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Equivalence systems and generalized wreath products

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1. Introduction

Wreath products of two groups have been constructed in various ways, and the different types of construction lead to different properties of the product (see, for example, [7]). Important types of construction are the complete and the restricted wreath product. If A is a group and B is a permutation group on a set X then the complete wreath product $A\,\mathrm{Wr}\,B$ is the semidirect product of A^X and B, where B acts on A^X by permuting the components. The restricted wreath product $A\,\mathrm{wr}\,B$ is the semidirect product of $D=\{a=(a_x)_{x\in X}\in A^X|a_x=1$ for all but finitely many $x\}$ with the action as above.

A generalization of the restricted wreath product to a set of permutation groups indexed by a totally ordered set was given by P. HALL [5], the same construction also works for a partially ordered index set (see, for example, [4], [8]). In the case of the complete wreath product there is more than one natural way to generalize it. One construction was given by W. CH. HOLLAND [6], a different one by CH. WELLS [9].

An equivalence system (X, E) is a pair consisting of a set X and a set E of equivalence relations on X. The automorphism group $\mathrm{Aut}\,(X, E)$ is the group of all permutations of X which leave each relation in E invariant, that is, $\mathrm{Aut}\,(X, E)= =\{g\in \mathrm{Sym}\,(X)|x\,e\,y$ if and only if $(xg)\,e\,(yg)$ for all $x, y\in X$ and $e\in E\}$. In [3], the author has shown that if (X, E) is an equivalence system with E totally ordered then $\mathrm{Aut}\,(X, E)$ is isomorphic to a generalized wreath product of full symmetric groups. In [2], the author considered equivalence systems where X is countable, E is totally ordered and $\mathrm{Aut}\,(X, E)$ is transitive on X. It should be possible to describe the automorphism groups which occur there as generalized wreath products (in a suitable sense) of full symmetric groups, however they can not be described as such products either in the sense of Holland or of Wells. This provides the motiva-

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tion to give a construction of a generalization of the complete wreath product which includes all the constructions above, and to investigate the properties of such wreath products.

2. Systematic subsets

If Λ is a partially ordered set (short: poset) and G_λ is a permutation group on a set X_λ for $\lambda\in\Lambda$ then the generalized wreath product will be a permutation group on a subset of $X=\prod_{\lambda\in\Lambda} X_\lambda$. The constructions of Holland and Wells use different subsets of X, and in this paper we shall see that there is a still greater choice of suitable subsets. It is, however, sensible to demand that all constructions should give the same group if the index set Λ is finite, and also that an associative law like Theorem 3.8 in [6] holds. This gives a certain restriction on the kind of subset of X which we shall consider.

A subset Σ of a poset Λ is called an ideal if whenever $\sigma\in\Sigma$ and $\lambda\in\Lambda$ such that $\lambda\leqq\sigma$ then $\lambda\in\Sigma$. The dual concept is called a filter. Note that the complement of any ideal is a filter and vice versa. In the rest of this section, let Λ be a poset, let X_λ be a non-empty set for $\lambda\in\Lambda$, and let $X=\prod_{\lambda\in\Lambda} X_\lambda$. A non-empty subset S of X is called systematic if the following two conditions hold. (1) For every ideal Σ of Λ and for all $x,y\in S$ if $z\in X$ is defined by $z_\lambda=x_\lambda$ if $\lambda\in\Sigma$ and $z_\lambda=y_\lambda$ if $\lambda\notin\Sigma$ then $z\in S$. (2) For all $\lambda\in\Lambda$ and all $r\in X_\lambda$ there exists $x\in S$ such that $x_\lambda=r$. A systematic subset S of X is called strongly systematic if (1) holds for any subset Σ of Λ.

A subset Φ of Λ is called convex if whenever $\varphi_1,\varphi_2\in\Phi$ and $\lambda\in\Lambda$ such that $\varphi_1\leqq\lambda\leqq\varphi_2$ then $\lambda\in\Phi$. A non-empty subset Φ of Λ is called an order block if for all $\lambda\in\Lambda\setminus\Phi$ we have either $\lambda>\varphi$ for all $\varphi\in\Phi$ or $\lambda<\varphi$ for all $\varphi\in\Phi$ or λ and φ are incomparable for all $\varphi\in\Phi$. It is easy to see that ideals, filters and order blocks are convex. We now show that in the definition of a systematic set we could replace (1) by a stronger condition.

Lemma 2.1. *Let $S\subseteqq X$ be systematic, $\Phi\subseteqq\Lambda$ convex, and let $x,y\in S$. If $z\in X$ is defined by $z_\lambda=x_\lambda$ for $\lambda\in\Phi$ and $z_\lambda=y_\lambda$ for $\lambda\notin\Phi$ then $z\in S$.*

Proof. Let $\Sigma_1=\{\lambda\in\Lambda \mid \text{there exists } \varphi\in\Phi \text{ such that } \lambda\leqq\varphi\}$, and let $\Sigma_2=\Sigma_1\setminus\Phi$. We claim that Σ_1, Σ_2 are ideals. This is trivial for Σ_1. So let $\sigma\in\Sigma_2$, $\lambda\in\Lambda$ such that $\lambda\leqq\sigma$. Clearly $\lambda\in\Sigma_1$, and there exists $\varphi\in\Phi$ such that $\sigma\leqq\varphi$. Suppose that $\lambda\in\Phi$. Then we have $\lambda\leqq\sigma\leqq\varphi$, and as Φ is convex it follows that $\sigma\in\Phi$, giving a contradiction. Hence $\lambda\in\Sigma_1\setminus\Phi=\Sigma_2$, which proves the claim. Now let $z'\in X$ be defined by $z'_\lambda=x_\lambda$ if $\lambda\in\Sigma_1$ and $z'_\lambda=y_\lambda$ if $\lambda\notin\Sigma_1$. Then $z'\in S$. Now note that $z_\lambda=y_\lambda$ if $\lambda\in\Sigma_2$ and $z_\lambda=z'_\lambda$ if $\lambda\notin\Sigma_2$. Therefore $z\in S$.

Lemma 2.2. *Let* $S \subseteqq X$ *be systematic, and let* $x \in S$. *Then if* $x' \in X$ *is such that* $x'_\lambda = x_\lambda$ *for all but finitely many* $\lambda \in \Lambda$ *then* $x' \in S$.

This follows easily from Lemma 2.1 and condition (2) in the definition. As a consequence of Lemma 2.2 we immediately get

Lemma 2.3. *Let* $x \in X$. *Then* $S(x) := \{x' \in X \mid x'_\lambda = x_\lambda$ for all but finitely many $\lambda \in \Lambda\}$ *is strongly systematic, and it is the unique minimal systematic subset of* X *containing* x.

Lemma 2.4. *An intersection of systematic subsets is either empty or systematic.*

The proof of this shall be left to the reader. Note that we can define the join of any collection of systematic subsets as the intersection of all systematic subsets which contain all members of the collection (this is well defined as obviously X itself is systematic). Then the set of all systematic subsets together with the empty set forms a complete lattice under set-theoretic intersection and the join as defined.

If Φ is any subset of the poset Λ and X_λ are non-empty sets for $\lambda \in \Lambda$ then there exists a canonic projection p_Φ from $X = \prod_{\lambda \in \Lambda} X_\lambda$ onto $\prod_{\lambda \in \Phi} X_\lambda$ where $p_\Phi(x) = (x_\lambda)_{\lambda \in \Phi}$. Note that if P is a partition of Λ then we get a natural bijection $X \to \prod_{\Phi \in P} \prod_{\lambda \in \Phi} X_\lambda$ by $x \mapsto (p_\Phi(x))_{\Phi \in P}$. With this notation we can see that a subset S of X is systematic if and only if the set S is mapped onto $p_\Phi(S) \times p_{\Lambda \setminus \Phi}(S)$ for every convex subset $\Phi \subseteqq \Lambda$ under the natural bijection $X \to p_\Phi(X) \times p_{\Lambda \setminus \Phi}(X)$ and $p_{\{\lambda\}}(S) = X_\lambda$ for all $\lambda \in \Lambda$. Also S is strongly systematic if and only if the above holds for every subset $\Phi \subseteqq \Lambda$ and $p_{\{\lambda\}}(S) = X_\lambda$ for all $\lambda \in \Lambda$.

3. Definition and elementary properties of the wreath product

Let Λ be a poset, let G_λ be a permutation group on a non-empty set X_λ for $\lambda \in \Lambda$, and let S be a systematic subset of $X = \prod_{\lambda \in \Lambda} X_\lambda$. For all $\lambda \in \Lambda$ we define equivalence relations $e(\lambda)$ and $e_L(\lambda)$ on S in the following way. Whenever $x, y \in S$ we have $x\, e(\lambda)\, y$ if and only if $x_\mu = y_\mu$ for all $\mu > \lambda$ and $x\, e_L(\lambda)\, y$ if and only if $x_\mu = y_\mu$ for all $\mu \geqq \lambda$. We let $E = \{e(\lambda), e_L(\lambda) \mid \lambda \in \Lambda\}$. Then we define the wreath product $S-\mathrm{WR}_{\lambda \in \Lambda} G_\lambda$ by $S-\mathrm{WR}_{\lambda \in \Lambda} G_\lambda = \{g \in \mathrm{Aut}\,(S, E) \mid$For all $x \in S$ and $\lambda \in \Lambda$ there exists $g_{\lambda,x} \in G_\lambda$ such that $(x'g)_\lambda = x' g_{\lambda,x}$ for all $x' \in S$ with $x'\, e(\lambda)\, x\}$.

Note that it is easy to see that $S-\mathrm{WR}_{\lambda \in \Lambda} G_\lambda$ is, in fact, a group, and that $(g^{-1})_{\lambda,x} = (g_{\lambda,xg^{-1}})^{-1}$, and $(gh)_{\lambda,x} = g_{\lambda,x} h_{\lambda,xg}$. Also note that $\mathrm{Aut}\,(S, E) = S-\mathrm{WR}_{\lambda \in \Lambda} \mathrm{Sym}\,(X_\lambda)$.

We say that a subset Φ of a poset Λ satisfies the maximal condition if every non-empty subset of Φ has a maximal element. If X_λ is a non-empty set for $\lambda \in \Lambda$

and $x \in X = \prod_{\lambda \in \Lambda} X_\lambda$ then let $H(x) = \{y \in X \mid \{\lambda \in \Lambda \mid y_\lambda \neq x_\lambda\}$ satisfies the maximal condition$\}$. It is not hard to see that $H(x)$ is strongly systematic. Then Holland's wreath products [6] are just the products $H(x) - \mathrm{WR}_{\lambda \in \Lambda} G_\lambda$. Wells's wreath products [9] are the products $X - \mathrm{WR}_{\lambda \in \Lambda} G_\lambda$, and the groups studied by the author in [2] are of the form $S(x) - \mathrm{WR}_{\lambda \in \Lambda} \mathrm{Sym}(X_\lambda)$. Therefore the construction we have given generalizes all those wreath products. Note that if Λ is finite then by Lemma 2.3 we get $S(x) = X$ for all $x \in X$, thus there is only one systematic subset, and hence only one wreath product. In particular, if $\Lambda = \{1, 2\}$ with the natural order, we get the ordinary complete wreath product.

4. The associative law

We shall now investigate in which way wreath products over large index sets Λ can be put together from products over certain subsets of Λ. In order to do this we need more properties of systematic subsets.

Lemma 4.1. *Let S be a systematic subset of X, and let Φ be a subset of Λ. Then $p_\Phi(S)$ is a systematic subset of $p_\Phi(X) = \prod_{\lambda \in \Phi} X_\lambda$.*

Proof. Let Σ be an ideal of Φ, and let $x, y \in p_\Phi(S)$, let $z \in p_\Phi(X)$ be defined by $z_\lambda = x_\lambda$ if $\lambda \in \Sigma$ and $z_\lambda = y_\lambda$ if $\lambda \notin \Sigma$. Now let $\bar{x}, \bar{y} \in S$ such that $x = p_\Phi(\bar{x})$, $y = p_\Phi(\bar{y})$, and let $\bar{\Sigma} = \{\lambda \in \Lambda \mid$there exists $\sigma \in \Sigma$ such that $\lambda \leqq \sigma\}$. Clearly $\bar{\Sigma}$ is an ideal of Λ. Now let $\bar{z} \in X$ be defined by $\bar{z}_\lambda = \bar{x}_\lambda$ if $\lambda \in \bar{\Sigma}$ and $\bar{z}_\lambda = \bar{y}_\lambda$ if $\lambda \notin \bar{\Sigma}$. Then $\bar{z} \in S$. It remains to prove that $p_\Phi(\bar{z}) = z$. Let $\lambda \in \Phi$. First suppose that $\lambda \notin \bar{\Sigma}$. Then also $\lambda \notin \Sigma$, and we have $\bar{z}_\lambda = \bar{y}_\lambda = y_\lambda = z_\lambda$. Now suppose that $\lambda \in \bar{\Sigma}$. Then there exists $\sigma \in \Sigma$ such that $\lambda \leqq \sigma$. But $\lambda \in \Phi$ and Σ is an ideal of Φ, hence $\lambda \in \Sigma$. But then $\bar{z}_\lambda = \bar{x}_\lambda = x_\lambda = z_\lambda$. Therefore we have $z = p_\Phi(\bar{z})$, and $p_\Phi(S)$ is systematic, as condition (2) is trivially fulfilled.

Lemma 4.2. *Let P be a partition of a poset Λ into order blocks. Let $\Phi_1, \Phi_2 \in P$. Then the following are equivalent:*

(i) *There exist $\varphi_1 \in \Phi_1$, $\varphi_2 \in \Phi_2$ such that $\varphi_1 \leqq \varphi_2$.*

(ii) *Either $\Phi_1 = \Phi_2$ or for all $\varphi_1 \in \Phi_1$, $\varphi_2 \in \Phi_2$ we have $\varphi_1 < \varphi_2$.*

The proof of this is easy. As a consequence, we can define a partial order on P in the following way. If $\Phi_1, \Phi_2 \in P$ then $\Phi_1 \leqq \Phi_2$ if and only if there exist $\varphi_1 \in \Phi_1$, $\varphi_2 \in \Phi_2$ such that $\varphi_1 \leqq \varphi_2$.

Lemma 4.3. *Let P be a partition of Λ into order blocks, and let S be a systematic subset of X. Then $\bar{S} := \{(p_\Phi(x))_{\Phi \in P} \mid x \in S\}$ is a systematic subset of $\bar{X} := \prod_{\Phi \in P} p_\Phi(S)$.*

Proof. Let I be an ideal of P, let $x, y \in \bar{S}$, and let $z \in \bar{X}$ be defined by $z_\Phi = x_\Phi$ for $\Phi \in I$ and $z_\Phi = y_\Phi$ for $\Phi \notin I$. Now let $x', y' \in S$ such that $x_\Phi = p_\Phi(x')$ and $y_\Phi = p_\Phi(y')$ for all $\Phi \in P$. Note that $\Sigma := \{\lambda \mid \text{there exists } \Phi \in I \text{ such that } \lambda \in \Phi\}$ is an ideal of Λ, and if $z' \in X$ is defined by $z'_\lambda = x'_\lambda$ if $\lambda \in \Sigma$ and $z'_\lambda = y'_\lambda$ if $\lambda \notin \Sigma$ then $z' \in S$. But then clearly $z_\Phi = p_\Phi(z')$ for all $\Phi \in P$, and hence $z \in \bar{S}$, thus we have condition (1). Condition (2) is satisfied by definition.

Theorem 4.4. *Let P be a partition of a poset Λ into order blocks. For $\lambda \in \Lambda$ let G_λ be a permutation group on a non-empty set X_λ, and let S be a systematic subset of $X = \prod_{\lambda \in \Lambda} X_\lambda$. Let $\bar{S} = \{(p_\Phi(x))_{\Phi \in P} \mid x \in S\}$. Then $S - \mathrm{WR}_{\lambda \in \Lambda} G_\lambda$ is permutationally isomorphic to $\bar{S} - \mathrm{WR}_{\Phi \in P}(p_\Phi(S) - \mathrm{WR}_{\lambda \in \Phi} G_\lambda)$.*

Proof. We have a partial order on P by Lemma 4.2. The set $p_\Phi(S)$ is a systematic subset of $\prod_{\lambda \in \Phi} X_\lambda$ by Lemma 4.1, and $\bar{S}$ is a systematic subset of $\bar{X} = \prod_{\Phi \in P} p_\Phi(S)$ by Lemma 4.3. Therefore the group $\bar{G} = \bar{S} - \mathrm{WR}_{\Phi \in P}(p_\Phi(S) - \mathrm{WR}_{\lambda \in \Phi} G_\lambda)$ is well-defined. Let $\beta \colon X \to \prod_{\Phi \in P} \prod_{\lambda \in \Phi} X_\lambda$ be the natural bijection, note that $\beta(S) = \bar{S}$, and denote $G = S - \mathrm{WR}_{\lambda \in \Lambda} G_\lambda$. We define a mapping $\varphi \colon G \to \mathrm{Sym}(\bar{S})$ by $y(g\varphi) = y\beta^{-1}g\beta$. Clearly φ is a monomorphism and G is permutationally isomorphic to its image under φ. Thus it remains to prove that $\varphi(G) = \bar{G}$.

It is easy to see that $g\varphi$ leaves the relations $e(\Phi)$ and $e_L(\Phi)$ invariant for $\Phi \in P$ and $g \in G$. Note that if $x \in S$, $g \in G$ then $(x\beta)(g\varphi) = xg\beta$.

Let $\beta(x) \in \bar{S}$, and let $\Phi \in P$. We claim that there exists $g_{\Phi,x} \in p_\Phi(S) - \mathrm{WR}_{\lambda \in \Phi} G_\lambda$ such that $p_\Phi(x'g) = p_\Phi(x')g_{\Phi,x}$ for all $x' \in S$ with $p_\Phi(x')e(\Phi)p_\Phi(x)$. We define $g_{\Phi,x}$ in the following way. Let $y \in p_\Phi(S)$, and let $\bar{y} \in S$ be such that $y = p_\Phi(\bar{y})$. Then from Lemma 2.1 it follows that if $z_x(y) \in X$ is defined by $z_x(y)_\lambda = \bar{y}_\lambda$ for $\lambda \in \Phi$ and $z_x(y)_\lambda = x_\lambda$ for $\lambda \notin \Phi$ then $z_x(y) \in S$ and $p_\Phi(z_x(y)) = y$. Also note that $z_x(y)$ is uniquely defined with respect to this property and independent of the choice of $\bar{y}$. Then let $yg_{\Phi,x} = p_\Phi(z_x(y)g)$. Now let $\lambda \in \Phi$ and $y' \in p_\Phi(S)$ such that $y'\,e(\lambda)\,y$ (where the relation $e(\lambda)$ is taken in $p_\Phi(S)$). Then let $\bar{g}_{\lambda,y} := g_{\lambda, z_x(y)}$. We then note that we get $(y'g_{\Phi,x}) = y'\bar{g}_{\lambda,y}$. Hence we have $g_{\Phi,x} \in p_\Phi(S) - \mathrm{WR}_{\lambda \in \Phi} G_\lambda$, and we have $p_\Phi(x'g) = p_\Phi(x')g_{\Phi,x}$, which establishes the claim.

Therefore φ maps G into $\bar{G}$, and it remains to prove that φ is surjective. Let $\bar{g} \in \bar{G}$. If $x \in X$ then we define g by $xg := x\beta\bar{g}\beta^{-1}$. Then clearly $g \in \mathrm{Sym}(X)$. Now let $\lambda \in \Lambda$ and $x, x' \in S$. Let $\Phi \in P$ be such that $\lambda \in \Phi$. Then $\{\mu \in \Lambda \mid \mu > \lambda\} = \{\mu \in \Lambda \mid \text{there exists } \Phi' \in P,\ \Phi' > \Phi \text{ and } \mu \in \Phi'\} \cup \{\mu \in \Phi \mid \mu > \lambda\}$. By definition of $\bar{G}$, it follows that $x_\mu = x'_\mu$ is equivalent to $(xg)_\mu = (x'g)_\mu$ for all $\mu \in \Lambda$ such that there exists $\Phi' \in P$, $\Phi' > \Phi$ with $\mu \in \Phi'$. But then by definition of $p_\Phi(S) - \mathrm{WR}_{\lambda \in \Phi} G_\lambda$ we get that $x_\mu = x'_\mu$ is equivalent to $(xg)_\mu = (x'g)_\mu$ for all $\mu \in \Phi$ with $\mu > \lambda$. Therefore g leaves $e(\lambda)$ invariant. Similarly, it also leaves $e_L(\lambda)$ invariant. Now let $x \in S$, $\lambda \in \Lambda$. We finally have to show that there exists $g_{\lambda,x} \in G_\lambda$ such that

$(x'g)_\lambda = x'g_{\lambda,x}$ for all $x' \in S$ with $x' e(\lambda) x$. We know that if $\Phi \in P$ such that $\lambda \in \Phi$ then there exists $g_{\Phi,x} \in p_\Phi(S) - \mathrm{WR}_{\lambda \in \Phi} G_\lambda$ such that $(x'g\beta)_\Phi = p_\Phi(x')g_{\Phi,x}$ for all $x'\beta\, e(\Phi)\, x\beta$ (note that $x' e(\lambda) x$ implies $x'\beta\, e(\Phi)\, x\beta$). But then we know that there exists $g_{\lambda,x} \in G_\lambda$ such that $(y'g_{\Phi,x})_\lambda = y'_\lambda g_{\lambda,x}$ for all $y' \in p_\Phi(S)$ with $y' e(\lambda) p_\Phi(x)$ (where $e(\lambda)$ is the relation in $p_\Phi(S)$). But note that if $x' e(\lambda) x$ (with $e(\lambda)$ in S) then also $p_\Phi(x') e(\lambda) p_\Phi(x)$ (with $e(\lambda)$ in $p_\Phi(S)$). Together we get that if $x' e(\lambda) x$ then $(x'g)_\lambda = p_\Phi(x')_\lambda g_{\lambda,x} = x'_\lambda g_{\lambda,x}$, which proves the theorem.

5. Embeddings and transitivity

We now want to see which wreath products are transitive. We first consider products of full symmetric groups.

Theorem 5.1. *Let Λ be a partially ordered set, let X_λ be a non-empty set for $\lambda \in \Lambda$, and let S be a strongly systematic subset of $X = \prod_{\lambda \in \Lambda} X_\lambda$. Then* $\mathrm{Aut}(S, E)$ *is transitive on S.*

Proof. Let $x, y \in S$. Define $g \colon S \to X$ in the following way. If $z \in S$ then

$$(zg)_\lambda = \begin{cases} y_\lambda & \text{if} \quad z_\lambda = x_\lambda \\ x_\lambda & \text{if} \quad z_\lambda = y_\lambda \\ z_\lambda & \text{otherwise.} \end{cases}$$

First note that g maps S into S, as S is strongly systematic. Next, clearly g^2 is the identity on S, and hence $g \in \mathrm{Sym}(S)$. Finally, it is obvious that g leaves the equivalence relations $e(\lambda)$ and $e_L(\lambda)$ invariant, hence $g \in \mathrm{Aut}(S, E)$. As $xg = y$, we get the transitivity of $\mathrm{Aut}(S, E)$.

Clearly for the transitivity of the wreath product of groups G_λ it is necessary that all G_λ are transitive. This, however, is not a sufficient condition.

Proposition 5.2. *There exist a poset Λ and transitive permutation groups G_λ on sets X_λ ($\lambda \in \Lambda$), and a strongly systematic subset S of $X = \prod_{\lambda \in \Lambda} X_\lambda$ such that $G := S - \mathrm{WR}_{\lambda \in \Lambda} G_\lambda$ is not transitive on S.*

Proof. Let $\Lambda = \mathbf{Z}$ with the trivial order (i.e. any two distinct elements are incomparable), let $X_\lambda = \{0, 1, 2\}$ and let G_λ be generated by the cyclic permutation (012) for $\lambda \in \Lambda$. Let $S = \{x \in \prod_{\lambda \in \Lambda} X_\lambda \mid x_\lambda \neq 2$ for all but finitely many $\lambda\}$. Let $u, v \in S$ be defined by $u_\lambda = 0$ and $v_\lambda = 1$ for all $\lambda \in \Lambda$. Suppose that G is transitive on S. Then there exists $g \in G$ such that $ug = v$. Now note that $(xg)_\lambda = x_\lambda g_{\lambda,u}$ for all $x \in S$ with $x\, e(\lambda)\, u$. In particular, we have $1 = v_\lambda = (ug)_\lambda = u_\lambda g_{\lambda,u} = 0 g_{\lambda,u}$. But then we must have $g_{\lambda,u} = (012)$ for all $\lambda \in \Lambda$. Now we also have $v\, e(\lambda)\, u$ for all $\lambda \in \Lambda$.

Hence it follows that $(vg)_\lambda = v_\lambda g_{\lambda,u} = 1(012) = 2$ for all $\lambda \in \Lambda$. But then $vg \notin S$, which is a contradiction. Therefore G is not transitive.

In contrast to this result, for some systematic subsets S the wreath product is always transitive whenever all groups G_λ are transitive. This is trivial for $S=X$, it holds for $S=H(x)$ (Thm. 3.9 in [6]), and also for $S=S(x)$ (as then the restricted wreath product is a subgroup which is already transitive). If S and T are systematic subsets with $S \subseteq T$ then it is natural to ask if the wreath product constructed on S can be embedded in a natural way into the wreath product constructed on T. However, this does not need to be the case in general.

Proposition 5.3. *There exists a poset Λ, non-empty sets X_λ for $\lambda \in \Lambda$ and systematic subsets $S, T \subseteq X = \prod_{\lambda \in \Lambda} X_\lambda$ with $S \subseteq T$ such that there does not exist a monomorphism φ:* $\mathrm{Aut}(S, E) \to \mathrm{Aut}(T, E)$ *such that $x(g\varphi) = xg$ for all $x \in S$, $g \in$* $\mathrm{Aut}(S, E)$.

Proof. Let $\Lambda = \mathbf{Z}$ with its natural order, let $X_z = \{0, 1, 2\}$ for $z \in \mathbf{Z}$. Let $X = \prod_{z \in \mathbf{Z}} X_z$, and let $S = \{x \in X \mid \text{there exist } z_U, z_L \in \mathbf{Z} \text{ such that } x_z = x_{z_U} \text{ for } z \geqq z_U$ and $x_z = x_{z_L}$ for $z \leqq z_L\}$, and $T = \{x \in X \mid \text{there exist } z_U, z_L \in \mathbf{Z} \text{ such that } x_z = x_{z_U}$ for $z \geqq z_U$ and $x_z = 2$ for $z \leqq z_L$ or $x_z \in \{0, 1\}$ for $z \leqq z_L\}$. Clearly, the sets S and T are both systematic. Let $G = \mathrm{Aut}(S, E)$ and $H = \mathrm{Aut}(T, E)$. Let $g \in G$ be defined by $(xg)_z = x_z(012)$ for all $x \in S$, $z \in \mathbf{Z}$. First note that, in fact, we have $g \in G$.

Suppose there exists a monomorphism $\varphi\colon G \to H$ such that $x(g\varphi) = xg$ for all $x \in S$, $g \in G$. Now let $x \in X$ be defined in the following way. Let $x_z = 1$ if $z < 0$ and $z \equiv 1 \pmod 2$ and let $x_z = 0$ otherwise. Note that $x \in T$. For $m \in \mathbf{Z}$ define $y(m) \in X$ in the following way. Let $y(m)_z = x_z$ if $z \geqq m$ and $y(m)_z = 0$ if $z < m$. Then clearly $y(m) \in S$ for all $m \in \mathbf{Z}$. Now $y(m)\, e(m-1)\, x$. Next note that with g defined as above we have $(y(m)g)_m = y(m)_m(012)$. Hence we must have $(x(g\varphi))_m = (y(m)(g\varphi))_m = (y(m)g)_m = y(m)_m(012)$ for all $m \in \mathbf{Z}$. But then $(x(g\varphi))_m = 2$ if $m < 0$ and $m \equiv 1 \pmod 2$ and $(x(g\varphi))_m = 1$ otherwise. But then $x(g\varphi) \notin T$, which is a contradiction.

6. Some wreath products of full symmetric groups

In this section we shall show how some wreath products of full symmetric groups can be decomposed into simpler products.

Lemma 6.1. *Let Λ be a poset, let X_λ be a non-empty set for $\lambda \in \Lambda$, and let $x \in X = \prod_{\lambda \in \Lambda} X_\lambda$. Let $M(x) = \{y \in X \mid$*for all $\lambda \in \Lambda$ there exists $\mu \in \Lambda$ with $\mu \geqq \lambda$ such that $y_\gamma = x_\gamma$ for all $\gamma > \mu\}$. *Then $M(x)$ is a systematic subset of X.*

The proof of this is similar to some proofs already given and shall therefore be omitted. Clearly, if $y\in M(x)$ then $M(y)=M(x)$, hence these sets $M(x)$ form a partition of X. We recall that a poset Λ is called upper directed if for $\lambda, \lambda'\in\Lambda$ there exists $\mu\in\Lambda$ such that $\mu\geqq\lambda$ and $\mu\geqq\lambda'$.

Theorem 6.2. *Let Λ be an upper directed poset and X_λ a non-empty set for $\lambda\in\Lambda$. Let S be a strongly systematic subset of $X=\prod_{\lambda\in\Lambda} X_\lambda$, let $C=\{M(y)|y\in S\}$ and $x\in S$. Then* $\mathrm{Aut}\,(S,E)$ *is permutationally isomorphic to* $\mathrm{Aut}\,(S\cap M(x),E)\,\mathrm{Wr}\,\mathrm{Sym}\,(C)$.

Proof. For $y\in S$ we define $M_S(y)=M(y)\cap S$. Choose a subset R of S with $x\in R$ and such that $C=\{M(r)|r\in R\}$ and $|R\cap M(y)|=1$ for all $y\in S$. For each $r\in R$ we define a mapping $\alpha_{M(r)}$: $M_S(x)\to M_S(r)$ in the following way. Let $z\in M_S(x)$. Then

$$(z\alpha_{M(r)})_\lambda=\begin{cases} r_\lambda & \text{if } z_\lambda=x_\lambda\\ x_\lambda & \text{if } z_\lambda=r_\lambda\\ z_\lambda & \text{otherwise.}\end{cases}$$

We first have to show that $\alpha_{M(r)}$ maps $M_S(x)$ indeed into $M_S(r)$. It is clear that $z\alpha_{M(r)}\in S$ as S is strongly systematic. So let $\lambda\in\Lambda$. As $z\in M(x)$ there exists $\mu\in\Lambda$ with $\mu\geqq\lambda$ such that $z_\gamma=z_\gamma$ for all $\gamma>\mu$. But then we have $(z\alpha_{M(r)})_\gamma=r_\gamma$ for all $\gamma>\mu$, and hence $z\alpha_{M(r)}\in M(r)$, and also $z\alpha_{M(r)}\in M_S(r)$. We now claim that $\alpha_{M(r)}$ is a bijection. It is easy to see that it is injective. Let $s\in M_S(r)$. Define $\bar{s}\in X$ by

$$\bar{s}_\lambda=\begin{cases} r_\lambda & \text{if } s_\lambda=x_\lambda\\ x_\lambda & \text{if } s_\lambda=r_\lambda\\ s_\lambda & \text{otherwise.}\end{cases}$$

Then, as above, it follows that $\bar{s}\in M_S(x)$, and it is clear that $\bar{s}\alpha_{M(r)}=s$, which establishes the claim.

Let $\bar{S}=M_S(x)\times C$. We define a mapping α: $\bar{S}\to S$ by $(z,M(y))\alpha=z\alpha_{M(y)}$ for $z\in M_S(x), M(y)\in C$. We note that α is bijective. Let $X_\tau:=C$ and let $\bar{\Lambda}=\Lambda\cup\{\tau\}$ where $\tau>\lambda$ for all $\lambda\in\Lambda$. Then $\bar{S}$ is a strongly systematic subset of $\bar{X}:=\prod_{\lambda\in\bar{\Lambda}} X_\lambda$: Let $\bar{E}$ be the set of relations induced by E on $M_S(x)$ together with $e(\tau)$ and $e_L(\tau)$. Then by Theorem 4.4 it follows that $\mathrm{Aut}\,(\bar{S},\bar{E})$ is permutationally isomorphic to $\mathrm{Aut}\,(M_S(x))\,\mathrm{Wr}\,\mathrm{Sym}\,(C)$. So it remains to prove that $\mathrm{Aut}\,(\bar{S},\bar{E})$ and $\mathrm{Aut}\,(S,E)$ are permutationally isomorphic.

We define a mapping a: $\mathrm{Aut}\,(S,E)\to\mathrm{Sym}\,(\bar{S})$ in the following way. If $\sigma\in\mathrm{Aut}\,(S,E)$, $(z,M(y))\in\bar{S}$ then $(z,M(y))(\sigma a)=(z,M(y))\alpha\sigma\alpha^{-1}$. Note that it is clear that a is a monomorphism and that $\mathrm{Aut}\,(S,E)$ is permutationally isomorphic to its image under a. Thus all we have to prove now is that $\mathrm{Aut}\,(\bar{S},\bar{E})$ is equal to the image of $\mathrm{Aut}\,(S,E)$ under a.

First we want to show that $\sigma a \in \mathrm{Aut}(\bar{S}, \bar{E})$ for all $\sigma \in \mathrm{Aut}(S, E)$. Let $\lambda \in \Lambda$ and let $(z, M(y))\, e(\lambda)\, (z', M(y'))$. It follows that $M(y)=M(y')$ and also $z\, e(\lambda)\, z'$. Therefore we get $(z\alpha_{M(y)})\, e(\lambda)\, (z'\alpha_{M(y)})$ and $(z\alpha_{M(y)}\sigma)\, e(\lambda)\, (z'\alpha_{M(y)}\sigma)$. As Λ is upper directed, it follows that $M(z\alpha_{M(y)}\sigma)=M(z'a_{M(y)}\sigma)$, and hence we also get $(z\alpha_{M(y)}\sigma\alpha^{-1}_{M(z\alpha_{M(y)}\sigma)})\, e(\lambda)\, (z'\alpha_{M(y)}\sigma\alpha^{-1}_{M(z'\alpha_{M(y)}\sigma)})$, and hence

$$((z, M(y))(\sigma a))\, e(\lambda)\, ((z', M(y'))(\sigma a)).$$

The converse, and the result for $e_L(\lambda)$ follow similarly. Note that $e(\tau)$ is the universal relation. Let $(z, M(y))\, e_L(\tau)\, (z', M(y'))$. This means that $M(y)=M(y')$, and as above we get $M(z\alpha_{M(y)}\sigma)=M(z'\alpha_{M(y)}\sigma)$, and hence

$$((z, M(y))(\sigma a))\, e_L(\tau)\, ((z', M(y'))(\sigma a)).$$

Again, the converse follows similarly, and hence $\sigma a \in \mathrm{Aut}(\bar{S}, \bar{E})$.

Finally we have to show that a is surjective. Let $\varrho \in \mathrm{Aut}(\bar{S}, \bar{E})$. Then we have to prove that $\alpha^{-1}\varrho\alpha \in \mathrm{Aut}(S, E)$. Let $z, z' \in S$, $\lambda \in \Lambda$ with $z\, e(\lambda)\, z'$. As Λ is upper directed, we have $M(z)=M(z')$, and as $z\alpha^{-1}=(z\alpha^{-1}_{M(z)}, M(z))$, we have $z\alpha^{-1}\, e(\lambda)\, z'\alpha^{-1}$, and therefore $z\alpha^{-1}\varrho\, e(\lambda)\, z'\alpha^{-1}\varrho$. Then again it follows that $z\alpha^{-1}\varrho\alpha\, e(\lambda)\, z'\alpha^{-1}\varrho\alpha$. The converse follows similarly, and so does the result for $e_L(\lambda)$. Therefore a is surjective, which concludes the proof of the theorem.

7. The normal structure of wreath products

We recall that the set of all ideals of a poset Λ is a complete distributive lattice with respect to set-theoretic intersection and union. If S is a systematic subset of $X=\prod_{\lambda\in\Lambda} X_\lambda$ and Σ is an ideal of Λ then we can define an equivalence relation $e(\Sigma)$ on S by $x\, e(\Sigma)\, x'$ if and only if $x_\lambda = x'_\lambda$ for all $\lambda \notin \Sigma$. Note that $e(\Sigma)$ is the infimum over all the relations $e_L(\lambda)$ with $\lambda \notin \Sigma$.

Proposition 7.1. *Let Λ be a poset and G_λ a permutation group on a non-empty set X_λ for $\lambda \in \Lambda$. Let S be a systematic subset of $X=\prod_{\lambda\in\Lambda} X_\lambda$, and let $G=S-\mathrm{WR}_{\lambda\in\Lambda}\, G_\lambda$. For every ideal Σ of Λ let $D(\Sigma)=\{g\in G \mid x\, e(\Sigma)\, xg$ for all $x \in S\}$. Then $D(\Sigma)$ is a normal subgroup of G and the mapping $\Sigma \mapsto D(\Sigma)$ is a monomorphism from the lattice of ideals of Λ into the normal subgroup lattice of G preserving arbitrary meets and finite joins.*

Proof. Trivially, $D(\Sigma)$ is a subgroup of G. Let $h \in G$, $g \in D(\Sigma)$. Note that h leaves all relations $e_L(\lambda)$ invariant, and hence also the relation $e(\Sigma)$. So if $x \in S$ then $(xh^{-1})\, e(\Sigma)\, (xh^{-1})g$, and hence also $(xh^{-1})h\, e(\Sigma)\, ((xh^{-1})g)h$, therefore

$x\,e(\Sigma)\,x(h^{-1}gh)$, and $h^{-1}gh \in D(\Sigma)$. Note that it is also trivial that if $\Sigma \subseteq \Sigma'$ then $D(\Sigma) \leqq D(\Sigma')$.

Let Σ_i $(i \in I)$ be a set of ideals. Then we have $D(\bigcap_{i\in I} \Sigma_i) \leqq D(\Sigma_j)$ for all $j \in I$, and hence $D(\bigcap_{i\in I} \Sigma_i) \leqq \bigcap_{i\in I} D(\Sigma_i)$. Conversely, let $g \in \bigcap_{i\in I} D(\Sigma_i)$. Then for all $i \in I$, $x \in S$ and all $\lambda \notin \Sigma_i$ we have $x_\lambda = (xg)_\lambda$. Hence, for all $x \in S$ and all $\lambda \notin \Sigma_i$ we have $x_\lambda = (xg)_\lambda$, therefore $x\,e(\bigcap_{i\in I} \Sigma_i)\,xg$ for all $x \in S$, and $g \in D(\bigcap_{i\in I} \Sigma_i)$.

Let Σ_1, Σ_2 be ideals and $g \in D(\Sigma_1 \cup \Sigma_2)$. We define $h, h'\colon S \to X$ in the following way. If $x \in S$, $\lambda \in \Lambda$ then $(xh)_\lambda = x_\lambda g_{\lambda,x}$ if $\lambda \in \Sigma_1 \setminus \Sigma_2$ and $(xh)_\lambda = x_\lambda$ otherwise. Also $(xh')_\lambda = x_\lambda g_{\lambda, xh^{-1}}$ if $\lambda \in \Sigma_2$ and $(xh')_\lambda = x_\lambda$ otherwise. First we claim that $h \in \mathrm{Sym}(S)$. Note that $(xh)_\lambda = (xg)_\lambda$ if $\lambda \in \Sigma_1 \setminus \Sigma_2$ and $(xh)_\lambda = x_\lambda$ otherwise, hence $xh \in S$, as S is systematic. Also h is clearly injective. Let $y \in S$, and let $y' \in X$ be defined by $y'_\lambda = (yg^{-1})_\lambda$ if $\lambda \in \Sigma_1 \setminus \Sigma_2$ and $y'_\lambda = y_\lambda$ otherwise. Then $y' \in S$, and $y'h = y$. Thus $h \in \mathrm{Sym}(S)$. Furthermore, it is not hard to see that $h \in D(\Sigma_1)$. Note that $h_{\lambda,x} = 1$ if $\lambda \notin \Sigma_1 \setminus \Sigma_2$ and $h_{\lambda,x} = g_{\lambda,x}$ if $\lambda \in \Sigma_1 \setminus \Sigma_2$.

Next we show that $h' \in \mathrm{Sym}(S)$. For this, we observe that $(xh')_\lambda = (xh^{-1}g)_\lambda$ if $\lambda \in \Sigma_2$ and $(xh')_\lambda = x_\lambda$ otherwise, hence we have $xh' \in S$. As above, it follows that $h' \in D(\Sigma_2)$, and note that $h'_{\lambda,x} = 1$ if $\lambda \notin \Sigma_2$ and $h'_{\lambda,x} = g'_{\lambda, xh^{-1}}$ if $\lambda \in \Sigma_2$.

Finally, we show that $g = hh'$. Let $x \in S$, $\lambda \in \Lambda$. If $\lambda \notin \Sigma_1 \cup \Sigma_2$ then $(xhh')_\lambda = (xh)_\lambda = x_\lambda = (xg)_\lambda$. If $\lambda \in \Sigma_1 \setminus \Sigma_2$ then $(xhh')_\lambda = (xh)_\lambda = x_\lambda g_{\lambda,x} = (xg)_\lambda$, and if $\lambda \in \Sigma_2$ then $(xhh')_\lambda = (xh)_\lambda g_{\lambda,(xh)h^{-1}} = x_\lambda g_{\lambda,x} = (xg)_\lambda$. Therefore $g = hh'$, which proves the proposition.

We shall finally show that the normal subgroups constructed in Proposition 7.1 are themselves generalized wreath products. We remark that a similar result holds for generalized restricted wreath products (Thm. 4.2 in [1]). Let Λ be a poset, G_λ a permutation group on the non-empty set X_λ for $\lambda \in \Lambda$, let S be a systematic subset of $X = \prod_{\lambda\in\Lambda} X_\lambda$, and let Σ be an ideal of Λ and $D(\Sigma)$ defined as in Proposition 7.1. For $\sigma \in \Sigma$ define $F(\sigma) = \{\lambda \in \Lambda \mid \lambda > \sigma$ and $\lambda \notin \Sigma\}$. Let $\bar{\Sigma} = \{(\sigma, y) \mid \sigma \in \Sigma,\ y \in p_{F(\sigma)}(S)\}$. We partially order $\bar{\Sigma}$ by $(\sigma_1, y_1) < (\sigma_2, y_2)$ if and only if $\sigma_1 < \sigma_2$ and $(y_1)_\lambda = (y_2)_\lambda$ for all $\lambda \in F(\sigma_2)$. Let $G_{(\sigma,y)} = G_\sigma$ and $X_{(\sigma,y)} = X_\sigma$ for all $y \in p_{F(\sigma)}(S)$ and $\sigma \in \Sigma$. Note that if $w \in p_{\Lambda\setminus\Sigma}(S)$ and $T(w) = \{(\sigma, y) \in \bar{\Sigma} \mid y_\lambda = w_\lambda$ for all $\lambda \in F(\sigma)\}$ then the mapping $(\sigma, y) \mapsto \sigma$ is an order-isomorphism $T(w) \to \Sigma$. Also note that this order-isomorphism induces a canonical bijection $\beta_w\colon \prod_{(\sigma,w)\in T(w)} X_{(\sigma,w)} \to \prod_{\sigma\in\Sigma} X_\sigma$. Let $\bar{X} = \prod_{(\sigma,y)\in\bar{\Sigma}} X_{(\sigma,y)}$ and $\bar{S} = \{x \in \bar{X} \mid$ for all $w \in p_{\Lambda\setminus\Sigma}(S)$ it follows that $\beta_w(p_{T(w)}(x)) \in p_\Sigma(S)\}$.

Theorem 7.2. *Given the notations and assumptions of the preceding paragraph, the set $\bar{S}$ is a systematic subset of $\bar{X}$, and $D(\Sigma)$ is isomorphic to $\bar{S}-\mathrm{WR}_{(\sigma,y)\in\bar{\Sigma}}\, G_{(\sigma,y)}$.*

Proof. We first show that $\bar{S}$ is systematic. Note that if $x \in S$ we can define $\bar{x} \in \bar{X}$ by $\bar{x}_{(\sigma,y)} = x_\sigma$ for all $\sigma \in \Sigma$, $y \in p_{F(\sigma)}(S)$. Then clearly $\bar{x} \in \bar{S}$. So we easily get condition (2). We now show (1). Let Δ be an ideal of $\bar{\Sigma}$, let $x, y \in \bar{S}$ and define $z \in \bar{X}$ by $z_{(\sigma,v)} = x_{(\sigma,v)}$ if $(\sigma, v) \in \Delta$ and $z_{(\sigma,v)} = y_{(\sigma,v)}$ otherwise. Now let $w \in p_{\Lambda \setminus \Sigma}(S)$, and consider $\beta_w(p_{T(w)}(z))$. We have $\beta_w(p_{T(w)}(x)), \beta_w(p_{T(w)}(y)) \in p_\Sigma(S)$. Note that if $\bar{\Delta} = \{\sigma \in \Sigma | (\sigma, v) \in \Delta$ where $v_\lambda = w_\lambda$ for all $\lambda \in F(\sigma)\}$ then clearly $\bar{\Delta}$ is an ideal of Σ, and hence also of Λ. Also note that we have $(\beta_w(p_{T(w)}(z)))_\sigma = (\beta_w(p_{T(w)}(x)))_\sigma$ if $\sigma \in \bar{\Delta}$ and $(\beta_w(p_{T(w)}(z)))_\sigma = (\beta_w(p_{T(w)}(y)))_\sigma$ if $\sigma \in \Sigma \setminus \bar{\Delta}$. Now let $\tilde{x}, \tilde{y} \in S$ such that $\beta_w(p_{T(w)}(x)) = p_\Sigma(\tilde{x})$ and $\beta_w(p_{T(w)}(y)) = p_\Sigma(\tilde{y})$. Define $\tilde{z}$ by $\tilde{z}_\lambda = \tilde{x}_\lambda$ if $\lambda \in \bar{\Delta}$ and $\tilde{z}_\lambda = \tilde{y}_\lambda$ otherwise. Then clearly $\tilde{z} \in S$, and we have $\beta_w(p_{T(w)}(z)) = p_\Sigma(\tilde{z}) \in p_\Sigma(S)$, hence $\bar{S}$ is systematic.

Let $H = \bar{S} - \mathrm{WR}_{(\sigma,y) \in \bar{\Sigma}} G_{(\sigma,y)}$, and define $\varphi\colon D(\Sigma) \to H$ as follows. If $g \in D(\Sigma)$, $x \in \bar{S}$, $(\sigma, y) \in \bar{\Sigma}$ then $(x(g\varphi))_{(\sigma,y)} = x_{(\sigma,y)} g_{\sigma,z}$ where $z \in S$ is such that $p_{F(\sigma)}(z) = y$ and $p_\Sigma(z) = \beta_{p_{\Lambda \setminus \Sigma}(z)}(p_{T(p_{\Lambda \setminus \Sigma}(z))}(x))$. We are going to prove that φ is the desired isomorphism.

First of all, we remark that $(g\varphi)\colon \bar{S} \to \bar{X}$ is a well-defined mapping. For this, note that such an element $z \in S$ exists, and that the definition is independent of the choice of z. Namely, if z' is another element with the same properties then $z_\lambda = z'_\lambda$ for all $\lambda > \sigma$, and hence $g_{\sigma,z} = g_{\sigma,z'}$.

Next, we want to show that $(g\varphi)(\bar{S}) \subseteq \bar{S}$. Let $x \in \bar{S}$, and let $w \in p_{\Lambda \setminus \Sigma}(S)$. We have to prove that $\beta_w(p_{T(w)}(x(g\varphi))) \in p_\Sigma(S)$. Define $u \in S$ by $p_{\Lambda \setminus \Sigma}(u) = w$ and $p_\Sigma(u) = \beta_w(p_{T(w)}(x))$. We claim that $\beta_w(p_{T(w)}(x(g\varphi))) = p_\Sigma(ug)$. Let $(\sigma, y) \in T(w)$. Then note that u has the property that $p_{F(\sigma)}(u) = y$ and as $p_{\Lambda \setminus \Sigma}(u) = w$, we also have $p_\Sigma(u) = \beta_{p_{\Lambda \setminus \Sigma}(u)}(p_{T(p_{\Lambda \setminus \Sigma}(u))}(x))$, hence we get $(x(g\varphi))_{(\sigma,y)} = x_{(\sigma,y)} g_{\sigma,u}$. On the other hand, we have $(ug)_\sigma = u_\sigma g_{\sigma,u} = x_{(\sigma,y)} x g_{\sigma,u}$, which establishes the claim.

We are now going to show that $g\varphi$ is bijective. For this, it is enough to prove that $(g\varphi)(g^{-1}\varphi)$ is the identity on $\bar{S}$. We recall that if $z \in S$, $\lambda \in \Lambda$ then $(g^{-1})_{\lambda,x} = (g_{\lambda,xg^{-1}})^{-1}$. Now let $x \in \bar{S}$, and let $\bar{x} = x(g\varphi)$. Then if $(\sigma, y) \in \bar{\Sigma}$ and $z \in S$ is such that $p_{F(\sigma)}(z) = y$ and $p_\Sigma(z) = \beta_{p_{\Lambda \setminus \Sigma}(z)}(p_{T(p_{\Lambda \setminus \Sigma}(z))}(x))$ then we have $\bar{x}_{(\sigma,y)} = x_{(\sigma,y)} g_{\sigma,z}$. Now consider $\bar{z} = zg \in S$. Note that $\bar{z}_\lambda = z_\lambda$ if $\lambda \notin \Sigma$, and hence $p_{F(\sigma)}(\bar{z}) = y$. Also note that $p_\Sigma(\bar{z}) = \beta_{p_{\Lambda \setminus \Sigma}(\bar{z})}(p_{T(p_{\Lambda \setminus \Sigma}(\bar{z}))}(\bar{x}))$. Hence we get $(\bar{x}(g^{-1}\varphi))_{(\sigma,y)} = \bar{x}_{(\sigma,y)}(g^{-1})_{\sigma,\bar{z}}$. Therefore $(x(g\varphi)(g^{-1}\varphi))_{(\sigma,y)} = (\bar{x}(g^{-1}\varphi))_{(\sigma,y)} = \bar{x}_{(\sigma,y)}(g^{-1})_{\sigma,\bar{z}} = \bar{x}_{(\sigma,y)}(g_{\sigma,\bar{z}g^{-1}})^{-1} = (x_{(\sigma,y)} g_{\sigma,z})(g_{\sigma,zgg^{-1}})^{-1} = x_{(\sigma,y)}$. Hence $(g\varphi)(g^{-1}\varphi)$ is the identity on $\bar{S}$.

In the same way, we can show that $(gh)\varphi = (g\varphi)(h\varphi)$, hence φ is a homomorphism $\varphi\colon D(\Sigma) \to \mathrm{Sym}(\bar{S})$. We then have to prove that $g\varphi \in H$. Let $x, x' \in \bar{S}$ with $x\, e(\sigma, y)\, x'$. Then we have $x_{(\sigma_1,y_1)} = x'_{(\sigma_1,y_1)}$ whenever $(\sigma_1, y_1) > (\sigma, y)$. Let $z \in S$ be such that $p_{F(\sigma)}(z) = y$ and $p_\Sigma(z) = \beta_{p_{\Lambda \setminus \Sigma}(z)}(p_{T(p_{\Lambda \setminus \Sigma}(z))}(x))$. We then also have $p_\Sigma(z) = \beta_{p_{\Lambda \setminus \Sigma}(z)}(p_{T(p_{\Lambda \setminus \Sigma}(z))}(x'))$, hence we get $(x'(g\varphi))_{(\sigma,y)} = x'_{(\sigma,y)} g_{\sigma,z}$. From this it follows easily that $g\varphi \in H$.

It now remains to prove that φ is bijective. We first want to show that it is injective, that is, if $1 \neq g \in D(\Sigma)$ then $g\varphi \neq 1$. Let $1 \neq g \in D(\Sigma)$. Then there exists $\tilde{x} \in S$ such that $\tilde{x}g \neq \tilde{x}$, hence there exists $\sigma \in \Sigma$ such that $(\tilde{x}g)_\sigma \neq \tilde{x}_\sigma$, that is, $\tilde{x}_\sigma g_{\sigma, \tilde{x}} \neq \neq \tilde{x}_\sigma$. Define $x \in \bar{S}$ by $x_{(\bar{\sigma}, y)} = \tilde{x}_{\bar{\sigma}}$ for all $(\bar{\sigma}, y) \in \bar{\Sigma}$. As S is systematic we have $x \in \bar{S}$. Let $y = p_{F(\sigma)}(\tilde{x})$. We then have $(x(g\varphi))_{(\sigma, y)} = x_{(\sigma, y)} g_{\sigma, \tilde{x}} \neq \tilde{x}_\sigma = x_{(\sigma, y)}$, as we have $p_{F(\sigma)}(\tilde{x}) = y$ and also $p_\Sigma(\tilde{x}) = \beta_{p_{A \setminus \Sigma}(\tilde{x})}(p_{T(p_{A \setminus \Sigma}(\tilde{x}))}(x))$. Therefore $g\varphi \neq 1$, and hence φ is injective.

Finally, we show that φ is surjective. Let $h \in H$. Then for $(\sigma, y) \in \bar{\Sigma}$, $\bar{x} \in \bar{S}$, we have $h_{(\sigma, y), \bar{x}}$ such that $(x''h)_{(\sigma, y)} = x''_{(\sigma, y)} h_{(\sigma, y), \bar{x}}$ for all $x'' \in \bar{S}$ such that $x'' \, e(\sigma, y) \, \bar{x}$. We define $g \in D(\Sigma)$ in the following way. If $x \in S$ then $(xg)_\lambda = x_\lambda$ whenever $\lambda \notin \Sigma$, and $(xg)_\lambda = x_\lambda h_{(\lambda, y), q}$ where $y = p_{F(\lambda)}(x)$ and $q \in \bar{S}$ is defined by $q_{(\sigma, y)} = x_\sigma$ for all $(\sigma, y) \in \bar{\Sigma}$ if $\lambda \in \Sigma$. Using the same techniques as above, it follows that $g \in D(\Sigma)$, and that $g\varphi = h$. This proves the theorem.

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