$d(T,\mathscr{U}n)=\max(1-m(T),\|T\|-1)\quad \text{if}\quad \operatorname{ind} T=0,$$
$$d(T,\mathscr{U}n)=\max(1+n(T),\|T\|-1)\quad \text{if}\quad \operatorname{ind} T<0.$$

In the case $\operatorname{ind} T>0$, *consider* T^*.

Appendix. Now we prove Lemma 3. From polar decomposition we infer the existence of a partial isometry V from $N_{T_1}^\perp$ to $N_{T_1^*}^\perp$ with $T_1=V|T_1|$. Further, there exists an $L_1\in\mathscr{L}(H)$ such that $\|L_1\|_e\leqq(m_e(T_1))^{-1}$ and $L_1|T_1|-I$ is compact. (Use the Spectral Theorem — see [8] Theorem 7.18.) Let $L:=L_1V^*$. Then LT_1-I is compact and $\|L\|_e\leqq(m_e(T_1))^{-1}$. Because I is Fredholm with $\operatorname{ind} I=0$, LT_1 is Fredholm and also $\operatorname{ind} LT_1=0$. (See [2] Theorem 5.20.) Then $\|I-LT_2\|_e=\|LT_1-LT_2\|_e\leqq\|L\|_e\|T_1-T_2\|_e<1$. Hence, there is a compact $K\in\mathscr{L}(H)$ such that $\|I-LT_2-K\|<1$. Then LT_2+K is invertible and therefore Fredholm with index 0. Using again Theorem 5.20 of [2] we have that LT_2 is Fredholm and $\operatorname{ind} LT_2=0$. Similarly to the proof of Theorem 5.17 in [2] it follows that dim $N_{T_1} < \aleph_0$, dim $N_{T_2} < \aleph_0$ and that $T_1(H)$ and $T_2(H)$ are closed because LT_1 and LT_2 are Fredholm. Then (Atkinson's Theorem, [2] Theorem 5.17): dim $N_{T_1^*} < \aleph_0$ iff T_1 is Fredholm, iff (for LT_1 is Fredholm) L is Fredholm, ..., iff dim $N_{T_2^*} < \aleph_0$.

If dim $N_{T_1^*} < \aleph_0$, then by the additivity of the index for Fredholm operators (see [1] Theorem 2.1) ind T_1=ind LT_1−ind L=−ind L=ind T_2.

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