Acta Sci. Math., 54 (1990), 159—169

Kernels of generalized derivations

TONG YUSUN

The concept of generalized derivations is a natural generalization of the inner derivations. In this paper, the kernels of the generalized derivations will be studied. It will be proved that the kernels of generalized derivations of any order coincide with each other for several special kinds of operators. An asymptotic form of this kind of results will be obtained and a related theorem concerning compact operators will be given.

Let $\mathfrak{H}$ be a Hilbert space and $B(\mathfrak{H})$ the Banach algebra of all linear bounded operators in $\mathfrak{H}$. For $A, B \in B(\mathfrak{H})$, the generalized derivation δ_{AB} is defined by $\delta_{AB}: B(\mathfrak{H}) \to B(\mathfrak{H})$,

(1) $$\delta_{AB}(X) = AX - XB.$$

Correspondingly, for any natural number n, the higher derivation $\delta_{AB}^{(n)}$ is

(2) $$\delta_{AB}^{(n)}(X) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} A^i X B^{n-i}.$$

In the case $A=B$, we denote $\delta_{AB}^{(n)}$ by $\delta_A^{(n)}$.

It is easily seen from the following discussion that all of the results in this paper still hold if A and B are defined in two different Hilbert spaces, and sometimes we shall deal with this case. But for brevity, we restrict our statement of theorems in one Hilbert space.

Obviously, $\ker \delta_{AB} \subset \ker \delta_{AB}^{(n)}$ for any $A, B \in B(\mathfrak{H})$ and $n \geqq 1$. In general, $\ker \delta_{AB} \neq \ker \delta_{AB}^{(n)}$. For example,

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in \ker \delta_A^{(2)} \setminus \ker \delta_A,$$

if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Received April 3, 1987.

1. The well-known Fuglede—Putnam theorem asserts that if $A, B \in B(\mathfrak{H})$ and A, B are normal operators, then the pair (A, B) of operators has the following property:

(FP) *If* $AX=XB$ *where* $X \in B(\mathfrak{H})$, *then* $A^*X=XB^*$.

Theorem 1. *Suppose that* $A, B \in B(\mathfrak{H})$, *the pair* (A, B) *of operators has property* (FP), *then*

$$\ker \delta_{AB}^{(n)} = \ker \delta_{AB}, \quad n = 1, 2, \dots. \tag{3}$$

Proof. It will suffice to show that $\ker \delta_{AB}^{(2)} = \ker \delta_{AB}$. Suppose that $X \in \ker \delta_{AB}^{(2)}$, then

$$AX - XB \in \operatorname{ran} \delta_{AB} \cap \ker \delta_{AB}.$$

For any $Y \in \ker \delta_{AB}$, it follows from [7] that $\overline{\operatorname{ran} Y}$ reduces A, $(\ker Y)^{\perp}$ reduces B, and the restrictions $A|\overline{\operatorname{ran} Y}$, $B|(\ker Y)^{\perp}$ are normal operators. Take two decompositions of $\mathfrak{H}$:

$$\mathfrak{H}_1 = \mathfrak{H} = \overline{\operatorname{ran} Y} \oplus (\operatorname{ran} Y)^{\perp}, \quad \mathfrak{H}_2 = \mathfrak{H} = (\ker Y)^{\perp} \oplus \ker Y.$$

Then we get decompositions of operators respectively:

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

where A_1, B_1 are normal operators. For linear operators X, Y from $\mathfrak{H}_2$ into $\mathfrak{H}_1$, we have

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that $A_1Y_1 - Y_1B_1 = 0$ from $AY - YB = \begin{bmatrix} A_1Y_1 - Y_1B_1 & 0 \\ 0 & 0 \end{bmatrix}$. On the other hand,

$$AX - XB - Y = \begin{bmatrix} A_1X_1 - X_1B_1 - Y_1 & * \\ * & * \end{bmatrix}.$$

From [1] Theorem 1.5, it follows that

$$\|AX - XB - Y\| \geqq \|A_1X_1 - X_1B_1 - Y_1\| \geqq \|Y_1\| = \|Y\|,$$

which implies that

$$\operatorname{ran} \delta_{AB} \cap \ker \delta_{AB} = \{0\}.$$

Hence $X \in \ker \delta_{AB}$.

Corollary. *Suppose that* $A, B \in B(\mathfrak{H})$, *and* A, B^* *are hyponormal operators then*

$$\ker \delta_{AB}^{(n)} = \ker \delta_{AB} \quad (n = 1, 2, \dots).$$

Proof. In this case, the pair (A, B) of operators has property (FP) [7].

Of course, property (FP) is not necessary for $\ker \delta_{AB}^{(n)} = \ker \delta_{AB}$. This fact can be shown by the next theorem.

Let $\{\beta(n)\}$ be a sequence of positive numbers. We consider the space $H^2(\beta)$ of sequences $f=\{\hat{f}(n)\}$ such that

$$\|f\|^2 = \sum |\hat{f}(n)|^2 \beta(n)^2.$$

$H^2(\beta)$ is a Hilbert space with the inner product

$$(f, g) = \sum \hat{f}(n)\overline{\hat{g}(n)}\beta(n)^2.$$

We use the notation $f(z) = \sum \hat{f}(n)z^n$ for $f \in H^2(\beta)$. Let $\hat{f}_k(n) = \delta_{nk}$. Conseqeuently, $f_k(z) = z^k$ and $\|f_k\| = \beta(k)$. Now consider the linear operator M_z of multiplication by z on $\mathfrak{H}$:

$$(M_z f)(z) = \sum \hat{f}(n) z^{n+1}.$$

If $\sup \beta(k+1)\beta(k)^{-1} < +\infty$, then M_z is bounded [6].

Theorem 2. *Suppose that there exist positive constants c_1, c_2 such that $c_1 < \beta(k) < c_2$, $k = 0, 1, 2, \ldots$, then*

$$\ker \delta_{M_z}^{(n)} = \ker \delta_{M_z}, \quad n = 1, 2, \ldots. \tag{4}$$

Proof. It will suffice to prove (4) for $n=2$. Assume that $X \in \ker \delta_{M_z}^{(2)}$, i.e.

$$M_z(M_z X - XM_z) = (M_z X - XM_z)M_z.$$

Since the commutant of M_z is $H^\infty(\beta)$, so there exists $\varphi \in H^\infty(\beta)$, such that

$$M_z X - XM_z = M_\varphi. \tag{5}$$

Denote $\psi = Xf_0 \in H^2(\beta)$. It follows from (5) that $M_z\psi - Xf_1 = M_\varphi f_0$, i.e.

$$Xf_1 = M_z\psi - M_\varphi f_0.$$

By induction, we obtain

$$Xf_n = M_z^n\psi - nM_\varphi f_{n-1},$$

which implies

$$\|Xf_n\| \geqq n\|M_\varphi f_{n-1}\| - \|M_z^n\psi\|. \tag{6}$$

Write $\psi = \sum \hat{\psi}(k)z^k$, $\varphi = \sum \hat{\varphi}(k)z^k$, then

$$\|M_\varphi f_{n-1}\|^2 = \sum \beta(k+n-1)^2|\hat{\varphi}(k)|^2 \geqq (c_1/c_2)^2\|\varphi\|^2,$$

$$\|M_z^n\psi\|^2 = \sum \beta(k+n)^2|\hat{\psi}(k)|^2 \leqq (c_2/c_1)^2\|\psi\|^2.$$

Hence $\|\varphi\| = 0$ by (6). In other words, by (5),

$$M_z X - XM_z = 0, \quad \text{i.e.} \quad X \in \ker \delta_{M_z}.$$

M_z is the unilateral shift if $\beta(n)=1$ for all n. In this case, the pair (M_z, M_z) does not possess property (FP), since the commutant of unilateral shift is the set of all analytic Toeplitz operators, and the adjoint operator of an analytic Toeplitz operator is no longer the same kind of operators.

By $K(\mathfrak{H})$ we denote the set of all compact operators on $\mathfrak{H}$.

Theorem 3. *Suppose that* $A, B \in B(\mathfrak{H})$, $\|Ax\| \geqq \|x\| \geqq \|Bx\|$ *hold for all* $x \in \mathfrak{H}$. *Then*

$$\ker \delta_{AB}^{(n)} \cap K(\mathfrak{H}) = \ker \delta_{AB} \cap K(\mathfrak{H}), \quad n = 1, 2, \ldots. \tag{7}$$

To prove this theorem, we should establish a lemma.

Lemma 1. *Under the conditions of theorem 3, if* $Y \in K(\mathfrak{H})$ *and* $AY=YB$, *then* $\overline{\operatorname{ran} Y}$ *reduces* A, $(\ker Y)^{\perp}$ *reduces* B, *and* $A|\overline{\operatorname{ran} Y}$, $B|(\ker Y)^{\perp}$ *are unitary operators.*

Proof. Suppose that the polar decomposition of Y is $Y=UP$ where $\ker Y = \ker P$. Then

$$U^*AUP-PB=0.$$

Denote the spectral representation of P by $P=\sum a_iP_i$, where $a_1>a_2>\ldots>a_n>\ldots$ are non-zero eigenvalues of P, $P_i\mathfrak{H}=\ker(P-a_iI)$. We claim that $P_i\mathfrak{H}$ reduce B, and $B|P_i\mathfrak{H}$ are unitary operators.

Take $x\in P_1\mathfrak{H}$. Then we have

$$\|a_1x\|^2 = \|a_1Ux\|^2 \leqq \|a_1AUx\|^2 = \left\|\sum a_iAUP_ix\right\|^2 = \|AUPx\|^2 = \|PBx\|^2 =$$
$$= \left\|\sum a_iP_iBx\right\|^2 \leqq a_1^2\|Bx\|^2 \leqq a_1^2\|x\|^2.$$

It is easily seen that the above estimations hold if and only if $Bx\in P_1\mathfrak{H}$, and $\|Bx\| = \|x\|$. Since $\dim P_1\mathfrak{H}<\infty$, so $B|P_1\mathfrak{H}$ is a unitary operator. In this case, from

$$(U^*AUP-PB)|P_1\mathfrak{H}=0$$

we get $U^*AU|P_1\mathfrak{H}=B|P_1\mathfrak{H}$, hence $U^*AU|P_1\mathfrak{H}$ is unitary. Repeating the same arguement, we get the same conclusion for $P_2\mathfrak{H}$. By induction, it can be seen that $(\ker P)^{\perp}=(\ker Y)^{\perp}$ reduces U^*AU and B, $U^*AU|(\ker Y)^{\perp}=B|(\ker Y)^{\perp}$ is a unitary operator. However, U is a unitary operator from $(\ker Y)^{\perp}$ onto $\overline{\operatorname{ran} Y}$, thus $\overline{\operatorname{ran} Y}$ reduces A, and $A|\overline{\operatorname{ran} Y}$ is a unitary operator, too.

Proof of Theorem 3. It will suffice to prove that $X\in\ker\delta_{AB}$ for any $X\in\ker\delta_{AB}^{(2)}\cap K(\mathfrak{H})$. In fact, from Lemma 1 and

$$A(AX-XB)=(AX-XB)B$$

it can be seen that $A_1=A|P_1\mathfrak{H}$, $B_1=B|P_2\mathfrak{H}$ are unitary operators where P_1, P_2 are projections from $\mathfrak{H}$ onto $\overline{\operatorname{ran}(AX-XB)}$, $(\ker(AX-XB))^{\perp}$ respectively. Moreover, it can be easily seen that

$$\delta^{(2)}_{A_1B_1}(P_1XP_2) = \delta^{(2)}_{AB}(X) = 0.$$

Thus from the corollary of Theorem 1, we obtain

$$\delta_{AB}(X) = \delta_{A_1B_1}(P_1XP_2) = 0.$$

Corollary 1. *Under the conditions of Theorem 3,*

$$\ker\delta^{(n)}_{BA}\cap K(\mathfrak{H}) = \ker\delta_{BA}\cap K(\mathfrak{H}), \quad n = 1, 2, \dots. \tag{8}$$

Proof. Since A^{-1} exists and it is a contraction, the adjoint operator of a contraction is still a contraction, so

$$\|A^*x\| \geqq \|x\|, \quad \|B^*x\| \leqq \|x\|$$

hold for all $x\in\mathfrak{H}$. It follows from Theorem 3 that

$$\ker\delta^{(n)}_{A^*B^*}\cap K(\mathfrak{H}) = \ker\delta_{A^*B^*}\cap K(\mathfrak{H}).$$

We note that the set of all the adjoint elements of $\ker\delta^{(n)}_{A^*B^*}$ is $\ker\delta^{(n)}_{BA}$. Therefore (8) holds.

Corollary 2. *Suppose that* $A, B\in B(\mathfrak{H})$, *A is invertible and* $\|A^{-1}\|\|B\|\leqq 1$. *then*

$$\ker\delta^{(n)}_{AB}\cap K(\mathfrak{H}) = \ker\delta_{AB}\cap K(\mathfrak{H}), \quad \ker\delta^{(n)}_{BA}\cap K(\mathfrak{H}) = \ker\delta_{BA}\cap K(\mathfrak{H})$$

hold for $n=1, 2, \dots$.

Proof. Consider the operators

$$\hat{A} = \|B\|^{-1}A, \quad \hat{B} = \|B\|^{-1}B.$$

We have $\|\hat{A}x\|\geqq\|x\|$, $\|\hat{B}x\|\leqq\|x\|$ for all $x\in\mathfrak{H}$. From the above theorem and Corollary 1, we come to the conclusion.

2. The Fuglede theorem tells us that if N is a normal operator, S is a Borel set in the plane, then $X\in\ker\delta_N$ implies $X\in\ker\delta_{E(S)}$, where $E(S)$ is the spectral projection of N corresponding to S. Does the asymptotic form of this theorem still hold? In other words, if $NX-XN$ is "small", is $E(S)X-XE(S)$ "small"? [2] showed the answer is "no" for the norm topology, by constructing a normal operator N, a Borel set S and a sequence $\{X_n\}$ for which $\|X_n\|=1$ for all n, and $\|NX_n-X_nN\|\to 0$ but $\|E(S)X_n-X_nE(S)\|=1$ for all n. However, we shall show that the answer is "yes" for the strong operator topology and weak operator topology. This is the following theorem.

Theorem 4. *Let $N\in B(H)$ be a normal operator, S be a Borel set in the plane, $\mathcal{U}\subset B(\mathfrak{H})$ be a neighborhood of 0 with repect to the strong (weak) operator topology, $K>0$, n be a natural number. Then there exists a neighborhood $\mathcal{V}$ of 0 with respect to the same topology such that*

$$\delta_{E(S)}(X)\in\mathcal{U}$$

if $\|X\|<K$, and $\delta_N^{(n)}(X)\in\mathcal{V}$.

Proof. First, we assume that $n=1$. It follows from a computation that

$$N^kX-XN^k=\sum_{i=0}^{k-1}N^{k-1-i}(NX-XN)N^i \tag{9}$$

$$N^{*l}X-XN^{*l}=\sum_{i=0}^{l-1}N^{*l-1-i}(N^*X-XN^*)N^{*i} \tag{10}$$

hold for any k, l. Multiply N^{*l} from the left to the both sides of (9), and N^k from the right to the both sides of (10), add these results, we obtain

$$N^kN^{*l}X-XN^kN^{*l}=N^{*l}\sum_i N^{k-1-i}(NX-XN)N^i+\sum_i N^{*l-1-i}(N^*X-XN^*)N^{*i}N^k$$

since N is normal. Thus for bivariable polynomial $p(z,\bar z)=\sum a_{kl}z^k\bar z^l$, we have

$$[X,p(N,N^*)]= \tag{11}$$
$$=\sum_{k,l}a_{kl}\Big(\sum_i N^{*l}N^{k-1-i}(NX-XN)N^i+\sum_i N^{*l-1-i}(N^*X-XN^*)N^{*i}N^k\Big),$$

where we use the notation $[A,B]=AB-BA$ for brevity. Since the addition of operators is continuous and the multiplication is separately continuous with respect to strong (weak) operator topology, it follows from (11) that there exists a neighborhood $\mathcal{V}_1$, such that $[X,p(N,N^*)]\in\mathcal{U}$ if $[X,N]\in\mathcal{V}_1$ and $[X,N^*]\in\mathcal{V}_1$.

For $\mathcal{V}_1$, it follows from asymptotic Fuglede—Putnam theorem [5], that there exists a neighborhood $\mathcal{V}\subset\mathcal{V}_1$ such that $[X,N^*]\in\mathcal{V}_1$ if $\|X\|<K$, $[X,N]\in\mathcal{V}$. In this case, clearly

$$[X,p(N,N^*)]\in\mathcal{U}.$$

Since the set of all $p(N,N^*)$ is dense in the w^*-algebra generated by N with respect to the strong operator topology, we obtain the desired conclusion for $n=1$.

For general n, take $K_1>0$ such that

$$\|\delta_N^{(n-k)}\delta_{E(S)}^{(k+1)}(X)\|\leqq K_1,\quad k=1,\dots,n-1$$

for all $\|X\|<K$. Take $\mathcal{U}_0\subset\mathcal{U}$ such that $\delta_{E(S)}^{(n+1)}(X)\in\mathcal{U}$ if $\delta_{E(S)}^{(n)}(X)\in\mathcal{U}_0$.

Now, take $\mathcal{V}_i$ $(i=0,1,\dots,n)$ as follows:

$$\mathcal{V}_0=\mathcal{U}_0,\quad \delta_{E(S)}(X)\in\mathcal{V}_i\ \text{ if }\ \|X\|\leqq K_1$$

and $\delta_N(X)\in\mathcal{V}_{i+1}$. Denote $\mathcal{V}=\mathcal{V}_n$. If $\|X\|<K$ and $\delta_N^{(n)}(X)\in\mathcal{V}_n$, then $\delta_{E(S)}\delta_N^{(n-1)}(X)\in\mathcal{V}_{n-1}$. But $E(S)N=NE(S)$, so

$$\delta_N^{(n-1)}\delta_{E(S)}(X)=\delta_{E(S)}\delta_N^{(n-1)}(X)\in\mathcal{V}_{n-1}.$$

By induction, we get

$$\delta_{E(S)}^{(n)}(X)\in\mathcal{U}_0.$$

It is easily seen that

$$\delta_{E(S)}(X)=\delta_{E(S)}^{(n)}(X)\in\mathcal{U}_0\subset\mathcal{U},\quad \text{for odd } n,$$

and

$$\delta_{E(S)}(X)=\delta_{E(S)}^{(n+1)}(X)\subset\mathcal{U},\quad \text{for even } n.$$

Corollary. *Let $N\in B(\mathfrak{H})$ be a normal operator, $K>0$, n be natural number $\mathcal{U}\subset B(\mathfrak{H})$ be a neighborhood of 0 with respect to the strong (weak) operator topology. Then there exists a neighborhood $\mathcal{V}$ with respect to the same topology such that $\delta_N(X)\in\mathcal{U}$ if $\|X\|<K$ and $\delta_N^{(n)}(X)\in\mathcal{V}$.*

Proof. First, take a neighborhood $\mathcal{U}_1$ of 0 such that $\mathcal{U}_1+\mathcal{U}_1\subset\mathcal{U}$. Take a partition of $\sigma(N)$: $\sigma(N)=\bigcup_{i=1}^{m}\sigma_i$, where σ_i are some disjoint Borel sets satisfying

$$\delta_N(X)-\delta_{N_1}(X)\in\mathcal{U}_1$$

if $\|X\|<K$, where $N_1=\sum c_iE(\sigma_i)$, $c_i\in\sigma_i$. Then for fixed N_1, using Theorem 4, we can get a neighborhood $\mathcal{V}$ of 0 such that $\delta_{N_1}(X)\in\mathcal{U}_1$ if $\|X\|<K$, $\delta_{N_1}^{(n)}(X)\in\mathcal{V}$. Hence

$$\delta_N(X)=\delta_{N_1}(X)-\big(\delta_N(X)-\delta_{N_1}(X)\big)\in\mathcal{U}$$

if $\|X\|<K$, $\delta_{N_1}^{(n)}(X)\in\mathcal{V}$.

3. Now, we generalized some results in Section 1 to an asymptotic form. First we should show that the corollary of Theorem 4 is still hold with respect to the norm topology.

Lemma 2. *Let $N\in B(\mathfrak{H})$ be a normal operator, $K>0$, n be a natural number. Then for any $\varepsilon>0$, there exists $\eta>0$ such that $\|\delta_N(X)\|<\varepsilon$ if $\|X\|<K$, $\|\delta_N^{(n)}(X)\|<\eta$.*

Proof. Take a partition of $\sigma(N)$ by a finite number of straight lines which are parallel to x-axis or y-axis, $\sigma(N)=\bigcup_{i=1}^{m}\sigma_i$, where σ_i are disjoint each other, such that

(12) $$\|\delta_{N_1}(X)-\delta_N(X)\|<\varepsilon/2$$

for $\|X\|\leqq K$, where $N_1=\sum c_iE(\sigma_i)$, $c_i\in\sigma_i$, and

(13) $$\Delta<\varepsilon/36K$$

where $\Delta = \max_i \operatorname{diam} \sigma_i$. Denote $N_i = NE(\sigma_i)$, we have

$$\delta_N^{(n)}(X) = [\delta_{N_iN_j}^{(n)}(X_{ij})]$$

where $X_{ij} = E(\sigma_i)XE(\sigma_j)$.

(14) $$\delta_{N_1}(X) = [(c_i - c_j)X_{ij}] = [(c_i - c_j)X_{ij}]_1 + [(c_i - c_j)X_{ij}]_2,$$

where the non-zero elements in $[\ ]_1$ are those corresponding to $\sigma(N_i) \cap \sigma(N_j) = \emptyset$.

First, consider $[(c_i - c_j)X_{ij}]_1$. Since $\delta_{N_iN_j}^{(n)}$ is invertible if $\sigma(N_i) \cap \sigma(N_j) = \emptyset$, [3], so we can take $\eta > 0$ such that

(15) $$\eta < (\varepsilon/4m^2) \min\left\{|c_i - c_j|^{-1} \|\delta_{N_iN_j}^{(n)-1}\|^{-1} \middle| \sigma(N_i) \cap \sigma(N_j) = \emptyset\right\}.$$

Obviously, $\|\delta_{N_iN_j}^{(n)}(X_{ij})\| < \eta$ if $\|\delta_N^{(n)}(X)\| < \eta$. It follows from (15) that

$$\|(c_i - c_j)X_{ij}\| \leqq |c_i - c_j| \|\delta_{N_iN_j}^{(n)-1}\| \|\delta_{N_iN_j}^{(n)}(X_{ij})\| \leqq |c_i - c_j| \|\delta_{N_iN_j}^{(n)-1}\| \eta < \varepsilon/4m^2$$

if $\|\delta_N^{(n)}(X)\| < \eta$. Thus in this case

(16) $$\|[(c_i - c_j)X_{ij}]_1\| < \varepsilon/4.$$

Next, consider $[(c_i - c_j)X_{ij}]_2$. We claim that $\|[\dots]_2\| < \varepsilon/4$. In fact for any $f \in \mathfrak{H}$, $\|f\| = 1$, $f = \sum f_i$ where $f_i = E(\sigma_i)f$, we have

$$\|[(c_i - c_j)X_{ij}]_2 f\|^2 = \sum_i{}_2 \Big\|\sum_j{}_2 (c_i - c_j)X_{ij}f_j\Big\|^2 \leqq \sum_i{}_2 \sum_j{}_2 |c_i - c_j|^2 \|X_{ij}\|^2 \|f_j\|^2.$$

For each fixed j, the number of i satisfying $\sigma_i \cap \sigma_j \neq \emptyset$ doesn't exceed nine. Hence each f_j appears in $\sum_i{}_2 \sum_j{}_2$ at most nine times. Since $\|X_{ij}\| \leqq \|X\| < K$, and $|c_i - c_j| < 2\Delta$ in this sum, therefore

$$\|[(c_i - c_j)X_{ij}]_2 f\|^2 \leqq 9(2\Delta)^2 K^2 \sum \|f_j\|^2 = 36\Delta^2 K^2 < \varepsilon^2/4^2$$

by (13). Thus

(17) $$\|[(c_i - c_j)X_{1j}]_2\| < \varepsilon/4.$$

From (14), (16), (17) we obtain $\|\delta_{N_1}(X)\| < \varepsilon/2$. Using (12), we get $\|\delta_N(X)\| < \varepsilon$ if $\|X\| < K$ and $\|\delta_N^{(n)}(X)\| < \eta$.

Lemma 3. *Let N_1, N_2 be normal operators, $K > 0$, n be a natural number $\mathscr{U} \subset B(\mathfrak{H})$ be a neighborhood of 0 with respect to the norm topology (or strong operator topology, weak operator topology). Then there exists a neighborhood $\mathscr{V}$ of 0 with respect to the same topology such that*

(18) $$\delta_{N_1N_2}(X) \in \mathscr{U}$$

if $\|X\| < K$ and $\delta_{N_1N_2}^{(n)}(X) \in \mathscr{V}$.

Proof. Using Putnam's technique, consider the normal operator $N=\begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$ on $\mathfrak{H}\oplus\mathfrak{H}$. Denote

$$\tilde{\mathscr{U}}=\left\{\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\middle| X_i\in\mathscr{U}\right\}.$$

$\tilde{\mathscr{U}}$ is a neighborhood of 0 in $B(\mathfrak{H}\oplus\mathfrak{H})$. From Lemma 2 and the corollary of Theorem 4, it follows that there exists a neighborhood $\tilde{\mathscr{V}}$ of 0 such that $\delta_{\tilde{N}}(\tilde{X})\in\tilde{\mathscr{U}}$ if $\|\tilde{X}\|<K$ and $\delta_{\tilde{N}}^{(n)}(\tilde{X})\in\tilde{\mathscr{V}}$. Define

$$\mathscr{V}=\left\{X\middle|\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}\in\tilde{\mathscr{V}}\right\}.$$

If $X\in B(\mathfrak{H})$ satisfies $\|X\|<K$, $\delta_{N_1N_2}^{(n)}(X)\in\mathscr{V}$, then for $\tilde{X}=\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ we have $\|\tilde{X}\|<K$, and $\delta_{\tilde{N}}^{(n)}(\tilde{X})=\begin{bmatrix} 0 & \delta_{N_1N_2}^{(n)}(X) \\ 0 & 0 \end{bmatrix}\in\tilde{\mathscr{V}}$. In this case

$$\delta_{\tilde{N}}(\tilde{X})=\begin{bmatrix} 0 & \delta_{N_1N_2}(X) \\ 0 & 0 \end{bmatrix}\in\tilde{\mathscr{U}},$$

which implies $\delta_{N_1N_2}(X)\in\mathscr{U}$.

Theorem 5. *Let $A, B^*\in B(\mathfrak{H})$ be subnormal operators, $K>0$, n be a natural number, $\mathscr{U}$ be a neighborhood of 0 with respect to the norm topology (or strong operator topology, weak operator topology), then there exists a neighborhood $\mathscr{V}$ of 0 with respect to the same topology such that $\delta_{AB}(X)\in\mathscr{U}$ if $\|X\|<K$ and $\delta_{AB}^{(n)}(X)\in\mathscr{V}$.*

Proof. With no loss of generality, we may assume that the normal dilations of A and B^* are

$$\tilde{A}=\begin{bmatrix} A & A_1 \\ 0 & A_2 \end{bmatrix},\quad \tilde{B}^*=\begin{bmatrix} B^* & B_1 \\ 0 & B_2 \end{bmatrix}.$$

Denote $\tilde{\mathscr{U}}=\left\{\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}\middle| X_i\in\mathscr{U}\right\}$. Then, by Lemma 3, there exists a neighborhood $\tilde{\mathscr{V}}$ of 0, such that $\delta_{\tilde{A}\tilde{B}}(\tilde{X})\in\tilde{\mathscr{U}}$ if $\|\tilde{X}\|<K$ and $\delta_{\tilde{A}\tilde{B}}^{(n)}(\tilde{X})\in\tilde{\mathscr{V}}$. Denote $\mathscr{V}=\left\{X\middle|\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\in\tilde{\mathscr{V}}\right\}$, then $\mathscr{V}\subset B(\mathfrak{H})$ is a neighborhood of 0. For $\|X\|<K$, $\delta_{AB}^{(n)}(X)\in\mathscr{V}$, consider $\tilde{X}=\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$. We have $\|\tilde{X}\|<K$, and

$$\delta_{\tilde{A}\tilde{B}}^{(n)}(\tilde{X})=\begin{bmatrix} \delta_{AB}^{(n)}(X) & 0 \\ 0 & 0 \end{bmatrix}\in\tilde{\mathscr{V}},$$

therefore

$$\delta_{\tilde{A}\tilde{B}}(\tilde{X}) = \begin{bmatrix} \delta_{AB}(X) & 0 \\ 0 & 0 \end{bmatrix} \in \tilde{\mathscr{U}},$$

which implies

$$\delta_{AB}(X) \in \mathscr{U}.$$

From Theorem 5, we can rewrite the asymptotic Fuglede—Putnam theorem [4], [5] as follows.

Corollary. *Let* $A, B^* \in B(\mathfrak{H})$ *be subnormal operators,* $K>0$, n, m *be two natural numbers,* $\mathscr{U} \subset B(\mathfrak{H})$ *be a neighborhood of* 0 *with respect to the norm topology (or strong operator topology, weak operator topology), then there exists a neighborhood* $\mathscr{V}$ *of* 0 *with respect to the same topology, such that*

$$\delta^{(n)}_{A^*B^*}(X) \in \mathscr{U} \quad if \quad \|X\| < K \quad and \quad \delta^{(m)}_{AB}(X) \in \mathscr{V}.$$

4. At last, we study the case when the generalized derivations are compact operators.

Theorem 6. *Let* $N, M \in B(\mathfrak{H})$ *be normal operators,* $X \in B(\mathfrak{H})$, n *be a natural number,* $\delta^{(n)}_{NM}(X)$ *be a compact operator. Then* $\delta_{NM}(X)$ *is a compact operator.*

Proof. First we assume that $N=M$. Suppose that $\{f_n\} \subset \mathfrak{H}$, $\|f_n\|=1$, w-lim $f_n = 0$. We shall prove that s-lim $\delta_N(X)f_n = 0$.

Let $\varepsilon>0$ be an arbitrary fixed positive number. Similar to the proof of Lemma 2, make a partition of $\sigma(N)$ by a finite number of straight lines which are parallel to x-axis or y-axis, $\sigma(N) = \bigcup_{i=1}^{m} \sigma_i$, where σ_i are disjoint each other, such that

$$\|\delta_N(X) - \delta_{N_1}(X)\| < \varepsilon/2 \tag{19}$$

where $N_1 = \sum c_i E(\sigma_i)$, $c_i \in \sigma_i$, and

$$\Delta < \varepsilon/24\|X\| \tag{20}$$

where $\Delta = \max_i \operatorname{diam} \sigma_i$. Denote $N_i = NE(\sigma_i)$, $X_{ij} = E(\sigma_i)XE(\sigma_j)$. We have

$$\delta^{(n)}_N(X) = [\delta^{(n)}_{N_iN_j}(X_{ij})] \tag{21}$$

(22)

$$\delta_{N_1}(X)f_k = [(c_i - c_j)X_{ij}]f_k = \left(\sum (c_1 - c_j)X_{1j}f^j_k, \ldots, \sum (c_n - c_j)X_{nj}f^j_k\right) = f_{k1} + f_{k2}$$

where $f^j_k = E(\sigma_j)f_k$, f_{k1} is the element each of its components is the sum of the terms corresponding to $\sigma(N_i) \cap \sigma(N_j) = \emptyset$.

By $\sum_2$ we denote the sum corresponding to f_{k2}. Apparently

$$\|f_{k2}\|^2 = \sum_i{}_2 \left\|\sum_j{}_2 (c_i - c_j)X_{ij}f^j_k\right\|^2 \leqq \sum_i{}_2 \sum_j{}_2 |c_i - c_j|^2 \|X_{ij}\|^2 \|f^j_k\|^2.$$

Since each f_k^j appears at most nine times in the sum of the right side, so we get

(23) $$\|f_{k2}\|^2 \leqq 9\cdot 4\cdot \Delta^2\|X\|^2 < (\varepsilon/4)^2$$

by (20).

Now, let us consider f_{k1}. We note that all of the $\delta^{(n)}_{N_iN_j}(X_{ij})$ are compact since $\delta^{(n)}_N(X)$ is compact. Besides, all of the $\delta_{N_iN_j}$ which relate to f_{k1} are invertible since $\sigma(N_i)\cap\sigma(N_j)=\emptyset$ in this case. Thus for these X_{ij}, $X_{ij}=\delta^{(n)-1}_{N_iN_j}\delta^{(n)}_{N_iN_j}(X_{ij})$ are compact operators. From the structure of f_{k1}, it follows that there exists K, such that

(24) $$\|f_{k1}\| < \varepsilon/4$$

for $k>K$. Therefore, from (19), (22), (23), (24), we obtain

$$\|\delta_N(X)f_k\| < \varepsilon \quad \text{if} \quad k>K.$$

In general, N and M are two different normal operators. Similar to the proof of Lemma 3, consider the operators

$$\tilde{N} = \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix}, \quad \tilde{X} = \begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}.$$

By the hypothesis,

$$\delta^{(n)}_{\tilde{N}}(\tilde{X}) = \begin{bmatrix} 0 & \delta^{(n)}_{NM}(X) \\ 0 & 0 \end{bmatrix}$$

is a compact operator, it follows from what has been proved for the case $N=M$ that

$$\delta_{\tilde{N}}(\tilde{X}) = \begin{bmatrix} 0 & \delta_{NM}(X) \\ 0 & 0 \end{bmatrix}$$

is compact, so is $\delta_{NM}(X)$.

References

[1] J. Anderson and C. Foias, Properties which normal operators share with normal derivations and related operators, *Pacific J. Math.*, **61** (1975), 313—326.

[2] J. J. Bastian and K. J. Harrison, Subnormal weighted shifts and asymptotic properties of normal operators, preprint.

[3] L. A. Fialkow, Generalized derivations, *Topics in modern operator theory*, Operator theory: Advances and applications, Vol. 2 (Birkhäuser, 1981).

[4] R. L. Moore, An asymptotic Fuglede theorem, *Proc. Amer. Math. Soc.*, **50** (1975), 138—142.

[5] D. D. Rogers, On Fuglede's theorem and operator topologies, *Proc. Amer. Math. Soc.*, **75** (1979), 32—36.

[6] A. L. Shields, Weighted shift operators and analytic function, in: *Topics in operator theory*, Mathematical Surveys, no. 13, (Providence, R. I., 1974), pp. 49—128.

[7] K. Takahashi, On the converse of the Fuglede—Putnam theorem, *Acta Sci. Math.*, **43** (1981), 123—125.

DEPARTMENT OF MATHEMATICS
FUDAN UNIVERSITY
SHANGHAI, P. R. CHINA