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An integral formula for a Riemannian manifold with three orthogonal distributions

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1. Introduction. The result of this paper is a generalization of theorems given in [1] and [3]. We calculate the sum of divergences of three orthogonal distributions D_i ($i=1, 2, 3$) on a Riemannian manifold such that $\dim D_1+\dim D_2+\dim D_3=\dim M$. Assuming M to be closed and applying the Green Theorem, we get global results concerning integrals of curvatures for foliated manifolds.

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Let D be a distribution on a Riemannian manifold $(M, \langle,\rangle)$. The second fundamental form B of D is defined in the following way. If X and Y are two tangent vector fields to D, then $B(X, Y)$ is the normal component of the field $\frac{1}{2}(\nabla_X Y+\nabla_Y X)$ (see [2]). The trace H of B is called the mean curvature vector of D.

Let D_1, D_2, D_3 be three orthogonal distributions on a Riemannian manifold $(M, \langle,\rangle)$ such that $\dim D_1+\dim D_2+\dim D_3=m=\dim M$. We consider the mean curvature vectors H_k of D_k, $k=1, 2, 3$, and calculate the sum $\operatorname{div} H_1+\operatorname{div} H_2+\operatorname{div} H_3$.

Throughout this paper manifolds, fields, metrics etc. are assumed to be C^∞-differentiable.

2. Results. Let us take a local orthonormal frame $e_1, \ldots, e_m$ adapted to D_1, D_2, D_3, i.e. assume that e_i is tangent to D_1 for $i=1, 2, \ldots, \dim D_1$, e_α is tangent to D_2 for $\alpha=\dim D_1+1, \ldots, \dim D_1+\dim D_2$, e_j is tangent to D_3 for $j=\dim D_1+\dim D_2+1, \ldots, m$. Hereafter, the indices i, p range over the set $\{1, \ldots, \dim D_1\}$, α, β — over $\{\dim D_1+1, \ldots, \dim D_1+\dim D_2\}$, j, q — over $\{\dim D_1+\dim D_2+1, \ldots, m\}$.

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If v is a vector tangent to M, we write

$$v = v^{\top 1} + v^{\top 2} + v^{\top 3}$$

where $v^{\top k}$ is tangent to D_k. By $v^{\perp k}$ we denote the component of v orthogonal to D_k.

The integrability tensors T_k of D_k are defined as follows:

$$T_k(X_k, Y_k) = \frac{1}{2}[X_k, Y_k]^{\perp k}$$

for vector fields X_k, Y_k tangent to D_k.

Let B_k be the second fundamental form of D_k. Then the mean curvature vectors H_k of D_k are given by

$$H_1 = \sum_i B_1(e_i, e_i) = \sum_i (\nabla_{e_i} e_i)^{\perp 1},$$

$$H_2 = \sum_\alpha B_2(e_\alpha, e_\alpha) = \sum_\alpha (\nabla_{e_\alpha} e_\alpha)^{\perp 2},$$

$$H_3 = \sum_j B_3(e_j, e_j) = \sum_j (\nabla_{e_j} e_j)^{\perp 3}.$$

Therefore

(1) $$\operatorname{div} H_1 + \operatorname{div} H_2 + \operatorname{div} H_3 =$$
$$= \sum_{k=1}^{3} \Big(\sum_i \langle \nabla_{e_i} H_k, e_i\rangle + \sum_\alpha \langle \nabla_{e_\alpha} H_k, e_\alpha\rangle + \sum_j \langle \nabla_{e_j} H_k, e_j\rangle\Big) =$$
$$= -|H_1|^2 - |H_2|^2 - |H_3|^2 + \sum_{i,\alpha} \langle \nabla_{e_\alpha}(\nabla_{e_i} e_i)^{\perp 1}, e_\alpha\rangle + \sum_{i,j} \langle \nabla_{e_j}(\nabla_{e_i} e_i)^{\perp 1}, e_j\rangle +$$
$$+ \sum_{i,\alpha} \langle \nabla_{e_i}(\nabla_{e_\alpha} e_\alpha)^{\perp 2}, e_i\rangle + \sum_{\alpha,j} \langle \nabla_{e_j}(\nabla_{e_\alpha} e_\alpha)^{\perp 2}, e_j\rangle +$$
$$+ \sum_{i,j} \langle \nabla_{e_i}(\nabla_{e_j} e_j)^{\perp 3}, e_i\rangle + \sum_{\alpha,j} \langle \nabla_{e_\alpha}(\nabla_{e_j} e_j)^{\perp 3}, e_\alpha\rangle.$$

From the definition of the curvature tensor R we get

$$2\langle R(e_i, e_\alpha)e_\alpha, e_i\rangle + \langle \nabla_{e_\alpha}\nabla_{e_i} e_\alpha, e_i\rangle + \langle \nabla_{e_i}\nabla_{e_\alpha} e_i, e_\alpha\rangle + 2\langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha\rangle =$$
$$= \langle \nabla_{e_i}\nabla_{e_\alpha} e_\alpha, e_i\rangle + \langle \nabla_{e_\alpha}\nabla_{e_i} e_i, e_\alpha\rangle,$$

(2) $$2\langle R(e_j, e_\alpha)e_\alpha, e_j\rangle + \langle \nabla_{e_j}\nabla_{e_\alpha} e_j, e_\alpha\rangle + \langle \nabla_{e_\alpha}\nabla_{e_j} e_\alpha, e_j\rangle + 2\langle \nabla_{[e_j, e_i]} e_i, e_j\rangle =$$
$$= \langle \nabla_{e_i}\nabla_{e_j} e_j, e_i\rangle + \langle \nabla_{e_j}\nabla_{e_i} e_i, e_j\rangle,$$

$$2\langle R(e_j, e_\alpha)e_\alpha, e_j\rangle + \langle \nabla_{e_j}\nabla_{e_\alpha} e_j, e_\alpha\rangle + \langle \nabla_{e_\alpha}\nabla_{e_j} e_\alpha, e_j\rangle + 2\langle \nabla_{[e_j, e_\alpha]} e_\alpha, e_j\rangle =$$
$$= \langle \nabla_{e_\alpha}\nabla_{e_j} e_j, e_\alpha\rangle + \langle \nabla_{e_j}\nabla_{e_\alpha} e_\alpha, e_j\rangle.$$

Let us put

$$K(D_1, D_2) = \sum_{i,\alpha} \langle R(e_i, e_\alpha)e_\alpha, e_i\rangle,$$

$$K(D_1, D_3) = \sum_{i,j} \langle R(e_j, e_i)e_i, e_j\rangle,$$

$$K(D_2, D_3) = \sum_{\alpha,j} \langle R(e_j, e_\alpha)e_\alpha, e_j\rangle.$$

$K(D_1, D_2)$, $K(D_1, D_3)$, $K(D_2, D_3)$ do not depend on the choice of the orthonormal frame $e_1, \ldots, e_m$. They depend only on distributions D_1, D_2, D_3. From (2) we have

$$\sum_{i,\alpha} \big(\langle \nabla_{e_i}(\nabla_{e_\alpha}e_\alpha)^{\perp 2}, e_i\rangle + \langle \nabla_{e_\alpha}(\nabla_{e_i}e_i)^{\perp 1}, e_\alpha\rangle\big) = 2K(D_1, D_2) + \sum_{i,\alpha} \big(\langle \nabla_{e_\alpha}\nabla_{e_i}e_\alpha, e_i\rangle +$$

$$+ \langle \nabla_{e_i}\nabla_{e_\alpha}e_i, e_\alpha\rangle + 2\langle \nabla_{[e_\alpha, e_i]}e_i, e_\alpha\rangle - \langle \nabla_{e_i}(\nabla_{e_\alpha}e_\alpha)^{\top 2}, e_i\rangle - \langle \nabla_{e_\alpha}(\nabla_{e_i}e_i)^{\top 1}, e_\alpha\rangle\big),$$

(3) $$\sum_{i,j} \big(\langle \nabla_{e_i}(\nabla_{e_j}e_j)^{\perp 3}, e_i\rangle + \langle \nabla_{e_j}(\nabla_{e_i}e_i)^{\perp 1}, e_j\rangle\big) = 2K(D_1, D_3) +$$

$$+ \sum_{i,j} \big(\langle \nabla_{e_j}\nabla_{e_i}e_j, e_i\rangle + \langle \nabla_{e_i}\nabla_{e_j}e_i, e_j\rangle + 2\langle \nabla_{[e_j, e_i]}e_i, e_j\rangle -$$

$$- \langle \nabla_{e_i}(\nabla_{e_j}e_j)^{\top 3}, e_i\rangle - \langle \nabla_{e_j}(\nabla_{e_i}e_i)^{\top 1}, e_j\rangle\big),$$

$$\sum_{\alpha,j} \big(\langle \nabla_{e_\alpha}(\nabla_{e_j}e_j)^{\perp 3}, e_\alpha\rangle + \langle \nabla_{e_j}(\nabla_{e_\alpha}e_\alpha)^{\perp 2}, e_j\rangle = 2K(D_2, D_3) +$$

$$+ \sum_{\alpha,j} \big(\langle \nabla_{e_j}\nabla_{e_\alpha}e_j, e_\alpha\rangle + \langle \nabla_{e_\alpha}\nabla_{e_j}e_\alpha, e_j\rangle + 2\langle \nabla_{[e_j, e_\alpha]}e_\alpha, e_j\rangle -$$

$$- \langle \nabla_{e_\alpha}(\nabla_{e_j}e_j)^{\top 3}, e_\alpha\rangle - \langle \nabla_{e_j}(\nabla_{e_\alpha}e_\alpha)^{\top 2}, e_j\rangle\big).$$

Now, we observe that

$$\sum_{i,\alpha} \langle \nabla_{e_i}\nabla_{e_\alpha}e_i, e_\alpha\rangle = \sum_{i,\alpha} \big(-e_i, \langle e_i, \nabla_{e_\alpha}e_\alpha\rangle - \langle \nabla_{e_\alpha}e_i, \nabla_{e_i}e_\alpha\rangle\big),$$

$$\sum_{i,\alpha} \langle \nabla_{e_\alpha}\nabla_{e_i}e_\alpha, e_i\rangle = \sum_{i,\alpha} \big(-e_\alpha\langle e_\alpha, \nabla_{e_i}e_i\rangle - \langle \nabla_{e_i}e_\alpha, \nabla_{e_\alpha}e_i\rangle\big),$$

(4) $$\sum_{i,j} \langle \nabla_{e_i}\nabla_{e_j}e_i, e_j\rangle = \sum_{i,j} \big(-e_i\langle e_i, \nabla_{e_j}e_j\rangle - \langle \nabla_{e_j}e_i, \nabla_{e_i}e_j\rangle\big),$$

$$\sum_{i,j} \langle \nabla_{e_j}\nabla_{e_i}e_j, e_i\rangle = \sum_{i,j} \big(-e_j\langle e_j, \nabla_{e_i}e_i\rangle - \langle \nabla_{e_i}e_j, \nabla_{e_j}e_i\rangle\big),$$

$$\sum_{\alpha,j} \langle \nabla_{e_\alpha}\nabla_{e_j}e_\alpha, e_j\rangle = \sum_{\alpha,j} \big(-e_\alpha\langle e_\alpha, \nabla_{e_j}e_j\rangle - \langle \nabla_{e_j}e_\alpha, \nabla_{e_\alpha}e_j\rangle\big),$$

$$\sum_{\alpha,j} \langle \nabla_{e_j}\nabla_{e_\alpha}e_j, e_\alpha\rangle = \sum_{\alpha,j} \big(-e_j\langle e_j, \nabla_{e_\alpha}e_\alpha\rangle - \langle \nabla_{e_\alpha}e_j, \nabla_{e_j}e_\alpha\rangle\big).$$

Comparing equalities (1), (3) and (4), we obtain

$$
\begin{aligned}
(5)\qquad \operatorname{div} H_1+\operatorname{div} H_2+\operatorname{div} H_3 &= -|H_1|^2-|H_2|^2-|H_3|^2+\\
&+2K(D_1, D_2)+2K(D_1, D_3)+2K(D_2, D_3)+\\
&+\sum_{i,\alpha,j}\big(2(-\langle\nabla_{e_\alpha}e_i, \nabla_{e_i}e_\alpha\rangle-\langle\nabla_{e_i}e_j, \nabla_{e_j}e_i\rangle-\langle\nabla_{e_j}e_\alpha, \nabla_{e_\alpha}e_j\rangle+\\
&+\langle\nabla_{[e_\alpha, e_i]}e_i, e_\alpha\rangle+\langle\nabla_{[e_i, e_j]}e_j, e_i\rangle+\langle\nabla_{[e_j, e_\alpha]}e_\alpha, e_j\rangle)-\\
&-e_i\langle e_i, \nabla_{e_\alpha}e_\alpha\rangle-e_\alpha\langle e_\alpha, \nabla_{e_i}e_i\rangle-e_j\langle e_j, \nabla_{e_i}e_i\rangle-\\
&-e_i\langle e_i, \nabla_{e_j}e_j\rangle-e_\alpha\langle e_\alpha, \nabla_{e_j}e_j\rangle-e_j\langle e_j, \nabla_{e_\alpha}e_\alpha\rangle+\\
&+\langle(\nabla_{e_\alpha}e_\alpha)^{\top 2}, \nabla_{e_i}e_i\rangle+\langle(\nabla_{e_i}e_i)^{\top 1}, \nabla_{e_\alpha}e_\alpha\rangle+\langle(\nabla_{e_j}e_j)^{\top 3}, \nabla_{e_i}e_i\rangle+\\
&+\langle(\nabla_{e_i}e_i)^{\top 1}, \nabla_{e_j}e_j\rangle+\langle(\nabla_{e_j}e_j)^{\top 3}, \nabla_{e_\alpha}e_\alpha\rangle+\langle(\nabla_{e_\alpha}e_\alpha)^{\top 2}, \nabla_{e_j}e_j\rangle\big).
\end{aligned}
$$

Since $[X, Y]=\nabla_X Y-\nabla_Y X$ for vector fields on M, therefore

$$
\begin{aligned}
(6)\qquad \langle\nabla_{[e_\alpha, e_\alpha]}e_i, e_j\rangle &= \sum_{p,\beta,q}\big(\langle\nabla_{e_\alpha}e_i, e_p\rangle\langle\nabla_{e_p}e_i, e_\alpha\rangle+\langle\nabla_{e_\alpha}e_i, e_\beta\rangle\langle\nabla_{e_\beta}e_i, e_\alpha\rangle+\\
&+\langle\nabla_{e_\alpha}e_i, e_q\rangle\langle\nabla_{e_q}e_i, e_\alpha\rangle-\langle\nabla_{e_i}e_\alpha, e_p\rangle\langle\nabla_{e_p}e_i, e_\alpha\rangle-\\
&-\langle\nabla_{e_i}e_\alpha, e_\beta\rangle\langle\nabla_{e_\beta}e_i, e_\alpha\rangle-\langle\nabla_{e_i}e_\alpha, e_q\rangle\langle\nabla_{e_q}e_i, e_\alpha\rangle\big),\\
\langle\nabla_{[e_j, e_i]}e_i, e_j\rangle &= \sum_{p,\beta,q}\big(\langle\nabla_{e_j}e_i, e_p\rangle\langle\nabla_{e_p}e_i, e_j\rangle+\langle\nabla_{e_j}e_i, e_\beta\rangle\langle\nabla_{e_\beta}e_i, e_j\rangle+\\
&+\langle\nabla_{e_j}e_i, e_q\rangle\langle\nabla_{e_q}e_i, e_j\rangle-\langle\nabla_{e_i}e_j, e_p\rangle\langle\nabla_{e_p}e_i, e_j\rangle-\\
&-\langle\nabla_{e_i}e_j, e_\beta\rangle\langle\nabla_{e_\beta}e_i, e_j\rangle-\langle\nabla_{e_i}e_j, e_q\rangle\langle\nabla_{e_q}e_i, e_j\rangle\big),\\
\langle\nabla_{[e_j, e_\alpha]}e_\alpha, e_j\rangle &= \sum_{p,\beta,q}\big(\langle\nabla_{e_j}e_\alpha, e_\beta\rangle\langle\nabla_{e_p}e_\alpha, e_j\rangle+\langle\nabla_{e_j}e_\alpha, e_\beta\rangle\langle\nabla_{e_\beta}e_\alpha, e_j\rangle+\\
&+\langle\nabla_{e_j}e_\alpha, e_q\rangle\langle\nabla_{e_q}e_\alpha, e_j\rangle-\langle\nabla_{e_\alpha}e_j, e_p\rangle\langle\nabla_{e_p}e_\alpha, e_j\rangle-\\
&-\langle\nabla_{e_\alpha}e_j, e_\beta\rangle\langle\nabla_{e_\beta}e_\alpha, e_j\rangle-\langle\nabla_{e_\alpha}e_j, e_q\rangle\langle\nabla_{e_q}e_\alpha, e_j\rangle\big).
\end{aligned}
$$

From (6) we have

$$
\begin{aligned}
\sum_{i,\alpha}\langle\nabla_{[e_\alpha, e_i]}e_i, e_\alpha\rangle &= \sum_{i,\alpha,j}\sum_{p,\beta}\big(\langle\nabla_{e_\alpha}e_i, (\nabla_{e_i}e_\alpha)^{\top 1}+\langle\nabla_{e_\alpha}e_\beta, (\nabla_{e_\beta}e_\alpha)^{\top 1}\rangle+\\
+\langle\nabla_{e_\alpha}e_j, (\nabla_{e_j}e_\alpha)^{\top 1}\rangle&+\langle\nabla_{e_i}e_p, (\nabla_{e_p}e_i)^{\top 2}\rangle+\langle\nabla_{e_i}e_\alpha, (\nabla_{e_\alpha}e_i)^{\top 2}\rangle+\langle\nabla_{e_i}e_j, (\nabla_{e_j}e_i)^{\top 2}\rangle\big),\\
(7)\qquad \sum_{i,j}\langle\nabla_{[e_j, e_i]}e_i, e_j\rangle &= \sum_{i,\alpha,j}\sum_{p,q}\big(\langle\nabla_{e_j}e_i, (\nabla_{e_i}e_j)^{\top 1}\rangle+\langle\nabla_{e_j}e_\alpha, (\nabla_{e_\alpha}e_j)^{\top 1}\rangle+\\
+\langle\nabla_{e_j}e_q, (\nabla_{e_q}e_j)^{\top 1}\rangle&+\langle\nabla_{e_i}e_p, (\nabla_{e_p}e_i)^{\top 2}\rangle+\langle\nabla_{e_i}e_\alpha, (\nabla_{e_\alpha}e_i)^{\top 2}\rangle+\langle\nabla_{e_i}e_q, (\nabla_{e_q}e_i)^{\top 2}\rangle\big),\\
\sum_{\alpha,j}\langle\nabla_{[e_j, e_\alpha]}e_\alpha, e_j\rangle &= \sum_{i,\alpha,j}\sum_{p,q}\big(\langle\nabla_{e_j}e_i, (\nabla_{e_i}e_j)^{\top 2}+\langle\nabla_{e_j}e_\alpha, (\nabla_{e_\alpha}e_j)^{\top 2}\rangle+\\
+\langle\nabla_{e_j}e_q, (\nabla_{e_q}e_j)^{\top 2}&+\langle\nabla_{e_\alpha}e_i, (\nabla_{e_i}e_\alpha)^{\top 3}\rangle+\langle\nabla_{e_\alpha}e_\beta, (\nabla_{e_\beta}e_\alpha)^{\top 3}\rangle+\langle\nabla_{e_\alpha}e_j, (\nabla_{e_j}e_\alpha)^{\top 3}\rangle\big).
\end{aligned}
$$

Now, we shall give some notations

$$2B_{12}(e_\gamma, e_\delta) = (\nabla_{e_\gamma} e_\delta)^{\top 3} + (\nabla_{e_\delta} e_\gamma)^{\top 3},$$
$$2T_{12}(e_\gamma, e_\delta) = (\nabla_{e_\gamma} e_\delta)^{\top 3} - (\nabla_{e_\delta} e_\gamma)^{\top 3},$$

for $\gamma, \delta \in \{1, \ldots, \dim D_1 + \dim D_2\} = A_1$,

$$2B_{13}(e_\gamma, e_\delta) = (\nabla_{e_\gamma} e_\delta)^{\top 2} + (\nabla_{e_\delta} e_\gamma)^{\top 2},$$
$$2T_{13}(e_\gamma, e_\delta) = (\nabla_{e_\gamma} e_\delta)^{\top 2} - (\nabla_{e_\delta} e_\gamma)^{\top 2},$$

for $\gamma, \delta \in \{1, \ldots, \dim D_1\} \cup \{\dim D_1 + \dim D_2 + 1, \ldots, m\} = A_2$,

$$2B_{23}(e_\gamma, e_\delta) = (\nabla_{e_\gamma} e_\delta)^{\top 1} + (\nabla_{e_\delta} e_\gamma)^{\top 1},$$
$$2T_{23}(e_\gamma, e_\delta) = (\nabla_{e_\gamma} e_\delta)^{\top 1} - (\nabla_{e_\delta} e_\gamma)^{\top 1},$$

for $\gamma, \delta \in \{\dim D_1 + 1, \ldots, m\} = A_3$, and notice that

(8)

$$\sum_{\gamma, \delta \in A_1} \langle (\nabla_{e_\gamma} e_\delta)^{\top 3}, (\nabla_{e_\delta} e_\gamma)^{\top 3} \rangle = \sum_{\gamma, \delta \in A_1} \left(|B_{12}(e_\gamma, e_\delta)|^2 - |T_{12}(e_\gamma, e_\delta)|^2 \right) = |B_{12}|^2 - |T_{12}|^2.$$

Similarly,

$$\sum_{\gamma, \delta \in A_2} \langle (\nabla_{e_\gamma} e_\delta)^{\top 2}, (\nabla_{e_\delta} e_\gamma)^{\top 2} \rangle = |B_{13}|^2 - |T_{13}|^2,$$

(9)

$$\sum_{\gamma, \delta \in A_3} \langle (\nabla_{e_\gamma} e_\delta)^{\top 1}, (\nabla_{e_\delta} e_\gamma)^{\top 1} \rangle = |B_{23}|^2 - |T_{23}|^2.$$

Since

$$(\nabla_{e_i} e_p)^{\perp 1} + (\nabla_{e_p} e_i)^{\perp 1} = 2B_1(e_i, e_p),$$
$$(\nabla_{e_i} e_p)^{\perp 1} - (\nabla_{e_p} e_i)^{\perp 1} = 2T_1(e_i, e_p),$$

therefore

(10) $$\sum_{i, p} \langle (\nabla_{e_i} e_p)^{\perp 1}, (\nabla_{e_p} e_i)^{\perp 1} \rangle = |B_1|^2 - T_1|^2,$$

and analogously,

$$\sum_{\alpha, \beta} \langle (\nabla_{e_\alpha} e_\beta)^{\perp 2}, (\nabla_{e_\beta} e_\alpha)^{\perp 2} \rangle = |B_2|^2 - |T_2|^2,$$

(11)

$$\sum_{j, p} \langle (\nabla_{e_j} e_q)^{\perp 3}, (\nabla_{e_q} e_j)^{\perp 3} \rangle = |B_3|^2 - |T_3|^2.$$

From (7)—(11) we get

(12) $$\sum_{i, \alpha, j} 2\big(-\langle \nabla_{e_\alpha} e_i, \nabla_{e_i} e_\alpha \rangle - \langle \nabla_{e_i} e_j, \nabla_{e_j} e_i \rangle - \langle \nabla_{e_j} e_\alpha, \nabla_{e_\alpha} e_j \rangle +$$
$$+ \langle \nabla_{[e_\alpha, e_i]} e_i, e_\alpha \rangle + \langle \nabla_{[e_i, e_j]} e_j, e_i \rangle + \langle \nabla_{[e_j, e_\alpha]} e_\alpha, e_j \rangle =$$
$$= |B_{12}|^2 + |B_{13}|^2 + |B_{23}|^2 + |B_1|^2 + |B_2|^2 + |B_3|^2 -$$
$$- |T_{12}|^2 - |T_{13}|^2 - |T_{23}|^2 - |T_1|^2 - |T_2|^2 - |T_3|^2.$$

Let us calculate $\operatorname{div} H_i$ for $i=1, 2, 3$.

$$\operatorname{div} H_1 = \sum_{a,i,j}\left(\langle\nabla_{e_i}H_1, e_i\rangle+\langle\nabla_{e_a}H_1, e_a\rangle+\langle\nabla_{e_j}H_1, e_j\rangle\right) =$$
$$= -|H_1|^2+\langle\nabla_{e_a}(\nabla_{e_i}e_i)^{\mathrm{T}2}, e_a\rangle+\langle\nabla_{e_a}(\nabla_{e_i}e_i)^{\mathrm{T}3}, e_a\rangle+$$
$$+\langle\nabla_{e_j}(\nabla_{e_i}e_i)^{\mathrm{T}2}, e_j\rangle+\langle\nabla_{e_j}(\nabla_{e_i}e_i)^{\mathrm{T}3}, e_j\rangle.$$

Similarly,

$$\operatorname{div} H_2 = -|H_2|^2+\langle\nabla_{e_i}(\nabla_{e_a}e_a)^{\mathrm{T}1}, e_i\rangle+\langle\nabla_{e_i}(\nabla_{e_a}e_a)^{\mathrm{T}3}, e_i\rangle+$$
$$+\langle\nabla_{e_j}(\nabla_{e_a}e_a)^{\mathrm{T}1}, e_j\rangle+\langle\nabla_{e_j}(\nabla_{e_a}e_a)^{\mathrm{T}3}, e_j\rangle,$$
$$\operatorname{div} H_3 = -|H_3|^2+\langle\nabla_{e_a}(\nabla_{e_j}e_j)^{\mathrm{T}1}, e_a\rangle+\langle\nabla_{e_a}(\nabla_{e_j}e_j)^{\mathrm{T}2}, e_a\rangle+$$
$$+\langle\nabla_{e_i}(\nabla_{e_j}e_j)^{\dashv 1}, e_i\rangle+\langle\nabla_{e_i}(\nabla_{e_j}e_j)^{\mathrm{T}2}, e_i\rangle.$$

Applying the formula

$$Z(X, Y) = \langle\nabla_Z X, Y\rangle+\langle X, \nabla_Z Y\rangle$$

for vector fields X, Y, Z on M, we get

$$(13)\qquad \sum_{i,a,j}\big(-e_i\langle e_i, \nabla_{e_a}e_a\rangle-e_a\langle e_a, \nabla_{e_i}e_i\rangle-e_j\langle e_j, \nabla_{e_i}e_i\rangle-$$
$$-e_i\langle e_i, \nabla_{e_j}e_j\rangle-e_a\langle e_a, \nabla_{e_j}e_j\rangle-e_j\langle e_j, \nabla_{e_a}e_a\rangle+$$
$$+\langle(\nabla_{e_a}e_a)^{\mathrm{T}2}, \nabla_{e_i}e_i\rangle+\langle(\nabla_{e_i}e_i)^{\mathrm{T}1}, \nabla_{e_a}e_a\rangle+\langle(\nabla_{e_j}e_j)^{\mathrm{T}3}, \nabla_{e_i}e_i\rangle+$$
$$+\langle(\nabla_{e_i}e_i)^{\mathrm{T}1}, \nabla_{e_j}e_j\rangle+\langle(\nabla_{e_j}e_j)^{\mathrm{T}3}, \nabla_{e_a}e_a\rangle+\langle(\nabla_{e_a}e_a)^{\mathrm{T}2}, \nabla_{e_j}e_j\rangle\big)=$$
$$= -\operatorname{div} H_1-\operatorname{div} H_2-\operatorname{div} H_3-|H_1|^2-|H_2|^2-|H_3|^2-$$
$$-2\big(\langle(\nabla_{e_i}e_i)^{\mathrm{T}3}, \nabla_{e_a}e_a\rangle+\langle(\nabla_{e_i}e_i)^{\mathrm{T}2}, \nabla_{e_j}e_j\rangle+\langle(\nabla_{e_a}e_a)^{\mathrm{T}3}, \nabla_{e_j}e_j\rangle\big) =$$
$$= -\operatorname{div} H_1-\operatorname{div} H_2-\operatorname{div} H_3-|H_1|^2-|H_2|^2-|H_3|^2-$$
$$-2\big(\langle H_1, H_2\rangle+\langle H_1, H_3\rangle+\langle H_2, H_3\rangle\big).$$

Equalities (5), (12) and (13) lead us to the following

Proposition. *If D_1, D_2, D_3 are three orthogonal distributions on a Riemannian manifold M such that* $\dim D_1+\dim D_2+\dim D_3=\dim M$, *then*

$$\operatorname{div} H_1+\operatorname{div} H_2+\operatorname{div} H_3 = K(D_1, D_2)+K(D_1, D_3)+K(D_2, D_3)-$$
$$-|H_1|^2-|H_2|^2-|H_3|^2-\langle H_1, H_2\rangle-\langle H_1, H_3\rangle-\langle H_2, H_3\rangle+$$
$$+(|B_{12}|^2+|B_{13}|^2+|B_{23}|^2+|B_1|^2+|B_2|^2+|B_3|^2-$$
$$-|T_{12}|^2-|T_{13}|^2-|T_{23}|^2-|T_1|^2-|T_2|^2-|T_3|^2)/2$$

where B_n, H_n, T_n $(n=1, 2, 3)$ denote, respectively, the second fundamental forms, mean curvature vectors and integrability tensors of D_n; B_{ij}, T_{ij} $(1\leqq i\leqq j\leqq 3)$ are the second fundamental forms of $D_i\oplus D_j$.

Corollary 1. *If D_2 and D_3 are parallel, we obtain the formula*

$$\operatorname{div} H_1 = K(D_1, D_2)+K(D_1, D_3)+K(D_2, D_3)-|H_1|^2+$$
$$+(|B_{12}|^2+|B_{13}|^2+|B_{23}|^2+|B_1|^2-|T_{12}|^2-|T_{13}|^2-|T_{23}|^2-|T_1|^2)/2.$$

Corollary 2. *In the case* $\dim D_1=\dim D_2=\dim D_3=1$ we have

$$\operatorname{div} H_1+\operatorname{div} H_2+\operatorname{div} H_3 =$$
$$= K(D_1, D_2)+K(D_1, D_3)+K(D_2, D_3)-\langle H_1, H_2\rangle-\langle H_1, H_3\rangle-\langle H_2, H_3\rangle+$$
$$+(|B_{12}|^2+|B_{13}|^2+|B_{23}|^2-|T_{12}|^2-|T_{13}|^2-|T_{23}|^2)/2.$$

The following theorem results immediately from our proposition.

Theorem. *If D_1, D_2, D_3 are three orthogonal distributions on a closed oriented Riemannian manifold M such that* $\dim D_1+\dim D_2+\dim D_3=\dim M$, *then*

$$\int_M \Big(\sum_{1\le i\le j\le 3} (K(D_i, D_j)-\langle H_i, H_j\rangle+(|B_{ij}|^2-|T_{ij}|^2)/2)+$$
$$+\sum_{i=1}^{3} (-|H_i|^2+(|B_i|^2-|T_i|^2)/2)\Big)\Omega = 0$$

where Ω is the volume form on M.

Corollary 3. *If $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$ are three orthogonal foliations on a closed oriented Riemannian manifold M such that* $\dim \mathscr{F}_1+\dim \mathscr{F}_2+\dim \mathscr{F}_3=\dim M$, *then*

$$\int_M \Big(\sum_{1\le i\le j\le 3} (K(\mathscr{F}_i, \mathscr{F}_j)-\langle H_i, H_j\rangle+(|B_{ij}|^2-|T_{ij}|^2)/2)+$$
$$+\sum_{i=1}^{3} (-|H_i|^2+|B_i|^2/2)\Big)\Omega = 0.$$

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